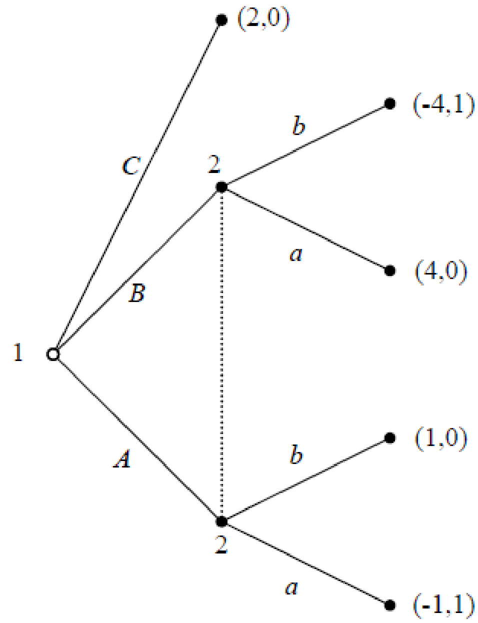
EconS 503 - Advanced Microeconomics - II
Midterm Exam #1 - Answer key

1. **Finding PSNE, MSNE and SPNE.** Consider the game represented in the next figure. Player 1 is the first move (see left-hand side of the figure) choosing between A, B or C. If player 1 chooses C the game is over. If player 1 chooses A or B, player 2 is called to move, but without observing whether player 1 chose A or B; as depicted in player 2's information set. Player 2 then must respond with a or b. The first payoff in every payoff pair corresponds to player 1, and the second payoff to player 2.



(a) Find all pure strategy Nash equilibria (psNE).

- We first represent the game tree in matrix form, as follows:

Player 2

Player 1		a	b
	A	$-1, 1$	$1, 0$
	B	$4, 0$	$-4, 1$
	C	$2, 0$	$2, 0$

Note that strategy A is strictly dominated by strategy C for player 1. Thus, we can delete this strategy and our reduced form game becomes

Player 2

Player 1		a	b
	B	$\underline{4}, 0$	$-4,\underline{1}$
	C	$2,\underline{0}$	$\underline{2},\underline{0}$

There is only one pure strategy Nash Equilibrium of this game, (C, b).

(b) Show that the game has no mixed strategy Nash equilibria (msNE). [*Hint*: A short answer suffices.]

- Player 2 cannot be made indifferent between a and b, because b yields a weakly higher payoff than a. But if player 2 does not randomize, playing b with pure probabilities, then player 1 will not have incentives to randomize either, playing C instead. Therefore, only (C, b) can be sustained as NE of the game, and no msNE exists.

(c) Find all subgame perfect Nash equilibria (SPE). [*Hint*: A short answer suffices.]

- The only proper subgame of this game is the game itself. The NE found in part (a), (C, b), is also subgame perfect. That is, in this game the set of NE coincides with the set of SPE.

2. **Stackelberg with two and three firms** Consider a market where three firms produce a homogeneous good, and face a linear demand function $p(Q) = 1 - Q$, where $Q \equiv q_1 + q_2 + q_3$ denotes aggregate output. All firms face a constant marginal cost of production given by c, where $1 > c > 0$. Firm 1 is the industry leader, choosing its output q_1 in the first stage; firm 2 observes q_1 and responds with its own output q_2 in the second stage; and, in the third stage, firm 3 observes the output pair (q_1, q_2) and responds with its own output q_3.

(a) Find the SPE of this game.

- Operating by backward induction, we need to start with firm 3, followed by firm 2, and followed by firm 1.
- *Firm 3.* Starting with firm 3, its profit maximization problem is

$$\max_{q_3} \ \pi_3 = (1 - q_1 - q_2 - q_3)q_3 - cq_3$$

Differentiating with respect to q_2, we obtain

$$1 - 2q_3 - q_1 - q_2 - c = 0$$

rearranging, we find $2q_3 = 1 - c - q_1 - q_2$, and solving for q_2, we obtain firm 3's best response function

$$q_3(q_1, q_2) = \frac{1-c}{2} - \frac{1}{2}(q_1 + q_2).$$

In this case, firm 3's best response function originates at a vertical intercept of $\frac{1-c}{2}$ units, and decreases at a rate of $\frac{1}{2}$ when either firm 1 or 2 marginal increase their production levels.

- *Firm 2.* Firm 2's profit maximization problem is

$$\max_{q_2} \ \pi_2 = [1 - q_1 - q_2 - q_3(q_1, q_2)]\, q_2 - cq_2$$

Simplifying, we find

$$\max_{q_2} \ \pi_2 = \left(\frac{1+c}{2} - \frac{1}{2}q_1 - \frac{1}{2}q_2\right) q_2 - cq_2$$

Differentiating with respect to q_2, we obtain

$$\frac{\partial \pi_2}{\partial q_2} = \frac{1+c}{2} - \frac{1}{2}q_1 - q_2 - c = 0$$

Solving for q_2, we obtain firm 2's best response function

$$q_2(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$$

Plugging this into firm 3's best response function will make firm 1's problem easier, and make firm 3's best response function only a function of firm 1's output:

$$\begin{aligned} q_3 &= \frac{1-c}{2} - \frac{1}{2}\left(q_1 + \overbrace{\left(\frac{1-c}{2} - \frac{1}{2}q_1\right)}^{q_2(q_1)} \right) \\ &= \frac{1-c}{4} - \frac{1}{4}q_1 \end{aligned}$$

- *Firm 1.* The leader's profit maximization problem is

$$\max_{q_1} \ \pi_1 = [1 - q_1 - q_2(q_1) - q_3(q_1)]\, q_1 - cq_1$$

substituting firm 2 and 3's best response functions, we obtain

$$\max_{q_1} \pi_1 = \left[1 - q_1 - \overbrace{\left(\frac{1-c}{2} - \frac{1}{2}q_1\right)}^{q_2(q_1)} - \overbrace{\left(\frac{1-c}{4} - \frac{1}{4}q_1\right)}^{q_3(q_1)} \right] q_1 - cq_1$$

Simplifying, we obtain

$$\max_{q_1} \ \pi_1 = \left(\frac{1+3c}{4} - \frac{1}{4}q_1\right) q_1 - cq_1$$

Differentiating with respect to q_1, yields $\frac{\partial \pi_1}{\partial q_1} = \frac{1+3c}{4} - \frac{1}{2}q_1 - c = 0$. Rearranging, we find that firm 1's optimal output is

$$q_1^* = \frac{1-c}{2}$$

Therefore, the SPNE of the Stackelberg game with three firms is

$$SPNE = \left\{ \frac{1-c}{2}, \frac{1-c}{2} - \frac{1}{2}q_1, \frac{1-c}{2} - \frac{1}{2}(q_1 + q_2) \right\}$$

where the first term represents firm 1's equilibrium output, the second term reflects firm 2's best response function, and the third term indicates firm 3's best response function.

(b) In a setting with only two firms, it is straightforward to show that the SPE of the game is $SPE = \left\{ \frac{1-c}{2}, \frac{1-c}{2} - \frac{1}{2}q_1 \right\}$, with firm 1 producing $q_1^* = \frac{1-c}{2}$ and firm 2 responding with best response function $q_2(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$. (You do not need to show this result.) Compare this SPE with two firms against your results in part (a) with three firms. How are firm 1's and 2's output decisions affected by the entry of firm 3? Interpret.

- Relative to the setting with two firms, the addition of a third firm does not change the output decision of the first two movers (firms 1 and 2). The third firm simply maximizes profit with respect to the *residual demand* left from the first two firms' output. This result holds in Stackelberg games with cost-symmetric firms, where the leader's output level is unaffected by entry and, similarly, the best response function of each follower is unaffected by entry either. It is generally known as "Stackelberg Independence." The result breaks down when firms are cost asymmetric.

3. **Temporary punishments in Bertrand competition**. Consider an industry with two firms competing in prices a la Bertrand, facing a linear inverse demand function $p(Q) = 100 - Q$, where Q denotes aggregate output. Firms face a common marginal cost $c = 10$. For simplicity, assume that both firms have the same discount factor $\delta \in [0, 1]$.

(a) *Bertrand equilibrium.* Find equilibrium prices in Nash equilibrium of the Bertrand game when firms interact only once.

- When firms i and j, where $i, j \in \{1, 2\}$, interact only once, each firm adopts marginal cost pricing, that is, $p^* = c = 10$, yielding a total sales of 90 units that is equally divided between the two firms, that is, $q_i^* = q_j^* = 45$. Equilibrium profits are zero in this setting.

(b) *Repeated game, Permanent reversion.* Consider now a grim-trigger strategy (GTS) where firms start setting a collusive price that maximizes their joint profits and continue to do so if both firms chose collusive prices in all previous periods. Otherwise, every firm permanently reverts to the Bertrand equilibrium you found in part (a). Under which conditions on discount factor δ this GTS can be sustained as a SPNE of the infinitely repeated game?

- *Cooperation.* Rearrange the inverse demand function $p(Q) = 100 - Q$ to obtain the demand function, $Q(p) = 100 - p$. The cartel comprising firms i and j chooses price p to solve the following joint profit maximization problem:

$$\begin{aligned} \max_{p \geq 0} \pi(p) &= p(Q)Q - c(Q) \\ &= p(100 - p) - 10(100 - p) \\ &= (p - 10)(100 - p) \end{aligned}$$

Differentiating with respect to the price p, and assuming interior solutions, that is, $p > 0$, we obtain

$$2(55 - p) = 0,$$

which yields a collusive price of $p^C = \$55$.
Substituting the collusive price, $p^C = 55$, into the demand function, we obtain that aggregate output becomes $Q^C = 100 - 55 = 45$ units. Since firms are symmetric, each firm would serve half of the market with $q_i^C = q_j^C = \frac{45}{2} = 22.5$ units of output. As a result, every firm i in the cartel earns a collusive profit of

$$\frac{(p-10)(100-p)}{2} = \frac{(55-10)(100-55)}{2} = 1,012.5.$$

If at any period t after a history of cooperation firm i charges the collusive price $p^C = 45$ (as prescribed by the grim-trigger strategy), every firm i obtains profits of $962\frac{1}{2}$ in every period, as follows

$$\begin{aligned}\pi_i^C &= 1,012.5 + (\delta \times 1,012.5) + \left(\delta^2 \times 1,012.5\right) + \ldots \\ &= \frac{1,012.5}{1-\delta}\end{aligned}$$

- *Optimal deviation.* Let us now analyze the payoff that firm i can obtain if it deviates from cooperation. If at any period t after a history of cooparation firm i deviates from the collusive price of $p^C = \$55$, its optimal deviation is to undercut firm j's price by $\varepsilon > 0$, such that it captures the entire market, selling 45 units of output, and earning a deviating profit of $55(45) - 10(45) = 2,025$. Firms i and j detect this deviation immediately, and revert to marginal cost pricing thereafter, which entails zero profits for all subsequent periods. In this context, the payoff that firm i obtains from deviating at any period t is

$$\begin{aligned}\pi_i^{Dev} &= 2,025 + (\delta \times 0) + \left(\delta^2 \times 0\right) + \ldots \\ &= 2,025\end{aligned}$$

- *Comparison.* Therefore, to sustain cooperation, it must be that the cooperation profit must be weakly higher than the deviation profit, that is,

$$\pi_i^C \geq \pi_i^{Dev} \iff \frac{1,012.5}{1-\delta} \geq 2,025$$

cross multiplying by $1-\delta$, we find $1,012.5 \geq 2,025(1-\delta)$. Solving for discount factor δ, we find that cooperation can be sustained as long as

$$\delta \geq \frac{1}{2}.$$

(c) *Repeated game, Temporary reversion.* Consider again the GTS of part (b), but assume that, upon a deviation, firms *temporary* revert to the Bertrand equilibrium of part (a) during $T = 2$ periods. Under which conditions on δ can this GTS be sustained as a SPNE of the infinitely repeated game? Can the GTS be sustained under any admissible value of δ?

- *Cooperation.* If at any period t after a history of cooperation firm i charges the collusive price $p^C = 55$ (as prescribed by the above GTS), firm i obtains

$$\begin{aligned}\pi_i^C &= 1,012.5 + (\delta \times 1,012.5) + \left(\delta^2 \times 1,012.5\right) + \ldots \\ &= \frac{1,012.5}{1-\delta}\end{aligned}$$

- *Optimal deviation.* Let us now analyze the payoff that firm i can obtain if it deviates from cooperation. By deviating, firm i earns all of the cartel profit in this period, nothing in the following 3 periods, and half of the cartel profits beginning the 4th period, yielding a deviation payoff of

$$\pi_i^{Dev} = \underbrace{2,025}_{\text{Deviation}} + \underbrace{(\delta \times 0) + \left(\delta^2 \times 0\right)}_{\text{Punishment for 2 periods}} + \underbrace{\left(\delta^3 \times 1,012.5\right) + \left(\delta^4 \times 1,012.5\right) + \ldots}_{\text{Back to cooperation}}$$
$$= 2,025 + 0 + \left[1,012.5 \times \delta^3 \left(1 + \delta + \delta^2 + ...\right)\right]$$
$$= 2,025 + \left(1,012.5 \times \delta^3 \frac{1}{1-\delta}\right)$$

- *Comparison.* Therefore, to sustain cooperation, it must be that the cooperation profit must be weakly higher than the deviation profit, that is,

$$\pi_i^C \geq \pi_i^{Dev}$$

or

$$\frac{1,012.5}{1-\delta} \geq 2,025 + \left(1,012.5 \frac{\delta^3}{1-\delta}\right)$$

which can be rearranged as follows

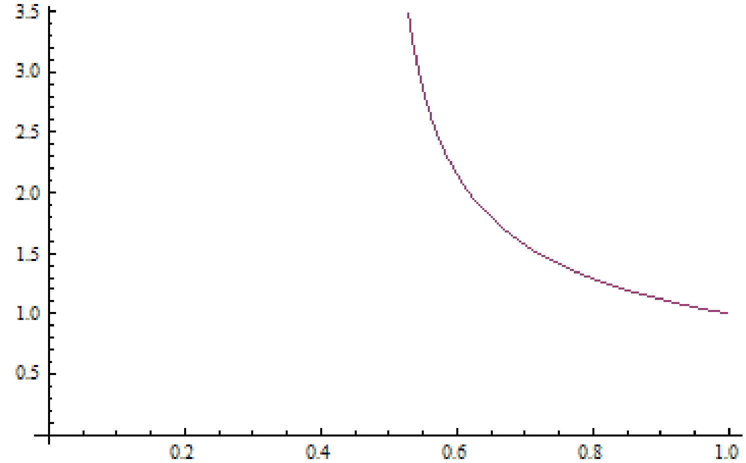
$$1,012.5 \geq 2,025(1-\delta) + 1,012.5\delta^3$$

This expression only produces one real-numbered root, $\delta \geq 0.62$. Relative to the minimal discount factor sustaining cooperation when firms permanently revert to the Nash equilibrium of the unrepeated game, $\delta \geq \frac{1}{2}$, as shown in part (b), the temporary punishment makes collusion more difficult to be sustained.

- As a practice, you can find the minimal discount factor with a temporary punishment that lasts:
  - for only one period, which is larger than one and implies that cooperation cannot be sustained in equilibrium; or
  - for three periods, which is $\delta \geq 0.54$.
  - More generally, when the temporary punishment lasts for T periods, the above condition for cooperation becomes

$$\frac{1,012.5}{1-\delta} \geq 2,025 + \left(1,012.5 \frac{\delta^{T+1}}{1-\delta}\right)$$

    Solving for discount factor δ is rather difficult, but solving for T is easier, yielding $T(\delta) = \frac{\ln(2d-1)}{\ln d} - 1$, which is depicted in the figure below with δ on the horizontal axis and T on the vertical axis. The most natural interpretation is that of the inverse $T^{-1}(\delta)$, which graphically means rotating the figure counter-clockwise 90 degrees. As T increases, the temporary punishment is longer, making deviations less attractive, and decreasing the minimal discount factor sustaining cooperation.



- After a history in which at least one firm deviated from cooperation, the GTS prescribes that every firm i implements the punishment during three rounds. This is firm i's best response to firm j implementing the punishment, so there are no further conditions on the discount factor, δ, that we need to impose.