EconS 424 - Strategy and Game Theory
Midterm Exam #1 - Answer key

1. **IDSDS**. Consider the following simultaneous-move game played by player 1 (in rows) and player 2 (in columns).

		Player 2		
		x	y	z
	a	$2, 3$	$1, 4$	$3, 2$
Player 1	b	$5, 1$	$2, 3$	$1, 2$
	c	$3, 7$	$4, 6$	$5, 4$
	d	$4, 2$	$1, 3$	$6, 1$

   (a) Which strategy pairs survive the application of iterative deletion of strictly dominated strategies (IDSDS)? For simplicity, assume that players can only use pure strategies (no randomizations).

   - For player 2 (column player), strategy z is strictly dominated by y. We can then delete column z, leaving us with the following reduced-form matrix.

		Player 2	
		x	y
	a	$2, 3$	$1, 4$
Player 1	b	$5, 1$	$2, 3$
	c	$3, 7$	$4, 6$
	d	$4, 2$	$1, 3$

   For player 1 (row player), strategy a is strictly dominated by b. After deleting the row corresponding to a, we obtain

		Player 2	
		x	y
	b	$5, 1$	$2, 3$
Player 1	c	$3, 7$	$4, 6$
	d	$4, 2$	$1, 3$

   At this point, we cannot delete any more strategies for players 1 or 2 if we restrict them to use pure strategies.

   (b) Let us now allow player 1 to use randomizations. Show that strategy d for player 1 is strictly dominated by a randomization between b and c. [*Hint:* You may assign probability p to strategy b and the remaining probability $1-p$ to strategy c.]

   - Assigning a probability p to strategy b and the remaining probability $1-p$ to strategy c, player 1's expected payoff when player 2 chooses strategy x (in the left-hand column of the above matrix) is

$$5p + 3(1-p) = 2p + 3$$

which is larger than player 1's payoff from strategy d, 4, as long as $2p+3 > 4$, or solving for p, if $p > \frac{1}{2}$. Similarly, when player 2 chooses straegy y (in the right-hand column of the above matrix), player 1's expected payoff from randomizing between b and c becomes

$$2p + 4(1-p) = 4 - 2p$$

which is larger than player 1's payoff from strategy d, 1, as long as $4-2p > 1$, or solving for p, if $p < \frac{3}{2}$. This condition holds by assumption since probability p must be a number between 0 and 1.

- Therefore, any randomization between strategies b and c that assigns more than 50% probability on strategy b (that is, $p > 1/2$) yields a expected utility larger than the utility player 1 receives from strategy d. We can therefore claim that strategy d is strictly dominated, and delete the bottom row of the above matrix, leaving us with the followed reduced-form matrix.

Player 2

Player 1		x	y
	b	$5, 1$	$2, 3$
	c	$3, 7$	$4, 6$

At this point, we cannot delete any further strategies for players 1 or 2. Then, the strategy profiles surviving IDSDS are those in the four cells of the above matrix:

$$IDSDS = \{(b, x), (b, y), (c, x), (c, y)\}.$$

2. **Pure and mixed-strategy Nash equilibrium.** Consider the following simultaneous-move game between players 1 and 2:

Player 2

Player 1		x	y
	b	$5, 1$	$2, 3$
	c	$3, 7$	$4, 6$

(a) Show that there is no pure strategy Nash equilibrium (psNE) in this game.

- Using the strategy profiles that survived IDSDS, we can next underline best response payoffs, as depicted in the matrix below.

Player 2

Player 1		x	y
	b	$\underline{5}, 1$	$2,\underline{3}$
	c	$3,\underline{7}$	$\underline{4}, 6$

Since there is no cell where both players' payoffs are underlined, we can claim that there is no pure strategy Nash equilibrium in this game.

(b) Find a mixed strategy Nash equilibrium (msNE) in this game.

- *Player 1.* If player 1 is randomizing, he must be indifferent between pure strategies b and c. His expected utility from choosing b (in the top row of the above matrix) is

$$EU_1(b) = 5q + 2(1-q) = 3q + 2$$

while his expected utility from selecting c (in the bottom row of the matrix) is

$$EU_1(c) = 5q + 4(1-q) = 4 - q.$$

Then, player 1 is indifferent between b and c if and only if $EU_1(b) = EU_1(c)$, which implies that

$$3q + 2 = 4 - q$$

and, after rearranging, $4q = 2$, or $q = \frac{1}{2}$.

- *Player 2.* If player 2 is randomizing, he must be indifferent between his pure strategies x and y. His expected utility from choosing x (in the left-hand column of the above matrix) is

$$EU_2(x) = 1p + 7(1-p) = 7 - 6p$$

while his expected utility from selecting y (in the right-hand column of the matrix) is

$$EU_2(y) = 3p + 6(1-p) = 6 - 3p.$$

Then, player 2 is indifferent between x and y if and only if $EU_2(x) = EU_2(y)$, which implies that

$$7 - 6p = 6 - 3p$$

or, after rearranging, $1 = 3p$, or $p = \frac{1}{3}$.

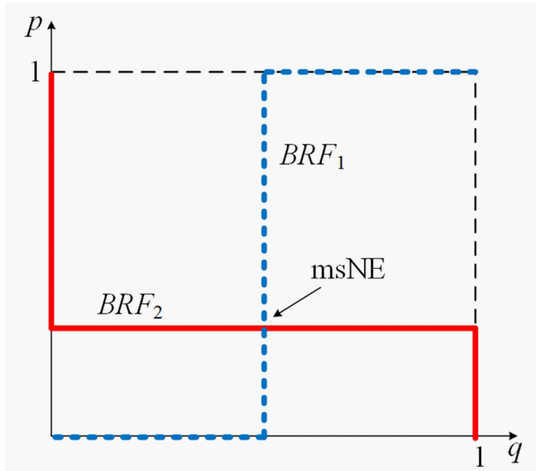
- Therefore, the mixed strategy Nash equilibrium of the game is

$$\left\{ \left(\frac{1}{3}b, \frac{2}{3}c \right), \left(\frac{1}{2}x, \frac{1}{2}y \right) \right\}$$

where the first pair indicates player 1's randomization between b and c with probabilities 1/3 and 2/3 respectively, while the second pair represents player 2's randomization between x and y, each with 50% probability.

(c) Depict the best response functions of each player in a figure with probability p on the vertical axis and probability q on the horizontal axis. Explain their shape and their crossing point/s.

- The next figure depicts the best response functions for each player. The best response functions only have a crossing point, which illustrates the mixed strategy Nash equilibrium of the game.



- For player 1, note that when $q = 1$ (player 2 chooses x), his best response is to choose b, implying that he assigns full probability to b, that is, $p = 1$ at the top right-hand corner of the figure. When $q = 0$ (player 2 selects y), player 1's best response is to choose c, implying that he assigns no probability to b, that is, $p = 0$, at the bottom left-hand corner of the figure.
- For player 2, note that when $p = 1$ (player 1 chooses b), his best response is to choose y, implying that he assigns no probability to x, that is, $q = 0$ at the top left-hand corner of the figure. When $p = 0$ (player 1 selects c), player 2's best response is to choose x, implying that he assigns full probability to x, that is, $q = 1$, at the bottom right-hand corner of the figure.

3. **Common pool resource.** Consider a common pool resource such as a fishing ground, a forest, or an aquifer, exploited by two firms, 1 and 2. Every firm takes prices as given (i.e., firms sell their appropriation in an international, perfectly competitive, market), where we can normalize this price to $p = \$1$, and every firm i's cost function is

$$C_i(q_i, q_j) = \frac{q_i(q_i + \theta q_j)}{S},$$

where $q_i \geq 0$ denotes firm i's appropriation, $q_j \geq 0$ represents its rival's appropriation, and S is the initial stock of the resource. Parameter $\theta \in [0, 1]$ denotes the severity of the cost externality. When $\theta = 0$, firm i's costs are unaffected by its rival's appropriation q_j, implying that the above cost function simplifies to $\frac{q_i^2}{S}$. When $\theta > 0$, firm i's costs are affected by q_j; and when $\theta = 1$, firm i's cost function becomes $\frac{q_i(q_i+q_j)}{S}$.

(a) Find every firm i's best response function, $q_i(q_j)$.

- Each firm chooses its appropriation level q_i to solve

$$\max_{q_i \geq 0} \pi_i = q_i - \frac{q_i(q_i + \theta q_j)}{S}$$

Following exercise 4.12, we normalize the price of the good to \$1. Differentiating with respect to q_i, we obtain

$$1 - \frac{2q_i + \theta q_j}{S} = 0.$$

To find the best response function, we first rearrange the equation to $S = 2q_i + \theta q_j$, and solving for q_i we find

$$q_i(q_j) = \frac{S}{2} - \frac{\theta}{2} q_j.$$

This indicates that appropriation levels are strategic substitutes, that is, an increase in firm j's appropriation induces a reduction in firm i's.

(b) Find firms' appropriation in the NE of this game, q_i^*. [*Hint*: At this point, you can invoke symmetry.]

- In a symmetric equilibrium, appropriation levels satisfy $q_i = q_j = q_i^*$. Every firm i's best response function is

$$q_i^* = \frac{S}{2} - \frac{\theta}{2} q_i^*$$

Rearranging yields $(1 + \frac{\theta}{2})q_i^* = \frac{S}{2}$, or, multiplying both sides by 2, $(2 + \theta)q_i^* = S$. Finally, solving for q_i^*, we obtain the equilibrium appropriation

$$q_i^* = \frac{S}{2 + \theta}.$$

Therefore, as the stock becomes more abundant (higher S), equilibrium appropriation increases at a rate of $\frac{1}{2+\theta}$.

(c) How is q_i^* affected by parameter θ? Interpret.

- An increase in θ decreases the equilibrium appropriation since $\frac{\partial q_i^*}{\partial \theta} = -\frac{S}{(2+\theta)^2} < 0$. Intuitively, a higher θ increases the negative externality from the other firm's exploitation of the resource, which increases the marginal cost for firm i. The increase in marginal cost reduces the equilibrium appropriation from the firm. (Recall that price is given, at the international price, implying that a change in firm i's appropriation does not change the price of the good and, thus, does not affect its marginal revenue either.)