

# EconS 503 - Microeconomic Theory II

## Homework #3 - Answer key

**7.2 Pareto coordination game.** Consider the following Pareto coordination game where every firm simultaneously choosing between technology  $A$  or  $B$ , and payoffs satisfy  $a_i > b_i > 0$  for every firm  $i = \{1, 2\}$ , implying that both firms regard technology  $A$  as superior. For generality, the premium that firm  $i$  assigns to technology  $A$ , as captured by  $a_i - b_i$ , can be different between firms 1 and 2. Firm  $i$ 's discount factor is  $\delta_i \in (0, 1)$ .

		<i>Firm 2</i>	
		Tech. $A$	Tech. $B$
<i>Firm 1</i>	Tech. $A$	$a_1, a_2$	$0, 0$
	Tech. $B$	$0, 0$	$b_1, b_2$

(a) *Twice-repeated game.* Consider that the stage game is repeated twice. Show that the following strategy profile can be sustained as a SPE of the game: in the first period, every firm  $i$  chooses  $A$ ; in the second period, if  $(A, A)$  was selected in the first period, every firm chooses  $A$ . Otherwise, every firm chooses  $B$ .

- By backward induction, let us start from the second period.
  - After a history of cooperation (outcome  $(A, A)$  was played in the first period), every firm  $i$  has incentives to choose  $A$ . Note that unilateral deviations are not profitable in this game since, conditional on firm  $j$  sticking to this strategy, firm  $i$  would earn 0 from unilaterally deviating to  $B$ , yielding outcome  $(B, A)$  when firm 1 is the deviator or  $(A, B)$  when firm 2 is the deviator.
  - A similar argument applies if outcome  $(A, A)$  was not played in the first period because one or both firms chose  $B$ . In this context, the above strategy profile prescribes every firm choosing  $B$  in the second period, and every firm has no unilateral incentives to deviate from  $B$ . If it did, its payoff would decrease to 0, producing outcomes  $(A, B)$  when firm 1 is the deviator or  $(B, A)$  when firm 2 is the deviator.
- In the first period, every firm  $i$  has incentives to choose  $A$ , as prescribed by this strategy profile. As shown for second-period incentives, firms have no unilateral incentives to choose a different technology than its rival.

(b) Still in the twice-repeated game, can you support a strategy profile where every firm chooses  $A$  in every period of the game for any previous history of play?

- Following our argument in part (a), every firm  $i$  has no unilateral incentives to deviate from  $A$  in the second period, and this applies both after  $(A, A)$  was played in the first period and when outcomes different than  $(A, A)$  were played. In other words, if firm  $j$  chooses  $A$  in the second period, as prescribed by this strategy, firm  $i$ 's best response is to choose  $A$  as well, earning  $a_i$ , rather than unilaterally deviate to  $B$  and earning zero.

- This entails that outcome  $(A, A)$  arises in every period of interaction. A similar argument would apply if the game is finitely repeated  $T \geq 2$  times.
- (c) Still in the twice-repeated game, can you support a strategy profile where every firm chooses  $B$  in every period of the game for any previous history of play?
- Following our argument in part (a), every firm  $i$  has no unilateral incentives to deviate from  $B$  in the second period, and this applies both after  $(B, B)$  was played in the first period and when outcomes different than  $(B, B)$  were played. In other words, if firm  $j$  chooses  $B$  in the second period, as prescribed by this strategy, firm  $i$ 's best response is to choose  $B$  as well, earning  $b_i$ , rather than unilaterally deviate to  $A$  and earning zero.
  - This entails that outcome  $(B, B)$  arises in every period of interaction. A similar argument would apply if the game is finitely repeated  $T \geq 2$  times.
- (d) *Infinitely-repeated game.* Assume now that the game is infinitely repeated. In addition, consider a GTS where, in the first period, every firm  $i$  chooses  $A$ . In all subsequent periods, every firm chooses  $A$  if outcome  $(A, A)$  was played in all previous periods, but deviates to  $B$  otherwise. Under which conditions on  $\delta_i$  can this GTS be sustained as a SPE of the infinitely-repeated game?

- At any period  $t$ , after a history of cooperation, firm  $i$ 's payoff from behaving as prescribed by the strategy profile is  $a_i$ , yielding a discounted stream of payoffs

$$a_i + \delta_i a_i + \delta_i^2 a_i + \dots = \frac{1}{1 - \delta_i} a_i.$$

If, instead, firm  $i$  unilaterally deviates to  $B$ , it earns a payoff of zero in the current period. This deviation is detected and firms choose  $B$  in all subsequent periods, yielding a payoff of  $b_i$  in all future periods. Therefore, firm  $i$ 's discounted stream of payoffs from deviating is

$$0 + \delta_i b_i + \delta_i^2 b_i + \dots = \frac{\delta_i}{1 - \delta_i} b_i.$$

Comparing the above payoff streams, we find that

$$\frac{1}{1 - \delta_i} a_i > \frac{\delta_i}{1 - \delta_i} b_i$$

simplifies to  $a_i > \delta_i b_i$ , which holds given that payoff satisfy  $a_i > b_i$  and  $\delta_i \in (0, 1)$  by assumption. In other words, firm  $i$  has no unilateral incentives to deviate from  $A$  after a history of cooperation.

- If, after a history of defection, firm  $i$  behaves as prescribed by the above strategy profile (choosing  $B$  thereafter), his payoff is  $b_i$  in all subsequent periods, yielding

$$b_i + \delta_i b_i + \delta_i^2 b_i + \dots = \frac{1}{1 - \delta_i} b_i.$$

If, instead, firm  $i$  unilaterally deviates to  $A$ , it earns a payoff of zero in the current period, yielding

$$0 + \delta_i b_i + \delta_i^2 b_i + \dots = \frac{\delta_i}{1 - \delta_i} b_i.$$

Comparing the above payoff streams, we find that

$$\frac{1}{1 - \delta_i} b_i > \frac{\delta_i}{1 - \delta_i} b_i$$

simplifies to  $b_i > \delta_i b_i$ , which holds given that  $\delta_i \in (0, 1)$  by assumption. In other words, firm  $i$  has no unilateral incentives to deviate after a history of defection.

**7.10 Collusion with imperfect monitoring (Discrete strategies).** Consider the following matrix representing a simultaneous-move game between firm 1 and 2, where each firm independently chooses its output level (low,  $L$ , medium,  $M$ , or high,  $H$ ).

		<i>Firm 2</i>		
		$L$	$M$	$H$
<i>Firm 1</i>	$L$	20, 20	5, 13	0, 35
	$M$	13, 5	15, 15	3, 23
	$H$	35, 0	23, 3	4, 4

(a) Find the NE in the stage game.

- The following matrix underlines best response payoffs, showing that  $H$  is a strictly dominant strategy for every player and  $(H, H)$  is the unique NE of the stage game.

		<i>Firm 2</i>		
		$L$	$M$	$H$
<i>Firm 1</i>	$L$	20, 20	5, 13	0, <u>35</u>
	$M$	13, 5	15, 15	3, <u>23</u>
	$H$	<u>35</u> , 0	<u>23</u> , 3	<u>4</u> , <u>4</u>

(b) *GTS with perfect monitoring.* For the remainder of the exercise, assume that firms interact in an infinitely-repeated game. Find the minimal discount factor  $\underline{\delta}$  sustaining the following (standard) GTS, namely, in the first period, every firm chooses the collusive output  $L$ ; in all subsequent periods, every firm chooses  $L$  if  $(L, L)$  was the outcome in all previous periods. Otherwise, every firm reverts to the NE of the stage game,  $(H, H)$ , thereafter. In this part of the exercise, assume that players can immediately observe deviations (perfect monitoring).

- If both firms chose collusive output  $L$  in all previous periods, both firms will choose  $L$  thereafter and receive a payoff of 20 in each period. Therefore, firm's lifetime profit will be

$$20 + \delta 20 + \delta^2 20 + \dots = \frac{20}{1 - \delta}$$

- However, if one firm choose to deviate, it will choose High in that period. Once the deviation is detected, every firm reverts to the NE of the stage game,  $(H, H)$ , thereafter. The profit if the firm deviates is

$$35 + \delta 4 + \delta^2 4 + \dots = 35 + \frac{4\delta}{1 - \delta}$$

- Therefore, to sustain collusion, we must have the lifetime discounted profits be weakly higher than deviating for one period and reverting to  $(H, H)$  afterwards, i.e.,

$$\frac{20}{1-\delta} \geq 35 + \frac{4\delta}{1-\delta} \implies \delta \geq \frac{15}{31} \equiv \underline{\delta}$$

(c) *GTS with imperfect monitoring.* Assume now that firms suffer from imperfect monitoring.

- If firm  $j$  colludes, choosing  $L$ , we assume that, for simplicity, firm  $i$  cannot detect a deviation from the GTS, that is, if firm  $j$  colludes by producing the lowest output, firm  $i$  does not incorrectly observe its rival producing a medium or high output.
- If, instead, firm  $j$  chooses  $M$ , the probability that player  $i$  observes  $j$  deviating from  $L$  is  $p \in [0, 1]$ .
- Similarly, if firm  $j$  selects  $H$ , the probability that player  $i$  observes  $j$  deviating from  $L$  is  $q \in [0, 1]$ . In addition, we assume that  $q > p$ , indicating that when firm  $j$  increases its deviating output (from  $M$  to  $H$ ) it is more likely that firm  $i$  detects such a deviation.

In this context, the GTS described in part (b) must be rewritten as follows: In the first period, every firm chooses  $L$ ; in all subsequent periods, every player chooses  $L$  if  $(L, L)$  was the *observed* outcome in all previous periods. Otherwise, every firm reverts to the NE of the stage game,  $(H, H)$ , thereafter. Find the minimal discount factor sustaining this GTS,  $\underline{\delta}(q)$ , and show that it only depends on probability  $q$ .

- If firm  $i$  colludes, then its lifetime payoff is

$$20 + \delta 20 + \delta^2 20 + \dots = \frac{20}{1-\delta}$$

- If, instead, firm  $i$  chooses  $M$ , in the next period player  $j$  observes  $i$  deviating from  $L$  with probability  $p$  and reverting to  $(H, H)$  with a payoff of 4 forever, or does not observe the deviation with probability  $1-p$  and chooses  $L$  afterwards.

$$\underbrace{13}_{\text{Current gain from dev. to } M} + \delta \left[ p \underbrace{\frac{4}{1-\delta}}_{\text{Detected}} + (1-p) \underbrace{\frac{20}{1-\delta}}_{\text{Undetected}} \right].$$

- Similarly, if firm  $i$  selects  $H$ , in the next period player  $j$  observes  $i$  deviating from  $L$  with probability  $q$  and reverting to  $(H, H)$  with a payoff of 4 forever, or does not observe the deviation with probability  $1-q$  and choose  $L$  afterwards.

$$\underbrace{35}_{\text{Current gain from dev. to } H} + \delta \left[ q \underbrace{\frac{4}{1-\delta}}_{\text{Detected}} + (1-q) \underbrace{\frac{20}{1-\delta}}_{\text{Undetected}} \right].$$

- Therefore, for firm  $j$  to cooperate, we need

$$\frac{20}{1-\delta} \geq 13 + \delta \left[ p \frac{4}{1-\delta} + (1-p) \frac{20}{1-\delta} \right]$$

and

$$\frac{20}{1-\delta} \geq 35 + \delta \left[ q \frac{4}{1-\delta} + (1-q) \frac{20}{1-\delta} \right]$$

Solving for  $\delta$  in the first inequality, yields

$$\delta \leq \frac{7}{7-16p}$$

which does not satisfy  $\delta \in (0, 1)$  since  $0 < \frac{7}{7-16p} < 1$  entails  $p < \frac{7}{16}$  and  $p < 0$  that do not hold for  $p \in (0, 1)$ .

Solving for  $\delta$  in the second inequality, we obtain that

$$\delta \geq \frac{15}{15+16q} \equiv \underline{\delta}(q)$$

Intuitively, this means that firm  $i$  only has incentives to deviate to  $H$ , and thus the only condition that we find for the discount factor  $\delta$  is a function of the detection probability when the firm deviates to  $H$ ,  $q$ .

(d) Evaluate  $\underline{\delta}(q)$  at  $q = 1$  to confirm that  $\underline{\delta}(1) = \underline{\delta}$ .

- Inserting  $q = 1$  into the cutoff we found in part (c), we obtain that  $\underline{\delta}(1) = \frac{15}{15+16} = \frac{15}{31}$ , which coincides with cutoff  $\underline{\delta}$  in part (b).

(e) Does cutoff  $\underline{\delta}(q)$  increase or decrease in  $q$ ? Interpret.

- Cutoff  $\underline{\delta}(q)$  decreases in probability  $q$  since

$$\frac{\partial \underline{\delta}(q)}{\partial q} = -\frac{240}{(15+16q)^2} < 0.$$

Intuitively, this means that, as the probability of detecting firm  $i$  choosing  $H$  is higher (approaching 1), the range of discount factors sustaining this GTS as a SPE expands, indicating that cooperation is easier to sustain. Otherwise, when defection is never detected,  $q = 0$ , cutoff  $\underline{\delta}(q)$  becomes  $\underline{\delta}(0) = \frac{15}{15} = 1$ , implying that cooperation cannot be sustained for any discount factors.

**7.14 Repeated public good game.** Consider the public good game in section 4.4, where two individuals simultaneously and independently submit their contribution to the public good,  $x_i$  and  $x_j$ , with utility function

$$u_i(x_i, x_j) = (w - x_i) \sqrt{mX}$$

where  $m$  denotes the return of aggregate contributions to the public good,  $X = x_i + x_j$ . For simplicity, assume that both players have the same wealth,  $w = 12$ , that the return of the public good is  $m = 2$ , and they exhibit a common discount factor,  $\delta \in (0, 1)$ .

(a) *Stage game.* Find the best responses and NE contributions in the stage game.

- Every player  $i$  chooses his donation  $x_i$  to solve the following utility maximization problem,

$$\max_{x_i \geq 0} u_i(x_i, x_j) = (w - x_i) \sqrt{m(x_i + x_j)}$$

where we use the property  $X = x_i + x_j$ , so the objective function is explicitly a function of  $x_i$ . Differentiating with respect to  $x_i$ , we obtain

$$-\sqrt{m(x_i + x_j)} + \frac{\sqrt{m}}{2} \frac{w - x_i}{\sqrt{x_i + x_j}} = 0$$

which simplifies to

$$\frac{\sqrt{m}}{2} \frac{w - 3x_i - 2x_j}{\sqrt{x_i + x_j}} = 0$$

Solving for  $x_i$ , we find player  $i$ 's best response function:

$$x_i(x_j) = \begin{cases} \frac{w}{3} - \frac{2}{3}x_j & \text{if } x_j < \frac{w}{2}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

- In a symmetric equilibrium, each player makes the same contribution,  $x_i = x_j = x$ . Inserting this property in the above best response function, yields

$$x = \frac{w}{3} - \frac{2}{3}x.$$

or  $3x = w - 2x$ . Solving for  $x$ , we find the equilibrium contribution of every player  $i$ :

$$x^* = \frac{w}{5}.$$

- Substituting the wealth  $w = 12$  and the return of the public good  $m = 2$  into the equilibrium donation, we have

$$x^* = \frac{12}{5} \simeq 2.4$$

Therefore, each player's equilibrium utility is

$$\begin{aligned} u^* &= \left(12 - \frac{12}{5}\right) \sqrt{2 \times \left(\frac{12}{5} + \frac{12}{5}\right)} \\ &= \frac{192\sqrt{15}}{25} \simeq 29.74. \end{aligned}$$

(b) *Socially optimal outcome.* If players could coordinate their contributions to maximize their joint utility, which levels of  $x_i$  and  $x_j$  would they choose?

- The joint utility maximization problem is

$$\max_{x_i, x_j \geq 0} (w - x_i) \sqrt{mX} + (w - x_j) \sqrt{mX}$$

Since  $x_i + x_j = X$ , the above maximization problem simplifies to

$$\max_{X \geq 0} (2w - X)\sqrt{mX}$$

where we changed the choice variables from  $(x_i, x_j)$  to aggregate donations,  $X$ . Differentiating with respect to  $X$ , yields

$$\frac{\sqrt{m}(2w - X)}{2\sqrt{X}} - \sqrt{mX} = 0$$

which simplifies to  $\frac{\sqrt{m}(2w - 3X)}{2\sqrt{X}} = 0$ . Solving for  $X$ , we obtain the aggregate socially optimal donation

$$X^{SO} = \frac{2w}{3}$$

Since players are symmetric, every player's socially optimal donation is

$$x_i^{SO} = \frac{X^{SO}}{2} = \frac{w}{3}$$

implying that every individual contributes a third of his wealth to the public good.

- Substituting the wealth  $w = 12$  into the equilibrium donation, we have

$$x^{SO} = 4$$

and the aggregation donation is

$$X^{SO} = 8$$

- Substituting the return of the public good  $m = 2$ , each player's equilibrium utility is

$$u^{SO} = (12 - 4)\sqrt{2 \times 8} = 32.$$

(c) *Twice-repeated game.* Assume that the game is repeated twice. Analyze equilibrium contributions in the second stage,  $(x_i^2, x_j^2)$ , for a given profile of contributions in the first stage,  $(x_i^1, x_j^1)$ . Then, operating by backward induction, find the equilibrium contributions in the first stage. Can you sustain contributions different from those in the NE found in part (a) at every stage of the game?

- The NE of the stage game is unique, implying that, when the game is finitely repeated (as in this case), players behave as in the NE of the stage game every period when they interact, each of them contributing  $x^* = 2.4$ . As a practice, however, we next show this result.
- *Second Period.* Every player  $i$  chooses his donation  $x_i^2$  to solve the following utility maximization problem,

$$\max_{x_i^2 \geq 0} u_i^2(x_i^2, x_j^2) = (w - x_i^2)\sqrt{m(x_i^2 + x_j^2)}$$

where we use the property  $X^2 = x_i^2 + x_j^2$ , so the objective function is explicitly a function of  $x_i^2$ . (Superscript 2 denotes second-period contributions.) Differentiating with respect to  $x_i^2$ , we obtain

$$-\sqrt{m(x_i^2 + x_j^2)} + \frac{\sqrt{m}}{2} \frac{w - x_i^2}{\sqrt{x_i^2 + x_j^2}} = 0$$

which simplifies to

$$\frac{\sqrt{m} w - 3x_i^2 - 2x_j^2}{2 \sqrt{x_i^2 + x_j^2}} = 0$$

Solving for  $x_i^2$ , we find player  $i$ 's best response function:

$$x_i^2(x_j^2) = \begin{cases} \frac{w}{3} - \frac{2}{3}x_j^2 & \text{if } x_j^2 < \frac{w}{2}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

In a symmetric equilibrium, each player makes the same contribution,  $x_i^2 = x_j^2 = x^2$ . Inserting this property in the above best response function, yields

$$x^2 = \frac{w}{3} - \frac{2}{3}x^2.$$

or  $3x^2 = w - 2x^2$ . Solving for  $x^2$ , we find the equilibrium contribution of every player  $i$ :

$$x^{2*} = \frac{w}{5}$$

Substituting the wealth  $w = 12$  and the return of the public good  $m = 2$  into the equilibrium donation, we have

$$x^{2*} = \frac{12}{5} \simeq 2.4$$

Hence, the equilibrium utility in second period for every player is

$$\begin{aligned} u^{2*} &= \left(12 - \frac{12}{5}\right) \sqrt{2 \times \left(\frac{12}{5} + \frac{12}{5}\right)} \\ &= \frac{192\sqrt{15}}{25} \simeq 29.74. \end{aligned}$$

- *First Period.* Every player  $i$  chooses his donation  $x_i^1$  to solve the following utility maximization problem,

$$\begin{aligned} \max_{x_i^1 \geq 0} u_i(x_i^1, x_j^1, x_i^2, x_j^2) &= u_i^1(x_i^1, x_j^1) + \delta u_i^2(x_i^2, x_j^2) \\ &= (w - x_i^1) \sqrt{m(x_i^1 + x_j^1)} + \delta \frac{192\sqrt{15}}{25} \end{aligned}$$

where we use the property  $X^1 = x_i^1 + x_j^1$ , so the objective function is explicitly a function of  $x_i^1$ . Differentiating with respect to  $x_i^1$ , we obtain

$$-\sqrt{m(x_i^1 + x_j^1)} + \frac{\sqrt{m}}{2} \frac{w - x_i^1}{\sqrt{x_i^1 + x_j^1}} = 0$$

which simplifies to

$$\frac{\sqrt{m} w - 3x_i^1 - 2x_j^1}{2 \sqrt{x_i^1 + x_j^1}} = 0$$

Solving for  $x_i^1$ , we find player  $i$ 's best response function:

$$x_i^1(x_j^1) = \begin{cases} \frac{w}{3} - \frac{2}{3}x_j^1 & \text{if } x_j^1 < \frac{w}{2}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

In a symmetric equilibrium, each player makes the same contribution,  $x_i^1 = x_j^1 = x^1$ . Inserting this property in the above best response function, yields

$$x^1 = \frac{w}{3} - \frac{2}{3}x^1.$$

or  $3x^1 = w - 2x^1$ . Solving for  $x^1$ , we find the equilibrium contribution of every player  $i$ :

$$x^{1*} = \frac{w}{5}$$

Substituting the wealth  $w = 12$  and the return of the public good  $m = 2$  into the equilibrium donation, we have

$$x^{1*} = \frac{12}{5} = 2.4.$$

Hence, the equilibrium utility in first period for every player is

$$\begin{aligned} u^{1*} &= \left(12 - \frac{12}{5}\right) \sqrt{2 \times \left(\frac{12}{5} + \frac{12}{5}\right)} \\ &= \frac{192\sqrt{15}}{25} \simeq 29.74. \end{aligned}$$

- (d) *Infinitely-repeated game.* If the stage game is infinitely repeated, can you sustain a GTS where, in the first period, every player chooses the cooperative (socially optimal) outcome of part (b) and, in all subsequent stages, every player keeps choosing the cooperative outcome if this outcome was played in all previous stages. Otherwise, he reverts to the NE of the stage game thereafter.

- *Cooperation.* If after a history of cooperation, player  $i$  behaves as prescribed by the GTS, choosing the socially optimal donation  $x_i^{SO}$ , his discounted stream of payoffs is

$$32 + \delta 32 + \delta^2 32 + \dots = \frac{32}{1 - \delta}$$

- *Optimal deviation.* If, instead, player  $i$  unilaterally deviates from  $x_i^{SO}$  while his opponent still cooperates by choosing  $x_j^{SO} = 4$ , his optimal deviation is found by inserting  $x_j^{SO} = 4$  into player  $i$ 's best response function,  $x_i(x_j) = \frac{12}{3} - \frac{2}{3}x_j$ , as follows

$$x_i(x_j^{SO}) = \frac{12}{3} - \frac{2}{3}(4) = \frac{4}{3}$$

which yields a deviating utility for player  $i$  of

$$\begin{aligned} u^{Dev} &= \left(12 - \frac{4}{3}\right) \sqrt{2 \times \left(\frac{4}{3} + 4\right)} \\ &= \frac{128\sqrt{6}}{9} \simeq 34.83. \end{aligned}$$

Once the deviation is detected, every player reverts to the NE of the stage game thereafter, receiving payoff of  $\frac{192\sqrt{15}}{25} \simeq 29.74$ , yielding a discounted stream of payoffs

$$\underbrace{\frac{128\sqrt{6}}{9}}_{\text{Deviation}} + \underbrace{\delta \frac{192\sqrt{15}}{25} + \delta^2 \frac{192\sqrt{15}}{25} + \dots}_{\text{Reversion to NE}} = \frac{128\sqrt{6}}{9} + \frac{192\sqrt{15}\delta}{25(1-\delta)}$$

Therefore, every player  $i$  keeps cooperating if

$$\frac{32}{1-\delta} \geq \frac{128\sqrt{6}}{9} + \frac{192\sqrt{15}\delta}{25(1-\delta)}$$

Solving for  $\delta$ , yields  $\delta \geq \frac{25(4\sqrt{6}-9)}{2(50\sqrt{6}-27\sqrt{15})} \simeq 0.557$ .

#### 8.4 Bayesian normal form - Allowing for both players to be privately informed.

Consider the game in the following payoff matrix.

		<i>Player 2</i>	
		<i>L</i>	<i>R</i>
<i>Player 1</i>	<i>U</i>	<i>x, y</i>	5, 10
	<i>D</i>	10, 0	10, 17

where, as in example 8.1, player 1 privately observes the realization of  $x$ , which takes values  $x = 20$  and  $x = 3$ , with equal probabilities; but now player 2 privately observes the realization of  $y$ , which takes values  $y = 15$  and  $y = 5$ , also with equal probabilities.

(a) Find the strategy set of each player.

- Player 1's strategy set is

$$S_1 = \{U^{20}U^3, U^{20}D^3, D^{20}U^3, D^{20}D^3\}$$

where the first (second) component in every pair describes player 1's action when he observes  $x = 20$  ( $x = 3$ , respectively). Similarly, player 2's strategy set is

$$S_2 = \{L^{15}L^5, L^{15}R^5, R^{15}L^5, R^{15}R^5\}$$

where the first (second) component in every pair denotes player 2's action when he observes  $y = 15$  ( $y = 5$ , respectively).

(b) Build the Bayesian normal form of the game with the expected utilities of each player in the appropriate cells.

- From part (a), we found that every player has 4 possible strategies, implying that the Bayesian normal form representation is as follows.

		<i>Player 2</i>			
		$L^{15}L^5$	$L^{15}R^5$	$R^{15}L^5$	$R^{15}R^5$
<i>Player 1</i>	$U^{20}U^3$	$\frac{23}{2}, 10$	$\frac{33}{4}, \frac{25}{2}$	$\frac{33}{4}, \frac{15}{2}$	$5, 10$
	$U^{20}D^3$	$15, 5$	$\frac{45}{4}, \frac{21}{2}$	$\frac{45}{4}, 8$	$\frac{15}{2}, \frac{27}{2}$
	$D^{20}U^3$	$\frac{13}{2}, 5$	$7, \frac{21}{2}$	$7, 8$	$\frac{15}{2}, \frac{27}{2}$
	$D^{20}D^3$	$10, 0$	$10, \frac{17}{2}$	$10, \frac{17}{2}$	$10, 17$

- If player 1 chooses  $U^{20}U^3$ , in the top row, expected payoffs are

1.  $(U^{20}U^3, L^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}20 + \frac{1}{2}20 \right) + \frac{1}{2} \left( \frac{1}{2}3 + \frac{1}{2}3 \right) = \frac{23}{2}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}15 + \frac{1}{2}5 \right) + \frac{1}{2} \left( \frac{1}{2}15 + \frac{1}{2}5 \right) = 10$$

2.  $(U^{20}U^3, L^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}20 + \frac{1}{2}5 \right) + \frac{1}{2} \left( \frac{1}{2}3 + \frac{1}{2}5 \right) = \frac{33}{4}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}15 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}15 + \frac{1}{2}10 \right) = \frac{25}{2}$$

3.  $(U^{20}U^3, R^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}20 \right) + \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}3 \right) = \frac{33}{4}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}5 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}5 \right) = \frac{15}{2}$$

4.  $(U^{20}U^3, R^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}5 \right) + \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}5 \right) = 5$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) = 10$$

- If player 1 chooses  $U^{20}D^3$ , in the second row, expected payoffs are

1.  $(U^{20}D^3, L^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}20 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}20 + \frac{1}{2}10 \right) = 15$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}15 + \frac{1}{2}0 \right) + \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}0 \right) = 5$$

2.  $(U^{20}D^3, L^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}20 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}10 \right) = \frac{45}{4}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}15 + \frac{1}{2}0 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}17 \right) = \frac{21}{2}$$

3.  $(U^{20}D^3, R^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}20 + \frac{1}{2}10 \right) = \frac{45}{4}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}17 \right) + \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}0 \right) = 8$$

4.  $(U^{20}D^3, R^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}5 + \frac{1}{2}10 \right) = \frac{15}{2}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}17 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}17 \right) = \frac{27}{2}$$

• If player 1 chooses  $D^{20}U^3$ , in the third row, expected payoffs are

1.  $(D^{20}U^3, L^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}3 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}3 \right) = \frac{13}{2}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}15 \right) + \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}5 \right) = 5$$

2.  $(D^{20}U^3, L^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}3 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}5 \right) = 7$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}15 \right) + \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}10 \right) = \frac{21}{2}$$

3.  $(D^{20}U^3, R^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}5 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}3 \right) = 7$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}5 \right) = 8$$

4.  $(D^{20}U^3, R^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}5 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}5 \right) = \frac{15}{2}$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}10 \right) = \frac{27}{2}$$

- If player 1 chooses  $D^{20}D^3$ , in the bottom row, expected payoffs are

1.  $(D^{20}D^3, L^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) = 10$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}0 \right) + \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}0 \right) = 0$$

2.  $(D^{20}D^3, L^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) = 10$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}0 \right) + \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}17 \right) = \frac{17}{2}$$

3.  $(D^{20}D^3, R^{15}L^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) = 10$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}17 \right) + \frac{1}{2} \left( \frac{1}{2}0 + \frac{1}{2}0 \right) = \frac{17}{2}$$

4.  $(D^{20}D^3, R^{15}R^5)$ :

$$EU_1 = \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) + \frac{1}{2} \left( \frac{1}{2}10 + \frac{1}{2}10 \right) = 10$$

$$EU_2 = \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}17 \right) + \frac{1}{2} \left( \frac{1}{2}17 + \frac{1}{2}17 \right) = 17$$

(c) Underline each player's best response payoffs in the Bayesian normal form from part (b). Then, identify the BNEs in this game.

- Underlining the best response for each player, we have

		<i>Player 2</i>			
		$L^{15}L^5$	$L^{15}R^5$	$R^{15}L^5$	$R^{15}R^5$
<i>Player 1</i>	$U^{20}U^3$	$\frac{23}{2}, 10$	$\frac{33}{4}, \frac{25}{2}$	$\frac{33}{4}, \frac{15}{2}$	$5, 10$
	$U^{20}D^3$	<u><math>15, 5</math></u>	$\frac{45}{4}, \frac{21}{2}$	$\frac{45}{4}, 8$	$\frac{15}{2}, \frac{27}{2}$
	$D^{20}U^3$	$\frac{13}{2}, 5$	$7, \frac{21}{2}$	$7, 8$	$\frac{15}{2}, \frac{27}{2}$
	$D^{20}D^3$	$10, 0$	$10, \frac{17}{2}$	$10, \frac{17}{2}$	<u><math>10, 17</math></u>

Therefore, there is a unique BNE in this game,  $(D^{20}D^3, R^{15}R^5)$ .

(d) How do equilibrium results differ from those in exercise 8.1? Interpret.

- In exercise 8.1, there is only one BNE,  $(D^{20}D^3, R)$ , when player 1 was privately informed about  $x$ , and similarly, when player 2 is also privately informed,  $(D^{20}D^3, R^{15}R^5)$ , which produces the same equilibrium outcome as  $(D^{20}D^3, R)$ .

**8.10 Cournot competition under incomplete information-A twist.** Consider the same industry as in section 8.5, but assume that the marginal production cost of firm 1 (2) is high,  $c_H = \frac{1}{2} > 0$ , with probability  $p$  ( $q$ , respectively), where  $p, q \in [0, 1]$ . Similarly, the marginal cost of firm 1 (2) is low,  $c_L = 0$ , with probability  $1 - p$  ( $1 - q$ , respectively).

(a) Find firm 1's best response function when its costs marginal costs are low,  $q_1^L(q_2^H, q_2^L)$ . Find firm 2's best response function when its marginal costs are low,  $q_2^L(q_1^H, q_1^L)$ .

- *Firm 1.* When firm 1 has low costs, it chooses  $q_1^L$  to solve the following expected profit maximization problem:

$$\begin{aligned} \max_{q_1^L \geq 0} \pi_1^L(q_1^L) &= \overbrace{(1 - q)(1 - q_1^L - q_2^L)q_1^L}^{\text{Profits if firm 2 is low cost}} + \overbrace{q(1 - q_1^L - q_2^H)q_1^L}^{\text{Profits if firm 2 is high cost}} \\ &= (1 - q_1^L - (1 - q)q_2^L - qq_2^H)q_1^L \end{aligned}$$

which does not include the production cost of firm 1 because  $c_L = 0$ . Differentiating with respect to  $q_1^L$  and assuming an interior solution, yields

$$\frac{\partial \pi_1^L(q_1^L)}{\partial q_1^L} = 1 - 2q_1^L - (1 - q)q_2^L - qq_2^H = 0.$$

Solving for  $q_1^L$ , we find the best response function of firm 1 when its costs are low, as follows

$$q_1^L(q_2^L, q_2^H) = \frac{1}{2} - \frac{(1 - q)q_2^L + qq_2^H}{2}$$

which originates at  $1/2$ , and decreases in its rival's output at a rate of  $\frac{1-q}{2}$  ( $\frac{q}{2}$ ) when its rival has low (high) costs. Alternatively, firm 1 decreases its output at a rate of  $1/2$  when its rival's expected output, as captured by  $(1 - q)q_2^L + qq_2^H$ , increases by one unit.

- *Firm 2.* When firm 2 has low costs, it chooses  $q_2^L$  to solve the following expected profit maximization problem:

$$\begin{aligned} \max_{q_2^L \geq 0} \pi_2^L(q_2^L) &= \overbrace{(1 - p)(1 - q_2^L - q_1^L)q_2^L}^{\text{Profits if firm 1 is low cost}} + \overbrace{p(1 - q_2^L - q_1^H)q_2^L}^{\text{Profits if firm 1 is high cost}} \\ &= (1 - q_2^L - (1 - p)q_1^L - pq_1^H)q_2^L \end{aligned}$$

which does not include the production cost of firm 2 because  $c_L = 0$ . Differentiating with respect to  $q_2^L$  and assuming an interior solution, yields

$$\frac{\partial \pi_2^L(q_2^L)}{\partial q_2^L} = 1 - 2q_2^L - (1 - p)q_1^L - pq_1^H = 0.$$

Solving for  $q_2^L$ , we find the best response function of firm 2 when its costs are low, as follows

$$q_2^L(q_1^L, q_1^H) = \frac{1}{2} - \frac{(1 - p)q_1^L + pq_1^H}{2}$$

which originates at  $1/2$ , and decreases in its rival's output at a rate of  $\frac{1-p}{2}$  ( $\frac{p}{2}$ ) when its rival has low (high) costs. Alternatively, firm 2 decreases its output at a rate of  $1/2$  when its rival's expected output, as captured by  $(1-p)q_1^L + pq_1^H$ , increases by one unit.

- (b) Find firm 1's best response function when its costs marginal costs are high,  $q_1^H(q_2^H, q_2^L)$ . Find firm 2's best response function when its marginal costs are high,  $q_2^H(q_1^H, q_1^L)$ .

- *Firm 1.* When firm 1 has high costs, it chooses  $q_1^H$  to solve the following expected profit maximization problem:

$$\begin{aligned} \max_{q_1^H \geq 0} \pi_1^H(q_1^H) &= \overbrace{(1-q)(1-q_1^H - q_2^L)q_1^H}^{\text{Profits if firm 2 is low cost}} + \overbrace{q(1-q_1^H - q_2^H)q_1^H}^{\text{Profits if firm 2 is high cost}} - \frac{1}{2}q_1^H \\ &= \left(1 - \frac{1}{2} - q_1^H - (1-q)q_2^L - qq_2^H\right) q_1^H \end{aligned}$$

Assuming interior solutions, that is,  $q_1^H > 0$ , the first order condition satisfies

$$\frac{\partial \pi_1^H(q_1^H)}{\partial q_1^H} = \frac{1}{2} - 2q_1^H - (1-q)q_2^L - qq_2^H = 0$$

such that the best response function of firm 1 when its costs are high becomes

$$q_1^H(q_2^L, q_2^H) = \frac{1}{4} - \frac{(1-q)q_2^L + qq_2^H}{2}$$

which originates at  $\frac{1}{4}$ , but decreases in its rival's output at a rate of  $\frac{1-q}{4}$  ( $\frac{q}{4}$ ) when its rival has low (high) costs.

- *Firm 2.* When firm 2 has high costs, it chooses  $q_2^H$  to solve the following expected profit maximization problem:

$$\begin{aligned} \max_{q_2^H \geq 0} \pi_2^H(q_2^H) &= \overbrace{(1-p)(1-q_2^H - q_1^L)q_2^H}^{\text{Profits if firm 1 is low cost}} + \overbrace{p(1-q_2^H - q_1^H)q_2^H}^{\text{Profits if firm 1 is high cost}} - \frac{1}{2}q_2^H \\ &= \left(1 - \frac{1}{2} - q_2^H - (1-p)q_1^L - pq_1^H\right) q_2^H \end{aligned}$$

Assuming interior solutions, that is,  $q_2^H > 0$ , the first order condition satisfies

$$\frac{\partial \pi_2^H(q_2^H)}{\partial q_2^H} = \frac{1}{2} - 2q_2^H - (1-p)q_1^L - pq_1^H = 0$$

such that the best response function of firm 2 when its costs are high becomes

$$q_2^H(q_1^L, q_1^H) = \frac{1}{4} - \frac{(1-p)q_1^L + pq_1^H}{2}$$

which originates at  $\frac{1}{4}$ , but decreases in its rival's output at a rate of  $\frac{1-p}{4}$  ( $\frac{q}{4}$ ) when its rival has low (high) costs.

- Comparing it with firm  $i$ 's best response function when its costs are low,  $q_i^L(q_j^L, q_j^H)$ , we can see that, for a given profile of firm  $j$ 's output,  $(q_j^L, q_j^H)$ , firm  $i$  responds producing a larger output when its own costs are low than when they are high. Graphically,  $q_i^L(q_j^L, q_j^H)$  originates at  $\frac{1}{2}$  while  $q_i^H(q_j^L, q_j^H)$  originates at  $\frac{1}{4}$ , and both best response functions have the same slope, thus being parallel to each other.
- (c) Use your results from parts (a) and (b) to find the BNE of the game. [*Hint*: You cannot invoke symmetry since best response functions are not symmetric in this case.]
- Substituting  $q_2^H(q_1^L, q_1^H)$  and  $q_2^L(q_1^L, q_1^H)$  into  $q_1^H(q_2^L, q_2^H)$  and  $q_1^L(q_2^L, q_2^H)$ , we obtain

$$\begin{aligned} q_1^H &= \frac{1}{4} - \frac{(1-q)}{2} \overbrace{\left[ \frac{1}{2} - \frac{(1-p)q_1^L + pq_1^H}{2} \right]}^{q_2^L} - \frac{q}{2} \overbrace{\left[ \frac{1}{4} - \frac{(1-p)q_1^L + pq_1^H}{2} \right]}^{q_2^H} \\ &= \frac{q}{8} + \frac{p(q_1^H - q_1^L)}{4} + \frac{q_1^L}{4} \end{aligned}$$

$$\begin{aligned} q_1^L &= \frac{1}{2} - \frac{(1-q)}{2} \overbrace{\left[ \frac{1}{2} - \frac{(1-p)q_1^L + pq_1^H}{2} \right]}^{q_2^L} - \frac{q}{2} \overbrace{\left[ \frac{1}{4} - \frac{(1-p)q_1^L + pq_1^H}{2} \right]}^{q_2^H} \\ &= \frac{2+q}{8} + \frac{p(q_1^H - q_1^L)}{4} + \frac{q_1^L}{4} \end{aligned}$$

Solving for the system of equations simultaneously, we obtain

$$q_1^{H*} = \frac{1-p+2q}{12}$$

$$q_1^{L*} = \frac{4-p+2q}{12}$$

Substituting  $q_1^{H*}$  and  $q_1^{L*}$  into firm 2's best response functions, we have

$$\begin{aligned} q_2^{H*} &= \frac{1}{4} - \frac{(1-p)}{2} \overbrace{\left[ \frac{4-p+2q}{12} \right]}^{q_1^{L*}} - \frac{p}{2} \overbrace{\left[ \frac{1-p+2q}{12} \right]}^{q_1^{H*}} \\ &= \frac{1-q+2p}{12} \end{aligned}$$

$$\begin{aligned} q_2^{L*} &= \frac{1}{2} - \frac{(1-p)}{2} \overbrace{\left[ \frac{4-p+2q}{12} \right]}^{q_1^{L*}} - \frac{p}{2} \overbrace{\left[ \frac{1-p+2q}{12} \right]}^{q_1^{H*}} \\ &= \frac{4-q+2p}{12} \end{aligned}$$

Therefore, the equilibrium output levels are

$$\begin{aligned} q_1^{L*} &= \frac{4 - p + 2q}{12} \\ q_1^{H*} &= \frac{1 - p + 2q}{12} \\ q_2^{L*} &= \frac{4 - q + 2p}{12} \\ q_2^{H*} &= \frac{1 - q + 2p}{12} \end{aligned}$$

(d) How are the equilibrium output levels  $(q_1^H, q_1^L, q_2^H, q_2^L)$  affected by a marginal increase in  $p$ ? And by a marginal increase in  $q$ ? Interpret.

- Differentiating the equilibrium output levels with respect to  $p$ , we obtain

$$\begin{aligned} \frac{\partial q_1^{L*}}{\partial p} &= -\frac{1}{12} < 0 \\ \frac{\partial q_1^{H*}}{\partial p} &= -\frac{1}{12} < 0 \\ \frac{\partial q_2^{L*}}{\partial p} &= \frac{1}{6} > 0 \\ \frac{\partial q_2^{H*}}{\partial p} &= \frac{1}{6} > 0 \end{aligned}$$

As firm 1's costs are more likely to be high ( $p$  increases), firm 1's output level decreases regardless of its own marginal costs. However, firm 2 will produce more units.

- Differentiating the equilibrium output levels with respect to  $q$ , we find that

$$\begin{aligned} \frac{\partial q_1^{L*}}{\partial q} &= \frac{1}{6} > 0 \\ \frac{\partial q_1^{H*}}{\partial q} &= \frac{1}{6} > 0 \\ \frac{\partial q_2^{L*}}{\partial q} &= -\frac{1}{12} < 0 \\ \frac{\partial q_2^{H*}}{\partial q} &= -\frac{1}{12} < 0 \end{aligned}$$

As firm 2's costs are more likely to be high ( $q$  increases), firm 2's output level decreases regardless of its own marginal costs. However, firm 1 will produce more units.

(e) *Symmetric probabilities.* Evaluate your equilibrium results in the special case where both firms' costs occur with the same probability,  $p = q$ . What if, in addition, these probabilities are both  $1/2$ ?

- When  $p = q$ , firms are symmetric, the equilibrium output levels become

$$q_1^{L*} = q_2^{L*} = q^{L*} = \frac{4+p}{12}$$

$$q_1^{H*} = q_2^{H*} = q^{H*} = \frac{1+p}{12}$$

- In addition, if  $p = q = \frac{1}{2}$ , the equilibrium output levels become

$$q_1^{L*} = q_2^{L*} = q^{L*} = \frac{3}{8}$$

$$q_1^{H*} = q_2^{H*} = q^{H*} = \frac{1}{8}$$

(f) *Special cases.* Evaluate your equilibrium results in the special case where both firms' types are certain, as under complete information: (1)  $p = q = 1$ , (2)  $p = 1$  and  $q = 0$ , (3)  $p = 0$  and  $q = 1$ , and (4)  $p = q = 0$ . Interpret.

- When  $p = q = 1$ , both firms have high marginal costs, the equilibrium output levels become

$$q_1^{H*} = q_2^{H*} = q^{H*} = \frac{1}{6}$$

which coincides with the results in the standard Cournot game with complete information when both firms face same marginal cost,  $c_H = \frac{1}{2}$ .

- When  $p = 1$  and  $q = 0$ , firm 1 has high marginal costs and firm 2 has low marginal costs, the equilibrium output levels are

$$\begin{aligned} q_1^{H*} &= 0 \\ q_2^{L*} &= \frac{1}{2} \end{aligned}$$

As expected, the firm with low marginal cost produces more units than the firm with high marginal cost, and firm 2 behaves as if it is a monopoly.

- When  $p = 0$  and  $q = 1$ , firm 2 has high marginal costs and firm 1 has low marginal costs, the equilibrium output levels are

$$\begin{aligned} q_1^{L*} &= \frac{1}{2} \\ q_2^{H*} &= 0 \end{aligned}$$

As expected, the firm with low marginal cost produces more units than the firm with high marginal cost, and firm 1 behaves as if it is a monopoly.

- When  $p = q = 0$ , both firms have low marginal costs, the equilibrium output levels become

$$q_1^{L*} = q_2^{L*} = q^{L*} = \frac{1}{3}$$

which is exactly the same results as in standard Cournot game with complete information when both firms face zero marginal costs.

8.17 **BNEs involving mixed strategies.** Consider the following Matching pennies game under incomplete information. First, nature determines whether players interact in the standard Matching pennies game (left panel, with probability 1/2) or in the same game but with “reversed” payoffs (right panel, with probability 1/2). Player 1 privately observes the move from nature (i.e., whether players interact in the left or right matrix) but player 2 does not.

		<i>Player 2</i>	
		<i>h</i>	<i>t</i>
<i>Player 1</i>	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

		<i>Player 2</i>	
		<i>h</i>	<i>t</i>
<i>Player 1</i>	<i>H</i>	-1, 1	1, -1
	<i>T</i>	1, -1	-1, 1

(a) As a benchmark, consider a complete information game where both players observe the matrix they are facing. Show that there is no NE in pure strategies.

- Underlining the best response payoffs, we have

		<i>Player 2</i>	
		<i>h</i>	<i>t</i>
<i>Player 1</i>	<i>H</i>	<u>1</u> , -1	-1, <u>1</u>
	<i>T</i>	-1, <u>1</u>	<u>1</u> , -1

		<i>Player 2</i>	
		<i>h</i>	<i>t</i>
<i>Player 1</i>	<i>H</i>	-1, <u>1</u>	<u>1</u> , -1
	<i>T</i>	<u>1</u> , -1	-1, <u>1</u>

where we find no mutual best responses for players 1 and 2, so that this game has no NE in pure strategies.

(b) Consider now the incomplete information game described above. Show that there is no BNE in pure strategies. [*Hint*: Fix player 2’s strategy at  $h$  (which is type-independent), find player 1’s best response to  $h$ , either choosing  $HH$ ,  $HT$ ,  $TH$ , or  $TT$ . Then, show that player 2 has incentives to deviate. Repeat the same approach but fixing player 2’s strategy at  $t$ .]

- Fixing player 2’s strategy at  $h$ , we find that player 1’s best response is  $HT$ , meaning that he selects  $H$  in the left matrix, but chooses  $T$  in the right matrix, earning 1 in both matrices. However,  $h$  is not player 2’s best response to  $HT$ . Indeed, player 2’s expected payoff from responding with  $h$  against  $HT$  is

$$EU_2(h, HT) = \frac{1}{2}(-1) + \frac{1}{2}(-1) = -1$$

while that from responding with  $t$  is higher because

$$EU_2(t, HT) = \frac{1}{2}(1) + \frac{1}{2}(1) = 1$$

where, in each case, the first (second) term denotes the expected payoff when interacting in the left (right) matrix. Therefore, choosing  $h$  in pure strategies cannot be part of a BNE.

- Fixing player 2’s strategy at  $t$ , we find that player 1’s best response is  $TH$ , meaning that he chooses  $T$  in the left matrix, but selects  $H$  in the right matrix, earning 1 in both matrices. However,  $t$  is not player 2’s best response

to  $TH$ . Indeed, player 2's expected payoff from responding with  $t$  against  $TH$  is

$$EU_2(t, TH) = \frac{1}{2}(-1) + \frac{1}{2}(-1) = -1$$

while that from responding with  $h$  is higher given that

$$EU_2(h, TH) = \frac{1}{2}(1) + \frac{1}{2}(1) = 1$$

where, in each case, the first (second) term denotes the expected payoff when interacting in the left (right) matrix. Therefore, choosing  $t$  in pure strategies cannot be part of a BNE either. In summary, player 2 must be mixing in the BNE of the game, as we examine in part (c) of this exercise.

(c) Find a BNE in mixed strategies. [*Hint*: For simplicity, focus on the strategy profile where player 1 chooses heads regardless of her type,  $HH$ , while player 2 mixes.]

- Let player 2 mix between  $h$  and  $t$  with probability  $p$  and  $1 - p$ , respectively. We seek to show that player 1 sticks to  $HH$  instead of deviating, either when playing the left- or right-hand matrix.
- *Player 1's best response*. If player 1's type is "left-hand matrix", her payoff from choosing  $H$ , is

$$EU_1(H|Left) = p(1) + (1 - p)(-1) = 2p - 1$$

while that from deviating to  $T$  is

$$EU_1(T|Left) = p(-1) + (1 - p)(1) = 1 - 2p.$$

Therefore, the probability  $p$  that makes player 1 indifferent must satisfy  $2p - 1 = 1 - 2p$ , which solving for  $p$  yields  $p = \frac{1}{2}$ .

Similarly, if player 1's type is "right-hand matrix", her payoff from choosing  $H$ , is

$$EU_1(H|Right) = p(-1) + (1 - p)(1) = 1 - 2p$$

while that from deviating to  $T$  is

$$EU_1(T|Right) = p(1) + (1 - p)(-1) = 2p - 1.$$

Hence, the probability  $p$  that makes player 1 indifferent must satisfy  $1 - 2p = 2p - 1$ , which solving for  $p$  yields  $p = \frac{1}{2}$ , as above. In summary,  $HH$  is player 1's best response to player 2 mixing between  $h$  and  $t$  with probability  $1/2$ , meaning that player 1 chooses  $H$  both in the left- and right-hand matrix.

- *Player 2's best response*. Finally, we need to check that this mixed strategy by player 2 is her best response to player 1 choosing  $HH$ . To confirm this point, note that player 2's expected payoff from  $h$  is

$$EU_2(h, HH) = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$$

where the first payoff comes from the left matrix (which occurs with probability  $1/2$ ) and the second payoff originates from the right matrix (which also

happens with probability  $1/2$ ). Recall that player 1 chooses  $HH$ , so we only need to pay attention to the upper row in all matrixes. Similarly, player 2's expected payoff from  $t$  is

$$EU_2(t, HH) = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0.$$

Therefore, player 2's best response to player 1 choosing  $HH$  is any randomization between  $h$  and  $t$ , such as  $p = \frac{1}{2}$ . Hence, the strategy profile  $(HH, (\frac{1}{2}, \frac{1}{2}))$  is a BNE of this game, where one of the players uses mixed strategies.