EconS 503 - Microeconomic Theory II
Homework #4 - Answer key

9.5 **Equilibrium bidding in the FPA using the envelope theorem approach.** Consider a FPA with $N \geq 2$ bidders, each privately drawing his valuation for the object, v_i, from a uniform distribution. In this exercise, we show that the equilibrium bidding function in section 9.3 can be found using an alternative approach: instead of bidder i directly differentiating his expected utility from participating in the auction with respect to his bid b_i, we assume that he uses bidding function $b_i(z_i)$ where valuation z_i is allowed to coincide with his real valuation, $z_i = v_i$, or differ, $z_i \neq v_i$, and bidder i differentiates his expected utility with respect to valuation z_i.

(a) Find bidder i's expected utility from participating in the FPA.

- Bidder i wins the auction with probability $F(z_i)^{N-1} = z_i^{N-1}$, so his expected utility from participating in the auction, submitting a bid $b(z_i)$ when his true valuation is v_i, is
$$EU(z_i|v_i) = (z_i)^{N-1} \times [v_i - b(z_i)]$$

(b) Differentiate the expected utility of part (a) with respect to z_i. In equilibrium, every bidder i must have no incentives to bid according a valuation different than his true valuation, implying that $z_i = v_i$. Evaluate your first-order condition at $z_i = v_i$, rearrange, and find the equilibrium bidding function $b(v_i)$. Does your result coincide with that in section 9.4 (example 9.1)?

- Differentiating the expected utility of part (a) with respect to z_i, we find that
$$(N-1)(z_i)^{N-2}[v_i - b(z_i)] - (z_i)^{N-1}b'(z_i) = 0.$$
- In equilibrium, $z_i = v_i$, which implies that $b(z_i) = b(v_i)$. Inserting this property in the above first-order condition, we obtain
$$(N-1)(v_i)^{N-2}[v_i - b(v_i)] - (v_i)^{N-1}b'(v_i) = 0.$$

Rearranging, yields
$$(N-1)(v_i)^{N-2}b(v_i) + (v_i)^{N-1}b'(v_i) = (N-1)(v_i)^{N-1}.$$

The left-hand side can be alternatively represented as the derivative $\frac{d\left[(v_i)^{N-1} \times b(v_i)\right]}{dv_i}$, so the above first-order condition can be more compactly expressed as
$$\frac{d\left[(v_i)^{N-1} \times b(v_i)\right]}{dv_i} = (N-1)(v_i)^{N-1}.$$

At this point, recall that we seek to solve for the equilibrium bidding function $b(v_i)$, which is only in the derivative on the left-hand side. To eliminate the derivative, we integrate both sides of the above equality to obtain
$$v_i^{N-1} \times b(v_i) = \int_0^{v_i} (N-1)x^{N-1}dx.$$

Solving for $b(v_i)$, we find that the equilibrium bidding function in the FPA is

$$\begin{aligned} b(v_i) &= \frac{N-1}{{v_i}^{N-1}} \int_0^{v_i} x^{N-1} dx \\ &= \frac{N-1}{{v_i}^{N-1}} \left[\frac{x^N}{N} \right]_0^{v_i} \\ &= \frac{N-1}{{v_i}^{N-1}} \frac{v_i^N}{N} \\ &= \frac{N-1}{N} v_i \end{aligned}$$

which coincides with the equilibrium bidding function in section 9.4 (see example 9.1).

9.9 **FPA with risk averse bidders.** Consider a FPA where every bidder's valuation is drawn from a uniform distribution, $F(v_i) = v_i$, and $v_i \in [0, 1]$. Every bidder i is risk averse and exhibits a constant relative risk-aversion (CRRA) utility function $u(x) = x^\alpha$, where x is the bidder's income, and α denotes the Arrow-Pratt coefficient of relative risk-aversion, where $0 < \alpha < 1$. When $\alpha \to 1$, the utility function becomes almost linear in x, indicating that the individual's utility approaches risk neutrality; but as α decreases, the utility function becomes more concave in x, implying that the individual is more risk averse.

(a) *Two bidders.* Consider first a setting with only $N = 2$ bidders. Assuming that every bidder i uses a symmetric bidding function $b_i(v_i) = sv_i$, where $s \in (0, 1)$ denotes his bid shading, find the equilibrium bidding function in this setting.

- Every bidder i's expected utility maximization problem is

$$\max_{b_i \geq 0} \ \Pr(win) \times (v_i - b_i)^\alpha.$$

If every bidder uses bidding strategy $b_i = sv_i$, then $v_i = \frac{b_i}{s}$. Therefore, the probability of winning is $\Pr\{v_j < v_i\} = \frac{b_i}{s}$ because valuations are uniformly distributed. Inserting this probability in the above maximization problem, we obtain

$$\max_{b_i \geq 0} \ \frac{b_i}{s} \times (v_i - b_i)^\alpha.$$

- Differentiating with respect to b_i, yields

$$\frac{1}{s}(v_i - b_i)^\alpha - \frac{b_i}{s}\alpha(v_i - b_i)^{\alpha-1} = 0.$$

Rearranging the above expression, we obtain

$$v_i - b_i = \alpha b_i$$

Solving for b_i, we find bidder i's equilibrium bidding function

$$b_i(v_i) = \frac{1}{1+\alpha} v_i$$

(b) How is the equilibrium bidding function you found in part (b) affected by parameter α? Interpret.

- When the bidder becomes more risk averse (α decreases), he submits more aggressive bids because

$$\frac{\partial b_i(v_i)}{\partial \alpha} = -\frac{1}{(1+\alpha)^2} v_i < 0$$

That is, α and $b_i(v_i)$ move in different directions. When $\alpha \to 1$, bidder i's bidding function approaches $b_i = \frac{1}{2} v_i$, as in the setting with risk-neutral bidders we considered in the chapter. In contrast, when $\alpha \to 0$, his bidding function approaches $b_i = v_i$, indicating that he does not shade his bid.
This occurs because the risk-averse bidder seeks to minimize the probability of losing the auction. To understand this point, consider that bidder i reduces his bid from b_i to $b_i - \varepsilon$:

  - If he wins the auction, he obtains an additional profit of ε, since he has to pay a lower price for the object.
  - However, by lowering his bid, he increases the probability of losing the auction. Importantly, for a risk-averse bidder, the positive effect of getting the object at a cheaper price is offset by the negative effect of increasing the probability of losing the auction.

For a risk-averse individual, the disutility he suffers from the downside of a lottery is larger than the utility he experiences from the upside of a lottery. Overall, the risk-averse bidder does not have incentives to reduce his bid, but rather to increase it, relative to a risk-neutral bidder. In other words, the more risk-averse the bidder becomes, the less will his bid shading be. When $\alpha = 0$, the bidder is infinitely risk averse so that he does not shade his bid, that is, $b(v_i) = v_i$, and the bidding function coincides with the 45^o-line as depicted in figure 3.2.

(c) *More than two bidders.* Find the equilibrium bidding function in a setting with $N \geq 2$ bidders.

- Bidder i's expected utility maximization problem in this context becomes

$$\max_{b_i \geq 0} \left(\frac{b_i}{s}\right)^{N-1} \times (v_i - b_i)^{\alpha}.$$

where $\left(\frac{b_i}{s}\right)^{N-1}$ denotes the probability of winning (i.e., the probability that bidder i's valuation, v_i, exceeds that of his $N-1$ rivals).

- Differentiating the above expression with respect to b_i, yields

$$s^{-(N-1)} \left[(N-1)\, b_i^{N-2} (v_i - b_i)^{\alpha} - \alpha b_i^{N-1} (v_i - b_i)^{\alpha-1}\right] = 0.$$

which is rearranged to yield

$$\frac{b_i^{N-2} (v_i - b_i)^{\alpha-1}}{s^{N-1}} \left[(N-1)(v_i - b_i) - \alpha b_i\right] = 0.$$

Solving for b_i, we find bidder i's equilibrium bidding function

$$b_i(v_i) = \frac{N-1}{N-1+\alpha} v_i$$

which coincides with that of a first-price auction with risk-neutral bidders when $\alpha \to 1$, that is, $b_i(v_i) = \frac{N-1}{N} v_i$.

(d) How is the equilibrium bidding function affected by a marginal increase in α? And by a marginal increase in N?

- Differentiating this bidding function with respect to α, we find that

$$\frac{\partial b_i(v_i)}{\partial \alpha} = -\frac{N-1}{(N-1+\alpha)^2} v_i < 0$$

implying that bidders submit more (less) aggressive bids when they are more (less) risk averse. This result goes in line with that found in part (b).

- Differentiating the bidding function with respect to N, we obtain

$$\frac{\partial b_i(v_i)}{\partial N} = \frac{\alpha}{(N-1+\alpha)^2} v_i > 0$$

implying that bidders submit more aggressive bids as they face more competitors bidding for the object.

9.12 **Equilibrium bidding function in the APA.** Consider an APA where bidders draw their valuations from a uniform distribution, $F(v_i) = v_i$ where $v_i \in [0, 1]$, and there are $N \geq 2$ bidders.

(a) Assume that every bidder i uses a symmetric and strictly increasing bidding function $b(z_i)$, where $z_i \in [0, 1]$ denotes the valuation according to which he bids that can satisfy $z_i = v_i$ if he bids according to his true valuation for the object or $z_i \neq v_i$ if he does not. Write bidder i's expected utility maximization problem.

- The expected utility from participating in the APA is

$$EU_i = \underbrace{(z_i)^{N-1}}_{\Pr(win)} [v_i - b_i(z_i)] + \underbrace{\left[1 - (z_i)^{N-1}\right]}_{\Pr(lose)} (0 - b_i(v_i))$$

which simplifies to

$$EU_i = (z_i)^{N-1} v_i - b_i(z_i)$$

Intuitively, if winning, bidder i enjoys his valuation for the object, v_i, but he must pay the bid he submitted both when winning and losing the auction.

(b) Instead of differentiating the expected utility expression found in part (a) with respect to bidder i's bid, differentiate it with respect to z_i. This indirect approach is often known as the "envelope theorem" approach, while directly differentiating with respect to b_i is known as the "direct approach."

- Differentiating with respect to z_i, yields

$$(N-1)(z_i)^{N-2} v_i - \frac{db_i(z_i)}{dz_i} = 0$$

(c) Evaluate the first-order conditions that you found in part (b) at $z_i = v_i$ since every bidder i must have no incentives to bid according to a different valuation than v_i. Rearrange and find the equilibrium bidding function in this auction format. (For simplicity, you can assume that $b(0) = 0$, meaning that the bidder with the lowest valuation submits a bid of zero.)

- For bidding function $b_i(v_i)$ to maximize bidder i's utility, z_i must satisfy $z_i = v_i$, implying that the above first-order condition simplifies to

$$(N-1)(v_i)^{N-2} v_i = \frac{db_i(v_i)}{dv_i}$$

Given that this equality holds for every valuation, v_i, we can apply integrals on both sides of the equality to obtain,

$$\begin{aligned} b_i(v_i) &= \int_0^{v_i} (N-1) x^{N-2} x dx + C \\ &= (N-1) \int_0^{v_i} x^{N-1} dx + C, \end{aligned}$$

where C is the constant of integration, which is equal to 0 since the bidding function satisfies $b(0) = 0$, i.e., every bidder submits a bid of zero when his valuation is the lowest. Hence, the optimal bidding function becomes

$$\begin{aligned} b_i^{APA}(v_i) &= (N-1) \int_0^{v_i} x^{N-1} dx \\ &= (N-1) \left[\frac{x^N}{N} \right]_0^{v_i} \\ &= \frac{N-1}{N} v_i^N. \end{aligned}$$

(d) *Comparative statics.* Evaluate the equilibrium bidding function at $N = 2$, $N = 3$, and $N = 4$. Interpret.

- When $N = 2$ bidders, the optimal bidding function for the APA is $b_i^{APA}(v_i) = \frac{1}{2} v_i^2$; when $N = 3$, the bidding function becomes $b_i^{APA}(v_i) = \frac{2}{3} v_i^3$; and when $N = 4$, it is $b_i^{APA}(v_i) = \frac{3}{4} v_i^4$. Intuitively, the bidding function becomes more convex as more bidders compete for the object, indicating that every bidder submits more conservative bids when his valuation is low but more aggressive bids when his valuation is high.

(e) *Comparison with FPA.* Compare the equilibrium bidding function you found in part (c) against that in a FPA with uniformly distributed values. In which auction format the bidder submits more aggressive bids?

- The equilibrium bid in a FPA with $N \geq 2$ bidders drawing their valuations from a uniform distribution is $b_i^{FPA}(v_i) = \frac{N-1}{N}v_i$, whereas in the APA is $b_i^{APA}(v_i) = \frac{N-1}{N}v_i^N$. Comparing these equilibrium bidding functions, we obtain that

$$b_i^{FPA}(v_i) = \frac{N-1}{N}v_i > \frac{N-1}{N}v_i^N = b_i^{APA}(v_i)$$

simplifies to $1 > v_i^{N-1}$, which holds given that $v_i \in [0,1]$ by definition. Intuitively, every bidder i must pay his bid in the APA, whether he wins the object or not, inducing him to bid more conservatively than in the FPA.

10.7 **Brinkmanship game.** Consider the sequential-move game between two players in a dispute, depicted in figure 10.15. In the first stage, player 1 privately observes whether it is wimpy or surely, and chooses whether to stand firm or cave to the other players' demands. Player 2 does not observe player 1's type, but knows that it is surely with probability $p \in [0,1]$ and wimpy otherwise. If player 1 caves, the game is over. If player 1 stands firm, player 2 updates his belief about player 1' type, and responds starting a fight or not.

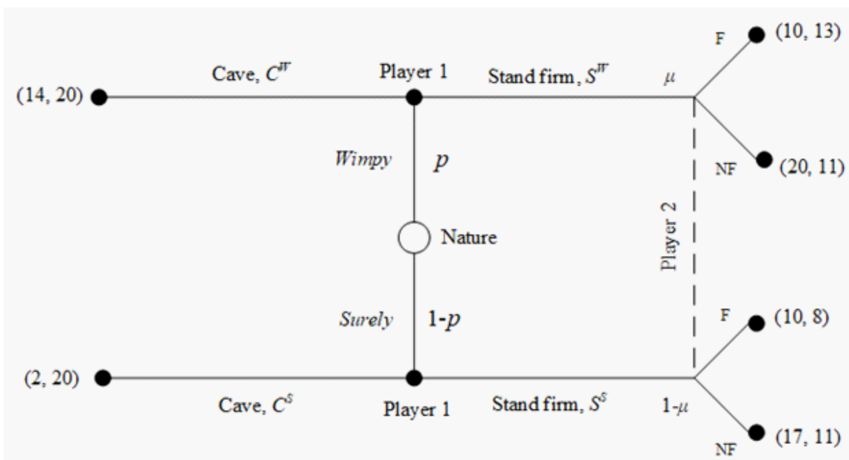


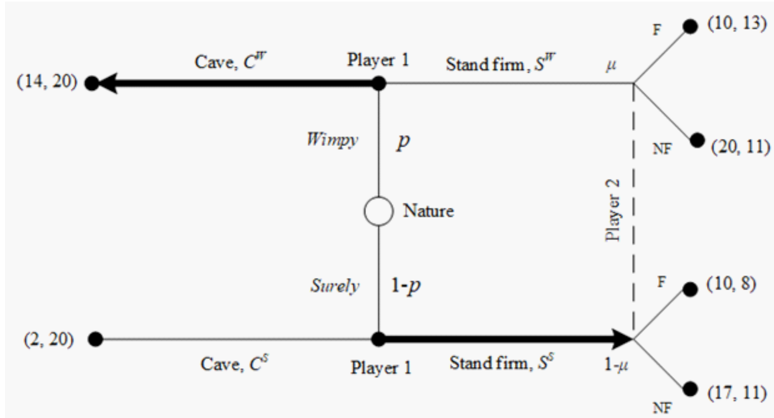
Figure 10.15 - Brinkmanship game.

(a) Can the separating strategy profile $S^{SU}C^{WI}$, where only the surely type of player 1 stands firm, be sustained as a PBE?

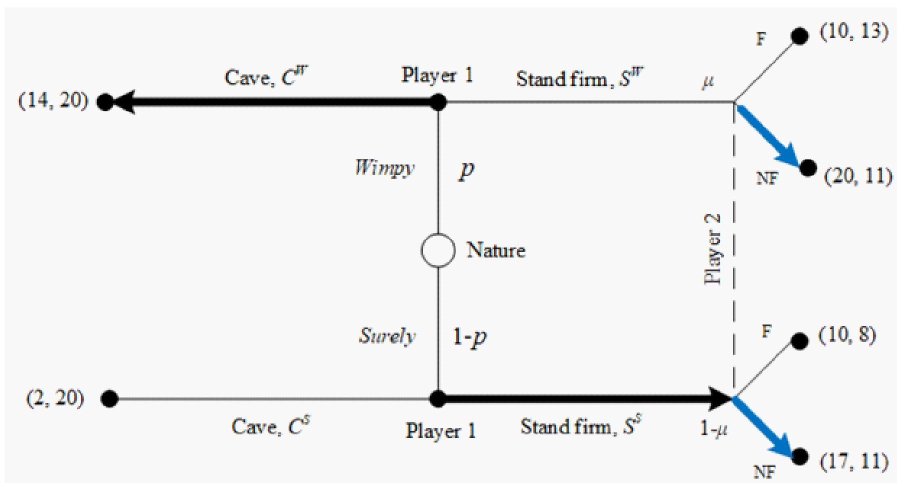
- *Step 1. Player 2's beliefs.* Player 2 (P2) updates his beliefs as follows:

$$\mu = \Pr(WI|S) = \frac{\Pr(S|WI)\Pr(WI)}{\Pr(S|WI)\Pr(WI) + \Pr(S|SU)\Pr(SU)} = \frac{p \times 0}{p \times 0 + (1-p) \times 1} = 0$$

which indicate that, when P2 observes P1 standing firm, it is convinced to deal with a surely P1.



- *Step 2. Player 2's response given his beliefs.*
  - After observing standing firm, P2 responds with no fight to earn a payoff of 11 since fighting entails a lower payoff of 8, as depicted at the bottom right-hand side of the figure.



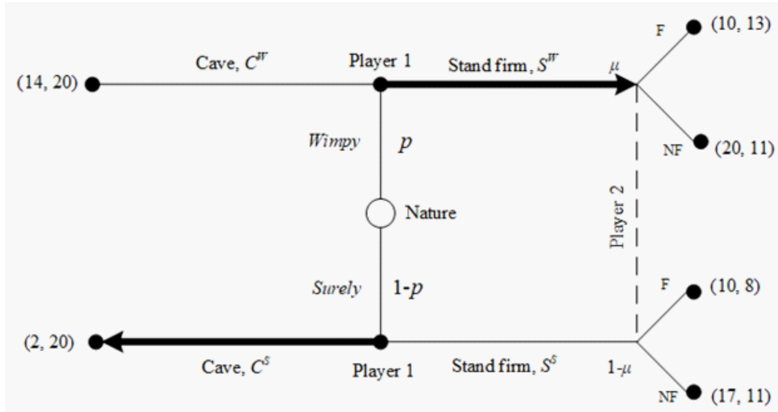
- *Step 3. Player 1's choice.* In this context, P1 behaves as follows:
  - The surely P1 (see bottom of the figure) has no incentives to deviate from standing firm to caving, as that would decrease his payoff from 17 to 2.
  - The wimpy P1 (see top of the figure) has incentives to deviate from caving to standing firm, as that would increase his payoff from 14 to 20.
- Therefore, the separating strategy profile, $\left(S^{SU}C^{WI}, NF\right)$, cannot be supported as a PBE of the game.

(b) Can the separating strategy profile $C^{SU}S^{WI}$, where only the wimpy type of player 1 stands firm, be sustained as a PBE?

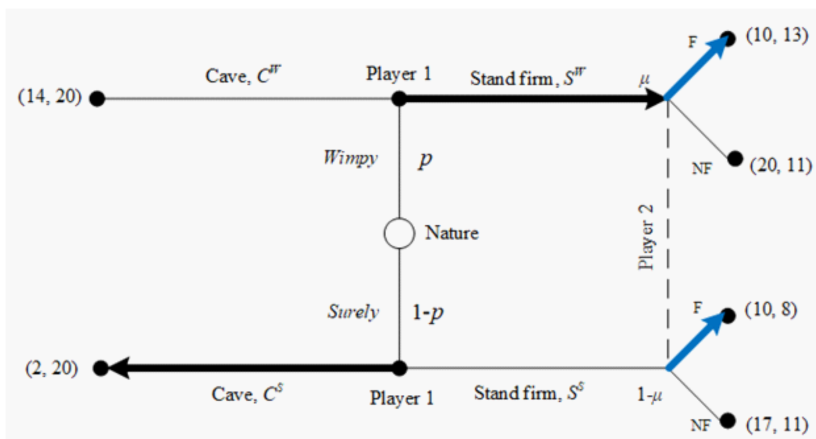
- *Step 1. Player 2's beliefs.* P2 updates its beliefs as follows.

$$\mu = \Pr\left(WI|\, S\right) = \frac{\Pr\left(S|\, WI\right)\Pr\left(WI\right)}{\Pr\left(S|\, WI\right)\Pr\left(WI\right) + \Pr\left(S|\, SU\right)\Pr\left(SU\right)} = \frac{p \times 1}{p \times 1 + (1-p) \times 0} = 1$$

which indicate that, when P2 observes P1 standing firm, it is convinced to deal with a wimpy type.



- *Step 2. Player 2's response given his beliefs.*
  - After observing standing firm, P2 responds fighting to earn a payoff of 13 since not fighting yields a lower payoff of 11, as depicted at the top right-hand side of the figure.



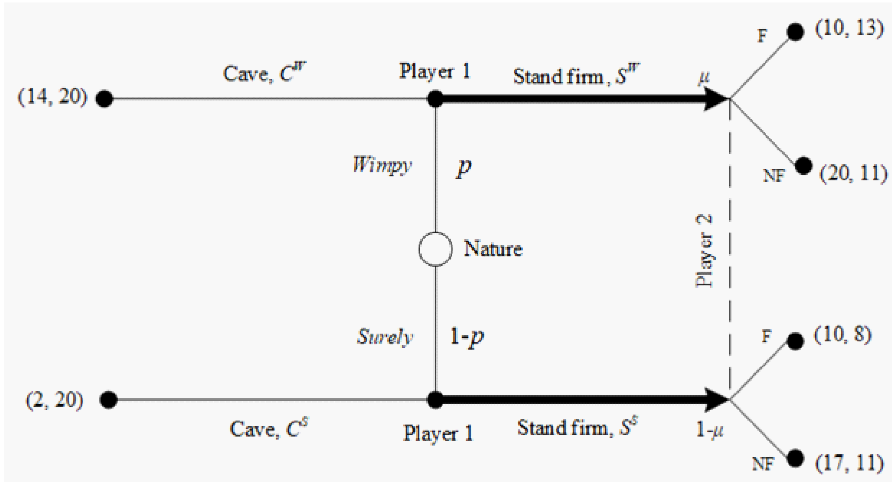
- *Step 3. Player 1's choice.* In this context, P1 behaves as follows:
  - The surely P1 (see bottom of the figure) has incentives to deviate from caving to standing firm, as that would increase his payoff from 2 to 10.
  - The wimpy P1 (see top of the figure) has incentives to deviate from standing firm to caving, as that would increase his payoff from 10 to 14.
- Therefore, the separating strategy profile, $\left(C^{SU}S^{WI}, F\right)$, cannot be supported as a PBE of the game.

(c) Can the pooling strategy profile $S^{SU}S^{WI}$, where both types of player 1 stand firm, be sustained as a PBE?

- *Step 1. Player 2's beliefs.* P2 updates its beliefs as follows.

$$\mu = \Pr\left(WI|\, S\right) = \frac{\Pr\left(S|\, WI\right)\Pr\left(WI\right)}{\Pr\left(S|\, WI\right)\Pr\left(WI\right) + \Pr\left(S|\, SU\right)\Pr\left(SU\right)} = \frac{p \times 1}{p \times 1 + (1-p) \times 1} = p$$

which indicate that, when P2 observes P1 standing firm, P2 cannot infer any addition information (i.e., posterior and prior beliefs coincide).



- *Step 2. Player 2's response given his beliefs.*
  - After observing standing firm, P2 responds fighting if its expected payoffs satisfy

$$E_2(F) = 13p + 8(1-p) \geq 11p + 11(1-p) = E_2(NF)$$

which simplifies to

$$8 + 5p \geq 11$$

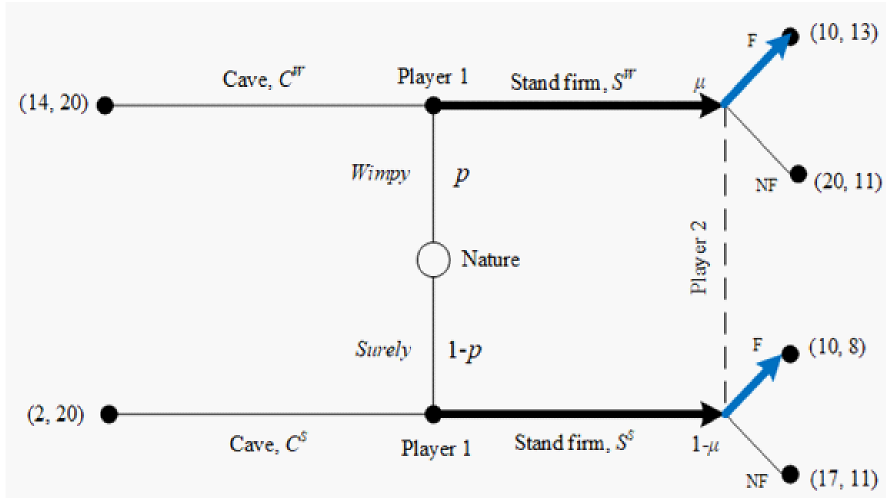
that, solving for p, yields

$$p \geq \frac{3}{5}$$

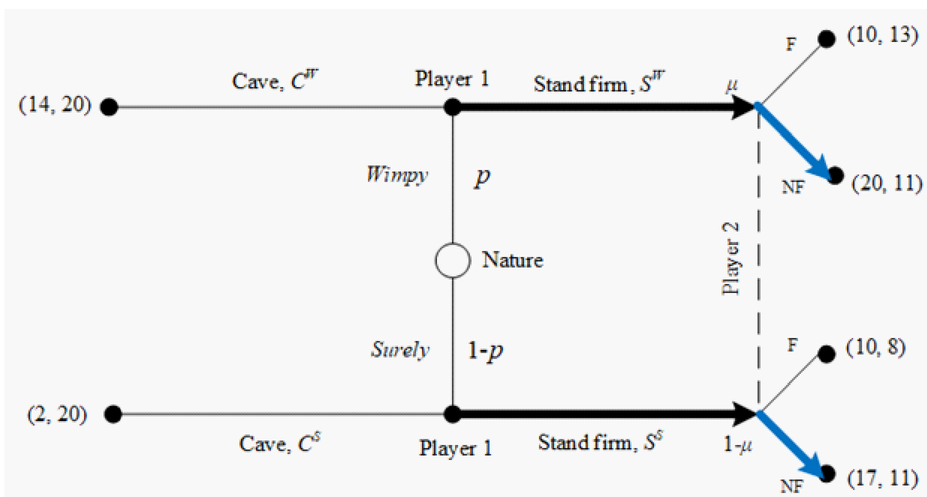
giving rise to two cases in Step 3, as we show next. Intuitively, P2 responds fighting when P1 is likely to be wimpy (relatively high p), but does not fight otherwise.

- *Step 3. Player 1's choice.* In this context, P1 behaves as follows:
  - When $p \geq \frac{3}{5}$, P2 responds fighting. As a consequence, the surely P1 (see bottom of the figure) has no incentives to deviate from standing firm to caving because that would decrease his payoff from 10 to 2. The wimpy P1 (see top of the figure), however, has incentives to deviate from standing

firm to caving as that increases his payoff from 10 to 14.



– When $p < \frac{3}{5}$, P2 responds not fighting. As a result, the surely P1 (see bottom of the figure) has no incentives to deviate from standing firm to caving since that would decrease his payoff from 17 to 2. Similarly, the wimpy P1 (see top of the figure) has no incentives to deviate from standing firm to caving as that would decrease his payoff from 20 to 14.



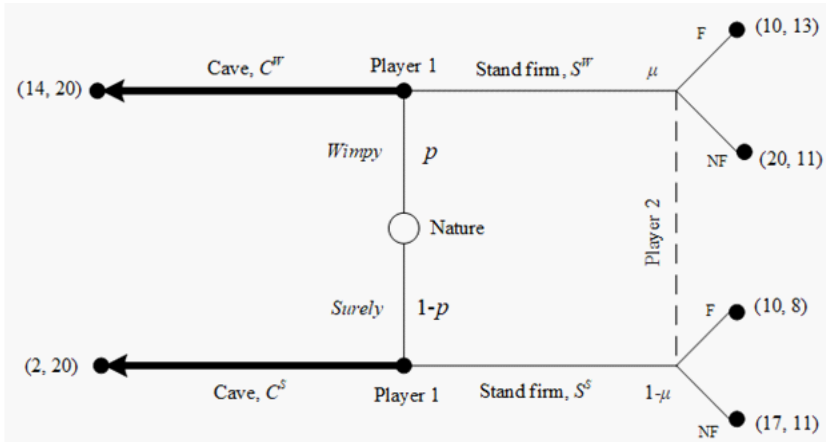
- Therefore, the pooling strategy profile $\left(S^{SU}S^{WI}, F\right)$ cannot be supported as a PBE of this game when prior beliefs satisfy $p \geq \frac{3}{5}$, but the pooling strategy profile $\left(S^{SU}S^{WI}, NF\right)$, can be supported as a PBE of this game when $p < \frac{3}{5}$.

(d) Can the pooling strategy profile $C^{SU}C^{WI}$, where both types of player 1 cave, be sustained as a PBE?

- *Step 1. Player 2's beliefs.* P2 updates its beliefs as follows.

$$\mu = \Pr\left(WI|\, S\right) = \frac{\Pr\left(S|\, WI\right)\Pr\left(WI\right)}{\Pr\left(S|\, WI\right)\Pr\left(WI\right) + \Pr\left(S|\, SU\right)\Pr\left(SU\right)} = \frac{p \times 0}{p \times 0 + (1-p) \times 0} = \frac{0}{0}$$

which is undefined, so we let $\mu \in [0, 1]$ unrestricted. Therefore, upon observing standing firm (which happens off-the-equilibrium path), P2's posterior belief, μ, is left unrestricted, $\mu \in [0, 1]$.



- *Step 2. Player 2's response given his beliefs.*
  - After observing standing firm, P2 responds fighting if his expected payoffs satisfy

    $$E_2(F) = 13\mu + 8(1-\mu) \geq 11\mu + 11(1-\mu) = E_2(NF)$$

    which simplifies to

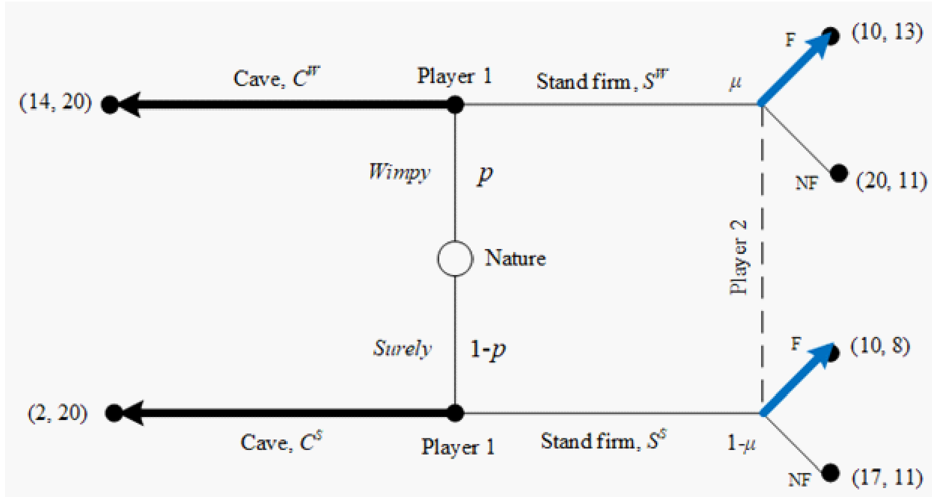
    $$8 + 5\mu \geq 11$$

    and, after solving for p, yields

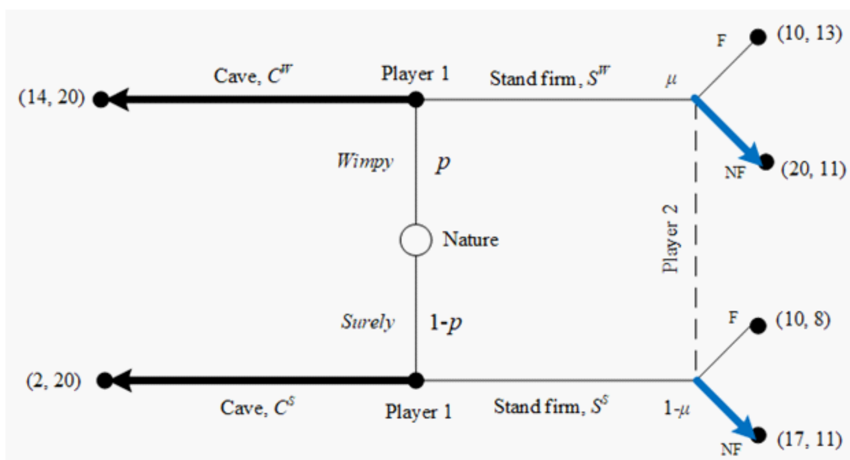
    $$\mu \geq \frac{3}{5}$$

    giving rise to two cases in Step 3, as we describe next. Intuitively, P2 responds fighting when it P1 is likely to be wimpy (relatively high μ), but does not fight otherwise.
- *Step 3. Player 1's choice.* In this context, P1 behaves as follows:
  - When $\mu \geq \frac{3}{5}$, P1 responds fighting. As a consequence, the surely P1 (see bottom of the figure) has incentives to deviate from caving to standing firm because that increases his payoff from 2 to 10. The wimpy P1 (see top of the figure), however, has no incentives to deviate from caving to

standing firm as that would decrease his payoff from 14 to 10.



– When $\mu < \frac{3}{5}$, P1 responds not fighting. As a result, the surely P1 (see bottom of the figure) has incentives to deviate from caving to standing firm as that increases his payoff from 2 to 17. Similarly, the wimpy P1 (see top of the figure) has incentives to deviate from caving to standing firm as that increases his payoff from 14 to 20.

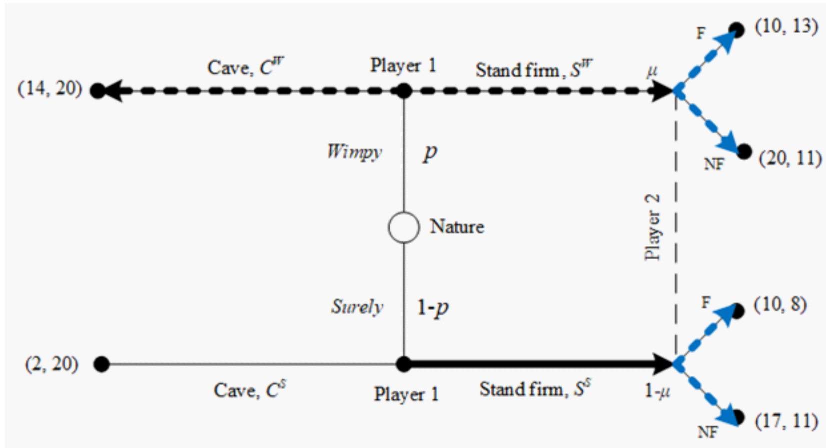


- Therefore, the pooling strategy profile $C^{SU}C^{WI}$ cannot be supported as a PBE regardless of P1's off-the-equilibrium beliefs (for all values of $\mu \in [0, 1]$).

(e) Find a semi-separating PBE where the surely type of player 1 stands firm with certainty (playing pure strategies) but the wimpy type randomizes between standing firm and caving (playing mixed strategies). [Hint: For the wimpy player 1 to standing firm with probability q, player 2 must also respond fighting with probability r.]

- *Step 1. Player 2's beliefs.*
  - Since the surely P1 finds it strictly dominant to stand firm, the wimpy P1 randomizes between standing firm and caving if and only if the payoff from caving, 14, equals the expected payoff of standing firm, $10r + 20(1 - r)$,

where r represents the probability that P2 responds fighting. Therefore, we need that $14 = 20 - 10r$ that simplifies to $r = \frac{3}{5}$. In other words, for the wimpy P1 to be indifferent between caving and standing firm, P2 must respond fighting with probability $r = \frac{3}{5}$. If, instead, $r > \frac{3}{5}$ ($r < \frac{3}{5}$), the wimpy P1 would prefer to cave (stand firm) using pure strategies.



– Since the wimpy P1 stands firm with probability q, P2 updates his posterior belief as follows

$$\mu = \Pr(WI|\,S) = \frac{\Pr(S|\,WI)\Pr(WI)}{\Pr(S|\,WI)\Pr(WI) + \Pr(S|\,SU)\Pr(SU)}$$
$$= \frac{p \times q}{p \times q + (1-p) \times 1} = \frac{pq}{1 - p(1-q)}$$

Intuitively, when $p = 0$, updated belief $\mu = \frac{pq}{1-p(1-q)}$ simplifies to $\mu = 0$, meaning that P2 is convinced of dealing with a surely P1. In contrast, when $p = 1$, his updated belief simplifies to $\mu = 1$, implying that P2 is convinced of facing a wimpy P1. More generally, μ increases in prior p since $\frac{\partial \mu}{\partial p} = \frac{q}{[1-p(1-q)]^2} > 0$, meaning that, as P1 is more likely to be wimpy, P2 assigns a larger probability weight on facing a wimpy P1.

- *Step 2. Player 2's response given his beliefs.*
  – After observing that P1 stands firm, P2 is indifferent between fighting and not fighting if and only if $EU_2(F) = EU_2(NF)$, or

$$13\mu + 8(1-\mu) = 11\mu + 11(1-\mu)$$

which simplifies to

$$8 + 5\mu = 11$$

and since $\mu = \frac{pq}{1-p(1-q)}$ from our above results in Step 1, we obtain that

$$8 + \frac{5pq}{1 - p(1-q)} = 11$$

that simplifies to

$$q = \frac{3(1-p)}{2p}$$

which is above 0 for all values of p, and does not exceed 1 if $3(1-p) \leq 2p$ that simplifies to $p \geq \frac{3}{5}$, implying that P1 must be sufficiently likely to be wimpy in order for P2 to randomize between fighting and not fighting.

- Therefore, P2 holds a posterior belief of

$$\mu = \frac{p\frac{3(1-p)}{2p}}{1-p\left(1-\frac{3(1-p)}{2p}\right)} = \frac{3}{5}$$

- *Step 3. Player 1's choice.* In this context, P1 behaves as follows:
  - When $p < \frac{3}{5}$, P1 is less likely to be wimpy, so that $E_2(NF) > E_2(F)$ for all values of p, implying that P2 does not fight. Consequently, both the surely and wimpy P1 will stand firm, resulting in the pooling strategy profile $\left(S^{SU}S^{WI}, NF\right)$ found in part (c).
  - When $p \geq \frac{3}{5}$, however, P1 is more likely to be wimpy, so with posterior belief $\mu = \frac{3}{5}$, P2 fights with probability $r = \frac{3}{5}$. In this context, the surely P1 stands firm while the wimpy P1 stands firm with probability $q = \frac{3(1-p)}{2p}$. In addition, this mixing is decreasing in the probability of P1 being wimpy because $\frac{\partial q}{\partial p} = -\frac{3}{2p^2} < 0$. Intuitively, when P1 is more likely to be wimpy, he cannot easily conceal his type by standing firm. Indeed, when $p \to 1$, the wimpy P1 stands firm with probability $q \to 0$.

10.12 **Selten's horse.** Consider the "Selten's Horse" game depicted in figure 10.17. Player 1 is the first mover in the game, choosing between C and D. If he chooses C, player 2 is called on to move between C' and D'. If player 2 selects C' the game is over. If player 1 chooses D or player 2 chooses D', then player 3 is called on to move without being informed whether player 1 chose D before him or whether it was player 2 who chose D'. Player 3 can choose between L and R, and then the game ends.

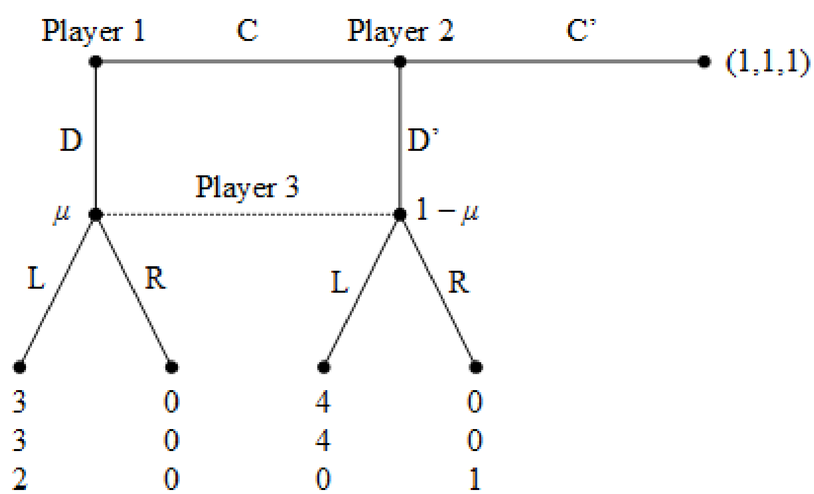


Figure 10.17. Selten's horse.

(a) Define the strategy spaces for each player. Then find all pure strategy Nash equilibria (psNE) of the game. [*Hint*: This is a three-player game, so you can consider that player 1 chooses rows, player 2 columns, and player 3 chooses matrices.]

- The strategy spaces of the players are as follows:

$$S_1 = \{C, D\}$$
$$S_2 = \{C', D'\}$$
$$S_3 = \{L, R\}$$

In Figure 2, we represent the strategies and payoffs of the three players in the following normal form representation of the game, where Player 1 chooses between the rows, Player 2 chooses between the columns, and Player 3 chooses between the matrixes.

Player 3 choosing L:

Player 1 \ Player 2	C'	D'
C	1, 1, 1	4, 4, 0
D	3, 3, 2	3, 3, 2

Player 3 choosing R:

Player 1 \ Player 2	C'	D'
C	1, 1, 1	0, 0, 1
D	0, 0, 0	0, 0, 0

Figure 2. Selten's horse - Matrix representation.

- We next underline the best responses of the three players in Figure 3, and identify that (C, C', R), and (D, C', L) are the pure strategy Nash equilibria of this game.

Player 3 choosing L:

Player 1 \ Player 2	C'	D'
C	$1, 1, \underline{1}$	$\underline{4}, \underline{4}, 0$
D	$\underline{3}, \underline{3}, \underline{2}$	$3, \underline{3}, \underline{2}$

Player 3 choosing R:

Player 1 \ Player 2	C'	D'
C	$\underline{1}, \underline{1}, \underline{1}$	$\underline{0}, 0, \underline{1}$
D	$0, \underline{0}, 0$	$\underline{0}, \underline{0}, 0$

Figure 3. Selten's horse - Underlining best response payoffs.

(b) Argue that one of the two psNEs you found in part (a) is not sequentially rational. A short verbal explanation suffices.

- (D, C', L) is not sequentially rational. If Player 1 chooses D, then Player 3's belief is $\mu = 1$, responding with L (see left-hand side at the bottom of the tree). Anticipating that Player 3 choosing L, Player 2 compares his payoff from C', 1, against that from D' (which is followed by Player 3 responding with L), 4, and thus chooses D'. Therefore, Player 2 choosing C' is not sequentially rational.

(c) Show that there is only one Perfect Bayesian equilibrium and it coincides with one of the pure strategy Nash equilibria you have identified in part (a).

- *Separating strategy profile* C, D'. First, we check the separating strategy profile, C, D', where Player 1 chooses C and Player 2 selects D.
As depicted in Figure 4, since player 1 chooses C (as illustrated by the blue horizontal arrow) and player 2 chooses D' (as illustrated by the green vertical arrow), Player 3's belief is totally concentrated on Player 2 choosing D' (right-hand node of his information set), entailing that $\mu = 0$ (that is, $1 - \mu = 1$). In this context, Player 3 is better off choosing R (as illustrated by the red arrows), which yields a payoff of 1, than choosing L, which yields a payoff of 0. However, given this response by Player 3, Player 2 maximizes his payoff by choosing *C'*, which provides him with a payoff of 1 (choosing D' gives him a payoff of 0). Therefore, the separating strategy profile C, D' cannot be supported as a PBE of this game.

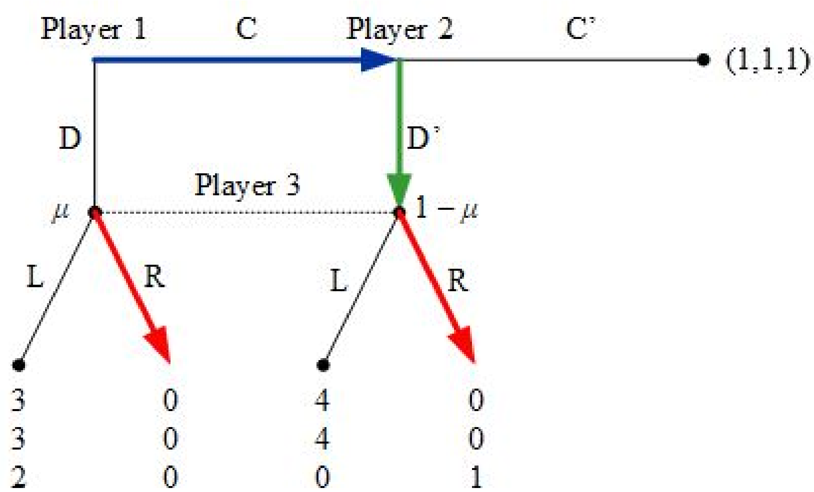


Figure 4. Separating Strategy Profile C, D'

- *Separating strategy profile* D, C'. Second, we check the separating strategy profile, D, C', where Players 1 chooses D and Player 2 selects *C'*.
As depicted in Figure 5 since player 1 chooses D (as illustrated by the blue vertical arrow) and player 2 chooses C' (as illustrated by the green horizontal arrow), Player 3's belief is totally concentrated on Player 1 choosing D (left-hand node of his information set), entailing that $\mu = 1$ (that is, $1 - \mu = 0$). In this context, Player 3 is better off responding with L (as illustrated by the red arrows), which yields a payoff of 2, than with R, which yields a payoff of 0. However, given this response, Player 2 is better off choosing *D'*, receiving a payoff of 4, than selecting *C'* (as prescribed in the above strategy profile). which gives him a payoff of only 1. Therefore, separating strategy profile

D, C' cannot be supported as a PBE of this game when Player 3's belief is $\mu = 1$ as in part (b).

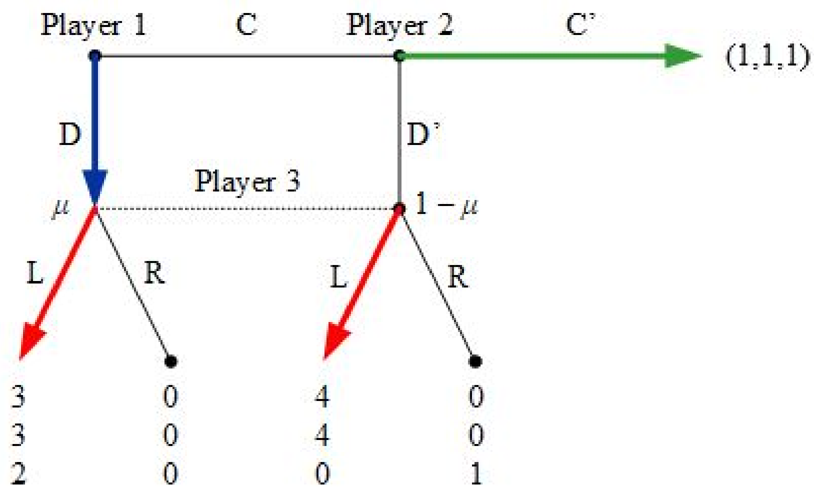


Figure 5. Separating Strategy Profile D, C'

- *Pooling strategy profile* C, C'. Third, we check the pooling strategy profile, C, C', where Player 1 chooses C and Player 2 selects C'.
  As depicted in Figure 6, since player 1 chooses C (as illustrated by the blue horizontal arrow) and player 2 chooses C' (as illustrated by the green horizontal arrow), messages D and D' are on the off-the-equilibrium path, leaving the beliefs of Player 3 unrestricted, that is, $\mu \in [0, 1]$. In other words, Player 3's information set should never be reached in this strategy profile.

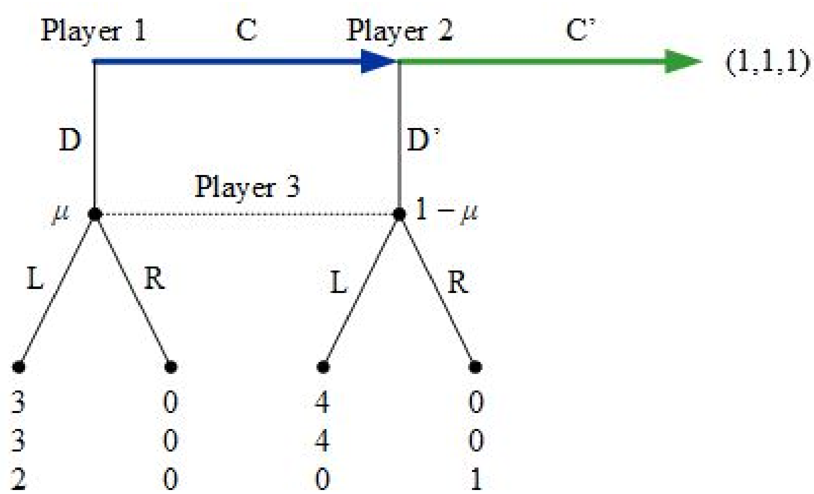


Figure 6. Pooling Strategy Profile C, C'

Therefore, if Player 3 is ever called out to move, he compares the expected payoff from responding with L and R, as follows:

$$EU_3(L) = 2 \times \mu + 0 \times (1-\mu) = 2\mu$$
$$EU_3(R) = 0 \times \mu + 1 \times (1-\mu) = 1-\mu$$

Player 3 then responds with L if $2\mu > 1-\mu$, which simplifies to $\mu > \frac{1}{3}$. Otherwise, he responds with R. This gives rise to two cases (one in which $\mu > \frac{1}{3}$, and Player 3 responds with L; and another in which $\mu \leq \frac{1}{3}$ and Player 3 responds with R), which we separately analyze below.

- *Case 1,* $\mu > \frac{1}{3}$. As depicted in Figure 7a, Player 3 responds with L (as illustrated by the red arrows) since $\mu > \frac{1}{3}$. In this context, Player 2 can improve his payoff by deviating from C', which yields a payoff of 1, to D', which yields a payoff of 4. Therefore, the pooling strategy profile C, C' cannot be supported as a PBE of this game when Player 3's beliefs satisfy $\mu > \frac{1}{3}$.

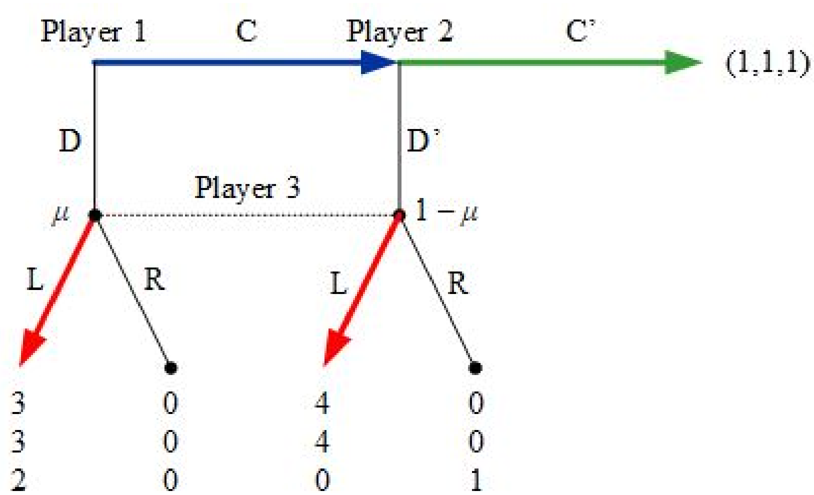


Figure 7a. Pooling Strategy Profile C, C' when $\mu > \frac{1}{3}$.

- *Case 2,* $\mu \leq \frac{1}{3}$. As depicted in Figure 7b, Player 3 responds with R (as illustrated by the red arrows) given that his beliefs are $\mu \leq \frac{1}{3}$. In this context, Player 2 does not deviate because his prescribed strategy, C', which gives him a payoff of 1, while deviating to D' would give him a payoff of 0. Similarly, Player 1 does not deviate because his prescribed strategy, C, which gives him a payoff of 1, exceeding his payoff from deviating to D, zero. Therefore, strategy profile C, C' can be supported as a PBE of this game when Player 3's beliefs satisfy $\mu \leq \frac{1}{3}$.

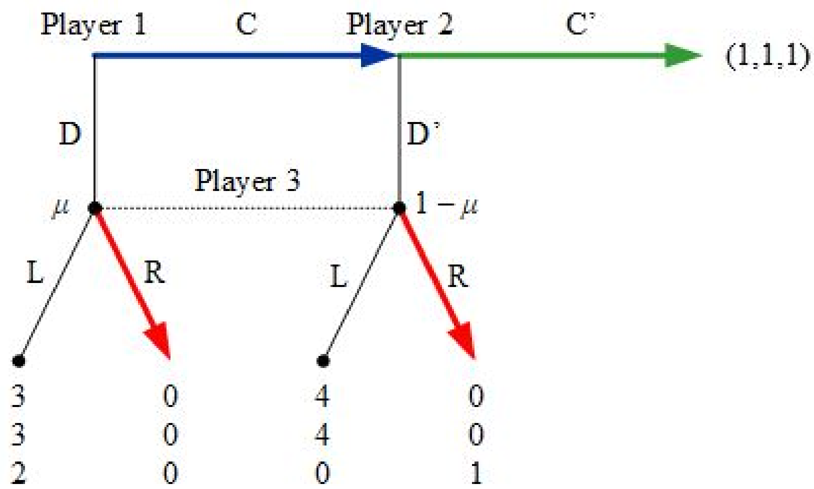


Figure 7b. Pooling Strategy Profile C, C' when $\mu \leq \frac{1}{3}$.

- *Pooling strategy profile* D, D'. Finally, we check the pooling strategy profile, D, D', when Player 1 chooses D and Player 2 selects D'.
  As depicted in Figure 8, since player 1 chooses D (as illustrated by the blue vertical arrow) and player 2 chooses D' (as illustrated by the green vertical arrow), messages D and D' are on the equilibrium path. In this setting, Player 3 being called out to move does not provide him with additional information about whether it is more likely that he is at the left- or right-hand side node on his information set. Therefore, if Player 3 is ever called out to move, he compares the expected payoff from responding with L and R, as follows:

$$EU_3(L) = 2 \times \mu + 0 \times (1 - \mu) = 2\mu$$
$$EU_3(R) = 0 \times \mu + 1 \times (1 - \mu) = 1 - \mu$$

  Player 3 then responds with L if $2\mu > 1 - \mu$, which simplifies to $\mu > \frac{1}{3}$. Otherwise, he responds with R. This gives rise to two cases (one in which $\mu > \frac{1}{3}$, and Player 3 responds with L; and another in which $\mu \leq \frac{1}{3}$ and Player 3 responds with R), which we separately analyze below.

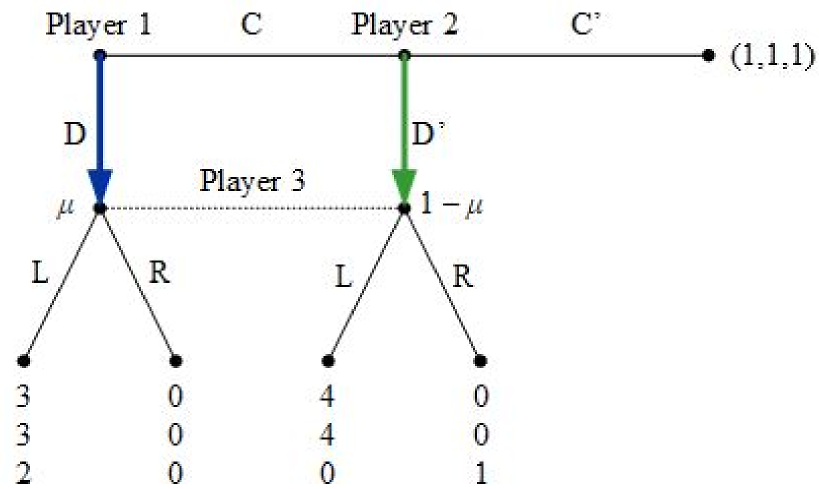


Figure 8. Pooling Strategy Profile D, D'.

- *Case 1,* $\mu > \frac{1}{3}$. As depicted in Figure 9a, Player 3 responds with L (as illustrated by the red arrows) given that his beliefs are $\mu > \frac{1}{3}$. Player 1 in this context can improve his payoff by deviating from the prescribed strategy of D, which yields a payoff of 3, to C, which yields a payoff of 4. Therefore, strategy profile $\{D, D', L\}$ cannot be supported as a PBE of this game when Player 3's beliefs satisfy $\mu > \frac{1}{3}$.

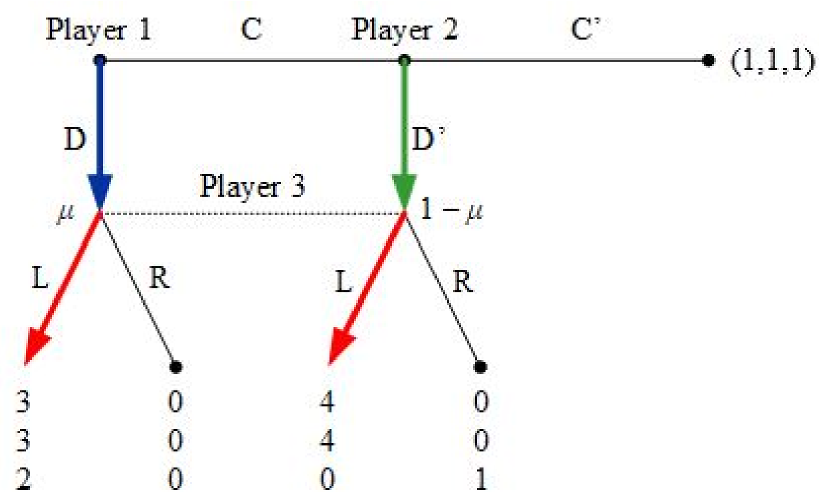


Figure 9a. Pooling Strategy Profile D, D' when $\mu > \frac{1}{3}$.

- *Case 2,* $\mu \leq \frac{1}{3}$. As depicted in Figure 9b, Player 3 responds with R (as illustrated by the red arrows) given that his beliefs are $\mu \leq \frac{1}{3}$. Player 2 in this context can improve his payoff by deviating from his prescribed strategy, D', which yields a payoff of 0, to C', which yields a higher payoff of 1. Therefore, strategy profile $\{D, D', R\}$ cannot be supported as a PBE of this game when Player 3's belief satisfies $\mu \leq \frac{1}{3}$. In other words, the pooling strategy profile D, D' cannot be supported as a PBE regardless of Player 3's beliefs since it couldn't be supported in cases 1 or 2.

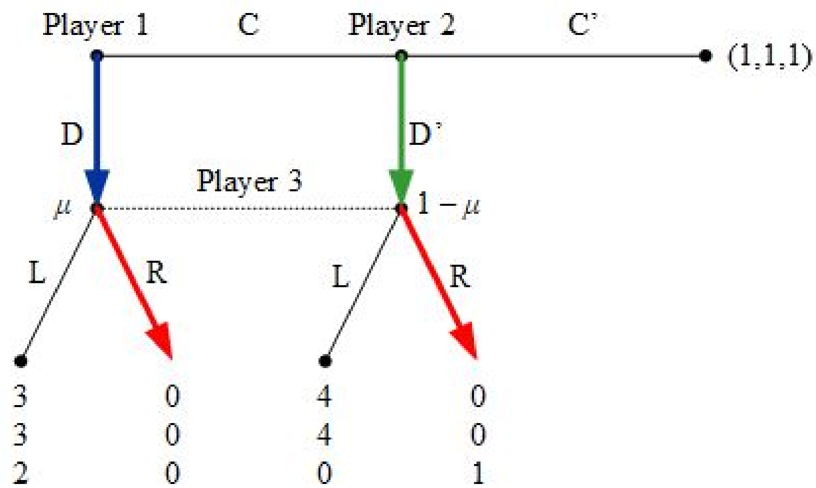


Figure 9b. Pooling Strategy Profile D, D' when $\mu \leq \frac{1}{3}$.

- In sum, $\{C, C', R\}$ is the unique PBE of this game, which can be sustained when Player 3's beliefs satisfy $\mu \leq \frac{1}{3}$.