

EconS 503 - Microeconomic Theory II

Homework #5 - Answer key

11.14 **PBEs when three players are uninformed.** Consider the game tree in figure 11.8, based on Gonzalez-Diaz and Melendez-Jimenez (2014), where player 3 has type I with probability p and type II with probability $1 - p$, where $p \in [0, 1]$. Once the type of player 3 is realized, but without observing it, player 1 chooses whether to continue the game (C) or to stop (S). If player 1 chooses S , the game is over; but if he chooses C , player 2 responds with U or D . Player 2 observes player 1's action (C), but does not observe player 3's type. Finally, player 3 is called to move if his type is I, responding with u or d , but without observing player 2's action.

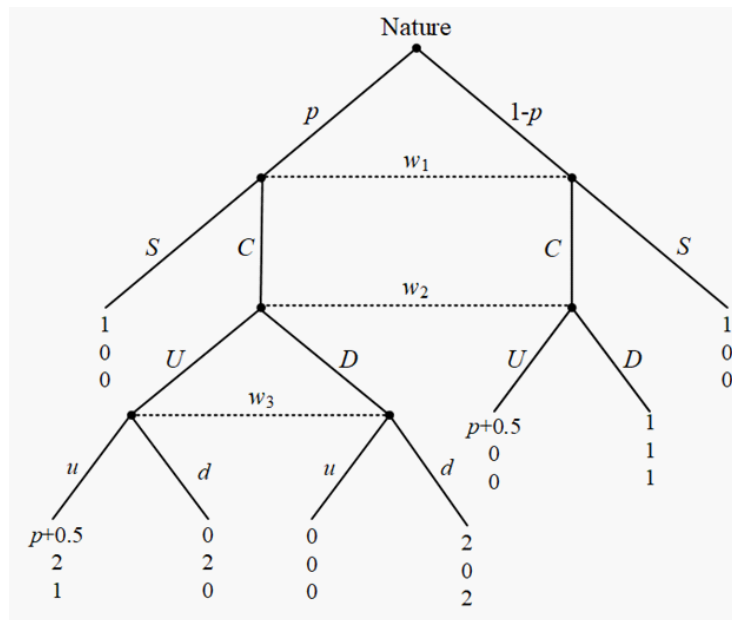
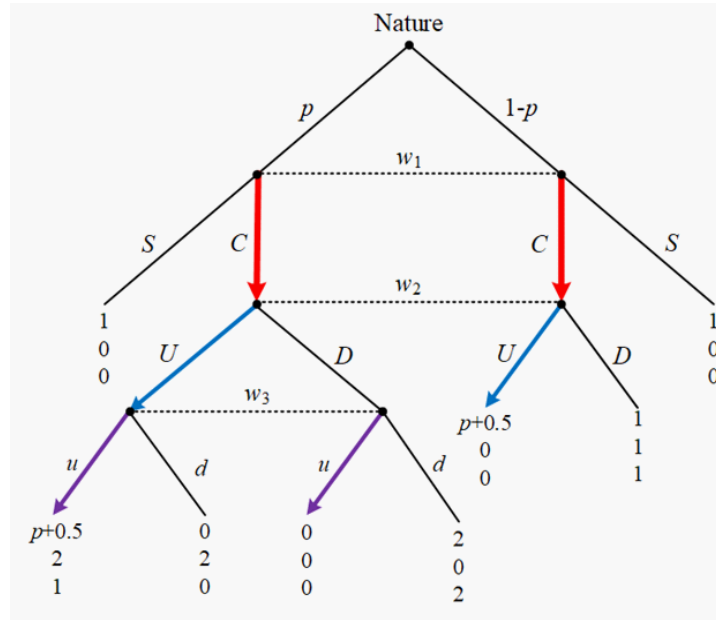


Figure 11.8 - PBEs when three players are uninformed.

(a) Show that strategy profile (C, U, u) can be supported as a PBE of the game.

- The figure below shades the branches associated with strategy profile (C, U, u) .



- *Last mover, player 3.* Because players 1 and 2 do not observe the move of nature (whether player 3's type is I or II), their actions cannot be used as messages that convey or conceal player 3's type. However, the fact that player 3 is called on to move means that player 3's type is I. Therefore, player 3 chooses u if and only if

$$EU_3(u) = 1\sigma_2(U) + 0(1 - \sigma_2(U)) > 0\sigma_2(U) + 2(1 - \sigma_2(U)) = EU_3(d)$$

where $\sigma_2(U)$ denotes the probability with which player 2 chooses U . This probability is not a function of player 3's type, since player 2 does not observe his type. However, player 2 observes whether player 1 chose to stop the game, S , or to continue it, C , since only in the latter case player 2 is called to respond. Similarly, $1 - \sigma_2(U)$ represents the probability with which player 2 chooses D .

The above inequality simplifies to $\sigma_2(U) > \frac{2}{3}$. Intuitively, player 3 responds with u , as required by this strategy profile, at the only information set where he is called upon to move in this game, if player 2 is likely to play U .

- *Second-to-last mover, player 2.* As described above, player 2 is only called to move when player 1 continues the game, playing C . However, player 2 anticipates player 3's response, which depends on $\sigma_2(U)$, giving rise to two cases:
 - If player 2 chooses U , $\sigma_2(U) = 1$, implying that player 3 responds with u since $\sigma_2(U) > \frac{2}{3}$, yielding an expected payoff of $EU_2(U) = 2p + 0(1 - p) = 2p$.
 - If, instead, player 2 chooses D , $\sigma_2(U) = 0$, entailing that player 3 responds with d since $\sigma_2(U) \leq \frac{2}{3}$, yielding an expected payoff of $EU_2(D) = 0p + 1(1 - p) = 1 - p$.

Therefore, player 2 chooses U if his expected payoffs satisfy $2p > 1 - p$ or, after rearranging, $p > \frac{1}{3}$; and D otherwise.

- *First mover, player 1.* Anticipating player 2's and 3's responses, but without observing player 3's type, player 1 chooses C , as prescribed in this strategy profile, if and only if

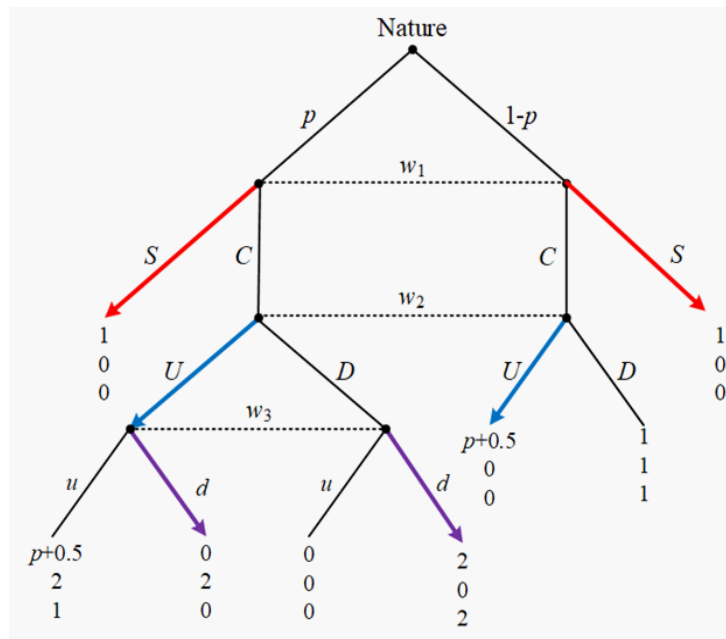
$$EU_1(C) = p(p + 0.5) + (1 - p)(p + 0.5) > p1 + (1 - p)1 = EU_1(S)$$

which simplifies to $p + 0.5 > 1$ or, solving for p , we obtain $p > 0.5$.

- *Summary.* In conclusion, the strategy profiles (C, U, u) can be supported as a PBE of this game if the prior probability satisfies $p > \frac{1}{2}$.

(b) Show that strategy profile (S, U, d) cannot be supported as a PBE of the game.

- The figure below depicts this strategy profile.



- *Last mover, player 3.* Following a similar approach as in part (a), we find that player 3 chooses d if and only if

$$EU_3(u) = 1\sigma_2(U) + 0(1 - \sigma_2(U)) < 0\sigma_2(U) + 2(1 - \sigma_2(U)) = EU_3(d)$$

where $\sigma_2(U)$ denotes the probability with which player 2 chooses U . The above inequality simplifies to $\sigma_2(U) < \frac{2}{3}$. Intuitively, player 3 responds with d , as required by this strategy profile, at the only information set where he is called upon to move in this game, if player 2 is likely to play D .

- *Second-to-last mover, player 2.* As described above, player 2 is only called to move when player 1 continues the game, playing C . However, player 2's information set in this strategy profile is only reached off-the-equilibrium path. His off-the-equilibrium beliefs must, nonetheless, coincide with prior

p since player 1 does not observe the move of nature, thus implying that a deviation from S to C cannot be type-dependent, that is,

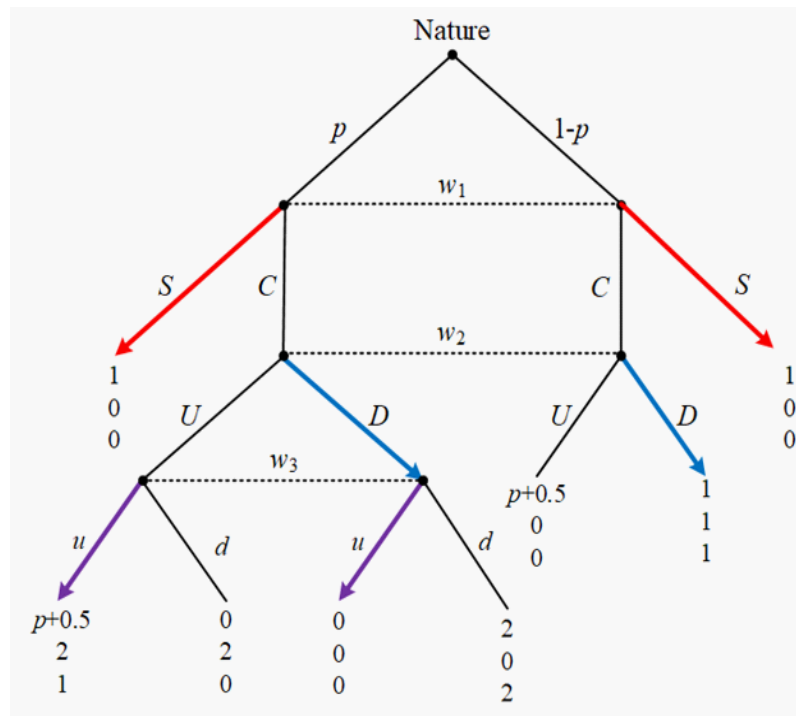
$$\mu(\text{type } I|C) = \frac{p\sigma_1(C)}{p\sigma_1(C) + (1-p)\sigma_1(C)} = \frac{p\sigma_1(C)}{[p + (1-p)]\sigma_1(C)} = p$$

This is an important point, as it helps us evaluate player 2's expected payoff from responding with U or D as functions of prior probability p , instead of a generic off-the-equilibrium belief $\mu(\text{type } I|C) \in [0, 1]$. Following our approach in part (a), we know that:

- If player 2 chooses U , $\sigma_2(U) = 1$, implying that player 3 responds with u since $\sigma_2(U) > \frac{2}{3}$, yielding an expected payoff of $EU_2(U) = 2p + 0(1-p) = 2p$.
- If, instead, player 2 chooses D , $\sigma_2(U) = 0$, entailing that player 3 responds with d since $\sigma_2(U) \leq \frac{2}{3}$, yielding an expected payoff of $EU_2(D) = 0p + 1(1-p) = 1-p$.

Therefore, player 2 chooses U if his expected payoffs satisfy $2p > 1-p$ or, after rearranging, $p > \frac{1}{3}$, but D otherwise. This implies that U and d are incompatible: when player 2 chooses U player 3 responds with u , and when player 2 chooses D player 3 responds with d .

- As a consequence, the strategy profile (S, U, d) cannot be supported as a PBE.
- (c) Show that strategy profile (S, D, u) cannot be supported as a PBE of the game.
- The next figure shades the branches of this strategy profile.



- Following the same logic as in part (b), player 3 responds with d when player 2 chose D , implying that strategy profile (S, D, u) cannot be supported as a PBE.

(d) Which of the PBEs found in parts (a)-(c) survive the Intuitive Criterion?

- The PBE in part (a) has all information sets being reached with positive probability, so there are no possible deviation to off-the-equilibrium messages that could help us update the beliefs of subsequent players. As a consequence, strategy profile (C, U, u) survives the Intuitive Criterion.

(e) Which of the PBEs found in parts (a)-(c) are sequential equilibria (SE)?

- We know that every SE is also a PBE, but the converse is not necessarily true. Therefore, the only strategy profile we need to consider is that from part (a), (C, U, u) , which we found to be a PBE, and we will now check if it is a SE as well.
- Consider a very small positive number $\varepsilon > 0$ applied to the following strategy profile,

$$(\varepsilon^{k_1} S (1 - \varepsilon^{k_1}) C, \varepsilon^{k_2} D (1 - \varepsilon^{k_2}) U, \varepsilon^{k_3} d (1 - \varepsilon^{k_3}) u),$$

for $k_1, k_2, k_3 \in \mathbb{N}$. Intuitively, the first component, $\varepsilon^{k_1} S (1 - \varepsilon^{k_1}) C$, indicates that player 1 chooses S with probability ε^{k_1} and C with probability $(1 - \varepsilon^{k_1})$. Similarly, $\varepsilon^{k_2} D (1 - \varepsilon^{k_2}) U$ denotes that player 2 chooses D with probability ε^{k_2} and U with the remaining probability; and $\varepsilon^{k_3} d (1 - \varepsilon^{k_3}) u$ represents that player 3 chooses d with probability ε^{k_3} and u with the remaining probability.

- *Player 3.* When $k = \max \{k_1, k_2, k_3\} \geq K$, where K is a sufficiently large positive integer, the expected payoff of player 3 choosing u is

$$\varepsilon^k 0 + (1 - \varepsilon^k) [\varepsilon^k 0 + (1 - \varepsilon^k) 1]$$

while that of choosing d is

$$\varepsilon^k 0 + (1 - \varepsilon^k) [\varepsilon^k 2 + (1 - \varepsilon^k) 0]$$

so that this player has no incentives deviating to d if and only if $1 - \varepsilon^k \geq 2\varepsilon^k$, which simplifies to $\varepsilon^k \leq \frac{1}{3}$, and holds when

$$\varepsilon \leq 3^{-\frac{1}{k}}$$

where we can always identify an ε within the neighborhood of 0, $B(0)$, that satisfies the above inequality for any finite positive integer k .

- *Player 2.* In the same vein, the expected payoff of player 2 choosing U is

$$p [\varepsilon^k 0 + (1 - \varepsilon^k) [\varepsilon^k 2 + (1 - \varepsilon^k) 2]] + (1 - p) [\varepsilon^k 0 + (1 - \varepsilon^k) 0]$$

while that of choosing D is

$$p [\varepsilon^k 0 + (1 - \varepsilon^k) [\varepsilon^k 0 + (1 - \varepsilon^k) 0]] + (1 - p) [\varepsilon^k 0 + (1 - \varepsilon^k) 1]$$

so that this player has no incentives deviating to D if and only if

$$2p(1 - \varepsilon^k) \geq (1 - p)(1 - \varepsilon^k)$$

that simplifies to

$$p \geq \frac{1}{3}$$

which holds in the PBE found in part (a), which is sustained for all $p > \frac{1}{2}$.

- *Player 1.* In this context, the expected payoff of player 1 choosing C is

$$p \left[\varepsilon^{2k} 2 + \varepsilon^k (1 - \varepsilon^k) (0 + 0) + (1 - \varepsilon^k)^2 (p + 0.5) \right] + (1 - p) \left[\varepsilon^k 1 + (1 - \varepsilon^k) (p + 0.5) \right]$$

$$= \frac{p(5 + 2p)\varepsilon^{2k} - (2p^2 + 5p - 1)\varepsilon^k + (2p + 1)}{2}$$

while that of choosing S is

$$p1 + (1 - p)1$$

so that this player has no incentives deviating to S if and only if

$$p(5 + 2p)\varepsilon^{2k} - (2p^2 + 5p - 1)\varepsilon^k + (2p - 1) \geq 0$$

which, when $\varepsilon \rightarrow 0$, the above inequality becomes

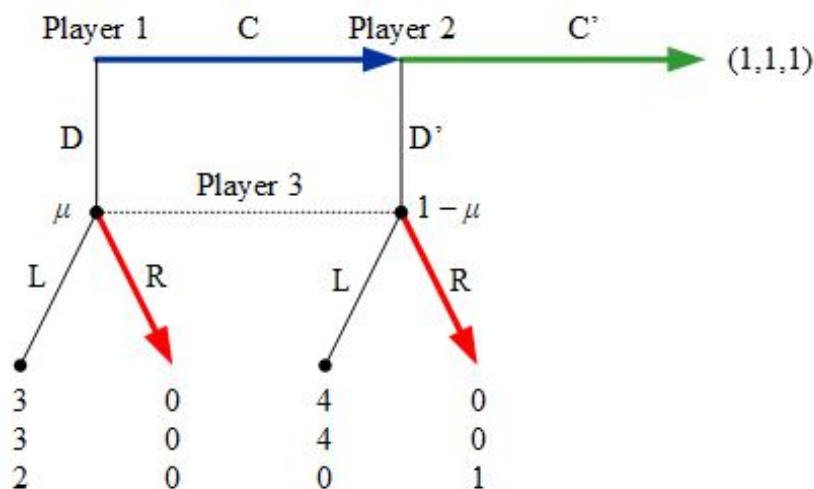
$$p \geq \frac{1}{2}$$

that holds in the PBE found in part (a), which is supported for all $p > \frac{1}{2}$.

- Since neither player has incentives to unilaterally deviate, the PBE we found in part (a), (C, U, u) , can be supported as a SE. For other refinement criteria in this type of sequential-move games, see Gonzalez-Diaz and Melendez-Jimenez (2014).

11.16 Finding sequential equilibria in the Selten's horse. Consider exercise 10.12, about the Selten's horse, where we found that $\{C, C', R\}$ is the unique PBE, which can be supported when player 3's off-the-equilibrium belief, μ , satisfies $\mu \leq \frac{1}{3}$. Show that this PBE is also a sequential equilibrium.

- We need to construct a sequence of totally mixed behavioral strategies, $\{b_1^k, b_2^k, b_3^k\}_{k=1}^\infty$, and their corresponding beliefs, $\{\mu^k\}_{k=1}^\infty$



- To show this result, recall that when players 1 and 2 choose a sequence of totally mixed behavioral strategies $\{b_1^k, b_2^k\}_{k=1}^\infty$, player 3's information set is reached with positive probability. Hence, we no longer face off-the-equilibrium information sets, and player 3's beliefs can be updated by Bayes' rule. In particular, consider

$$\{b_1^k, b_2^k\}_{k=1}^\infty = \left(\underbrace{(\varepsilon^k, 1 - \varepsilon^k)}_{\text{Player 1}}, \underbrace{(4\varepsilon^k, 1 - 4\varepsilon^k)}_{\text{Player 2}} \right)$$

where $\varepsilon^k > 0$ denotes a small perturbation. Using Bayes' rule to update player 3's beliefs, we obtain that

$$\mu^k = \frac{\varepsilon^k}{\varepsilon^k + (1 - \varepsilon^k) 4\varepsilon^k} = \frac{1}{5 - 4\varepsilon^k}$$

which converges to $\frac{1}{5}$ when $k \rightarrow +\infty$.

- Given this updated belief, player 3 responds with R since $2\mu \leq 1 - \mu$ holds for all $\mu \leq \frac{1}{3}$. Anticipating this response, player 2 chooses C' . Finally, anticipating (C', R) , player 1's optimal strategy is C . As a consequence, the PBE $\{C, C', R\}$ is also a sequential equilibrium.
- *Remark.* Note that other sequences of totally mixed behavioral strategies also work. Consider, for instance,

$$\{b_1^k, b_2^k\}_{k=1}^\infty = \left(\underbrace{(\varepsilon^k, 1 - \varepsilon^k)}_{\text{Player 1}}, \underbrace{(a\varepsilon^k, 1 - a\varepsilon^k)}_{\text{Player 2}} \right)$$

where parameter a satisfies $a > 1$. Using Bayes' rule to update player 3's beliefs, we obtain that

$$\mu^k = \frac{\varepsilon^k}{\varepsilon^k + (1 - \varepsilon^k) a\varepsilon^k} = \frac{1}{1 + a(1 - \varepsilon^k)}$$

which converges to $\frac{1}{1+a}$ when $k \rightarrow +\infty$. Since we know that player 3 responds with R when $2\mu \leq 1 - \mu$, which holds for all $\mu \leq \frac{1}{3}$, we require that $\frac{1}{1+a} \leq \frac{1}{3}$, which simplifies to $a \geq 2$. Therefore, any sequence of totally mixed behavioral strategies satisfying $a \geq 2$ can be used to show that the PBE $\{C, C', R\}$ is a sequential equilibrium.

13.3 A binary version of Crawford and Sobel. Consider the cheap talk model in section 13.4, but assume that $\theta = \theta_H$ with probability p and $\theta = \theta_L$ with probability $1 - p$. The sender (lobbyist) privately observes the realization of θ (whether it is θ_H or θ_L) and sends a binary message, either H or L , to the politician. The politician observes this message, updates her beliefs about θ , and responds with a binary response as well (h or l upon observing H , and h' or l' upon observing L).

(a) Describe the payoffs of each player in each terminal node.

- When the state of nature is θ_H , and:

- The politician responds with h or h' , whose payoff is $-(\theta_H - \theta_H)^2 = 0$, and the lobbyist's is $-(\theta_H - (\theta_H + \delta))^2 = -\delta^2$.
 - The politician responds with l or l' , whose payoff is $-(\theta_L - \theta_H)^2 = -(\theta_H - \theta_L)^2$, and the lobbyist's is $-(\theta_L - (\theta_H + \delta))^2 = -(\theta_L - \theta_H - \delta)^2$. Since $\theta_L - \theta_H < 0$, we can rewrite this expression as $-(\theta_L - \theta_H - \delta)^2 = -(\theta_H - \theta_L + \delta)^2$.
 - When the state of nature is θ_L , and:
 - The politician responds with l or l' , whose payoff is $-(\theta_L - \theta_L)^2 = 0$, and the lobbyist's is $-(\theta_L - \theta_L - \delta)^2 = -\delta^2$.
 - The politician responds with h or h' , whose payoff is $-(\theta_H - \theta_L)^2$, and the lobbyist's is $-(\theta_H - (\theta_L + \delta))^2 = -(\theta_H - \theta_L - \delta)^2$
 - $-(\theta_H - \theta_L - \delta)^2$.
- (b) Under which condition on the preference divergence parameter, δ , can you sustain a separating PBE where the sender chooses message H (L) when his type is θ_H (θ_L , respectively)?
- Beliefs are $\mu(\theta_H|H) = 1$ after observing H and $\mu(\theta_L|L) = 1$ after observing L .
 - The politician's responses, given the above beliefs, are:
 - Upon observing message H , she responds with policy h since $0 > -(\theta_H - \theta_L)^2$.
 - Upon observing message L , she responds with policy l' since $0 > -(\theta_H - \theta_L)^2$.
 - The lobbyist (sender) chooses the following messages:
 - When his type is θ_H , he chooses message H , as prescribed in this strategy profile, if $-\delta^2 > -(\theta_H - \theta_L + \delta)^2$, which simplifies to $\delta > -\frac{(\theta_H - \theta_L)}{2}$, which unambiguously holds.
 - Similarly, when his type is θ_L , he chooses message L , as prescribed in this strategy profile, if $-\delta^2 > -(\theta_H - \theta_L - \delta)^2$, which simplifies to $\delta < \frac{\theta_H - \theta_L}{2}$.
 - In summary, the separating strategy profile (H, L) can be supported when the politician's beliefs are $\mu(\theta_H|H) = 1$ after observing H , and $\mu(\theta_L|L) = 1$ after observing L , inducing responses (h, l') , if and only if the preference divergence parameter, δ , satisfies $\delta < \frac{\theta_H - \theta_L}{2}$.
- (c) If θ_H increases, is the above separating PBE more likely to hold? Interpret.
- An increase (decrease) in θ_H (θ_L) increases ratio $\frac{\theta_H - \theta_L}{2}$, thus relaxing the condition $\delta < \frac{\theta_H - \theta_L}{2}$. Intuitively, as the upward policy shift that the lobbyist induces by misreporting the state of nature becomes more severe, the condition on δ that sustains the separating strategy profile becomes less stringent, ultimately facilitating the emergence of this PBE.