EconS 503 - Microeconomic Theory II
Homework #2 - Answer key

1. **Exercise 4.5 - Cournot competition with only one firm benefiting from a cost advantage.** Consider an industry like that in section 4.2.2, namely, $N \geq 2$ firms selling a homogeneous good, competing à la Cournot, with inverse demand function $p(Q) = 1 - Q$, where Q denotes aggregate output. One of the firms (firm 1) faces marginal cost $c \in (0, 1)$, while its $N - 1$ rivals face a higher marginal cost c', where $1 > c' \geq c$. This cost advantage may be explained because firm 1 had more years of experience in the industry or operates in similar markets.

   (a) Find firm 1's best response function and the best response function of each of its rivals.

   - Firm 1 chooses its output, q_1, to solve

     $$\max_{q_1 \geq 0} (1 - q_1 - Q_{-1})q_1 - cq_1$$

     where Q_{-1} denotes the aggregate output by firm 1's rivals. Differentiating with respect to q_1, we obtain

     $$1 - 2q_1 - Q_{-1} - c = 0$$

     and solving for q_1, yields best response function

     $$q_1(Q_{-1}) = \frac{1-c}{2} - \frac{1}{2}Q_{-1}$$

     which is decreasing in the aggregate output of firm 1's rivals.
   - Every firm $i \neq 1$ chooses its output q_i to solve

     $$\max_{q_i \geq 0} (1 - q_i - q_1 - Q_{-i})q_i - c'q_i$$

     where Q_{-i} denotes the aggregate output by firm i's rivals (without firm 1's output, q_1). Differentiating with respect to q_i, we obtain

     $$1 - 2q_i - q_1 - Q_{-i} - c' = 0$$

     and solving for q_i, yields best response function

     $$q_i(q_1, Q_{-i}) = \frac{1-c'}{2} - \frac{1}{2}q_1 - \frac{1}{2}Q_{-i}$$

   (b) Find equilibrium output levels for each firm.

- Every firm other than firm 1 chooses the same output, so $q_i = q_j$ for all $i \neq j \neq 1$. Therefore, $Q_{-i} = (N-2)q_i$. Inserting this property in firm i's best response function, we find that

$$q_i = \frac{1-c'}{2} - \frac{1}{2}q_1 - \frac{1}{2}(N-2)q_i$$

Solving for q_i, yields

$$q_i = \frac{1-c'-q_1}{N}$$

implying that the aggregate output of firm 1's rivals is

$$Q_{-1} = (N-1)\frac{(1-c'-q_1)}{N}$$

We can now insert this expression into firm 1's best response function, to obtain

$$q_1 = \frac{1-c}{2} - \frac{1}{2}\underbrace{\frac{(N-1)(1-c'-q_1)}{N}}_{Q_{-1}}$$

Rearranging and solving for q_1, we find firm 1's equilibrium output

$$q_1^* = \frac{N(1-c)}{N+1} - \frac{(N-1)(1-c')}{N+1}.$$

We can finally insert this output level into firm i's best response function to find its equilibrium output, that is,

$$q_i^* = \frac{1-c'}{N} - \frac{1}{N}\left(\frac{N(1-c)}{N+1} - \frac{(N-1)(1-c')}{N+1}\right)$$

(c) Evaluate the equilibrium output for firm 1 and its $N-1$ rivals when all firms are symmetric, that is, $c = c'$.

- Evaluating firm 1's equilibrium output at $c = c'$, we obtain

$$\begin{aligned} q_1^* &= \frac{N(1-c)}{N+1} - \frac{(N-1)(1-c)}{N+1} \\ &= \frac{2(1-c)}{N+1} \end{aligned}$$

Similarly, evaluating its rivals equilibrium output at $c = c'$, we find

$$q_i^* = \frac{1-c}{N+1}$$

which coincides with the equilibrium output in a standard Cournot model with N symmetric firms.

(d) Under which conditions on firm 1's cost, c, will all firms produce a positive output in equilibrium? Under which conditions will only firm 1 be active? Interpret.

- Firm 1's equilibrium output, q_1^*, is positive if, solving for c,

$$c < 1 - \frac{(N-1)(1-c')}{N} \equiv c_A$$

where cutoff c_A satisfies $c_A < 1$ since $c \in (0,1)$. Similarly, the equilibrium output of every firm $i \neq 1$, q_i^*, is positive if, solving for c,

$$c > 2c' - 1 \equiv c_B$$

where cutoffs c_A and c_B satisfy $c_A > c_B$ since $1 - \frac{(N-1)(1-c')}{N} > 2c' - 1$ reduces to $(N+1)(1-c') > 0$. Comparing these two cutoffs of c, we find that there are three possible regions of c:

  - When c is relatively low, $0 < c < c_B$, only firm 1 produces. Intuitively, firm 1 is extremely efficient relative to its $N-1$ rivals, inducing them to stay inactive.
  - When c is intermediate, $c_B \leq c < c_A$, all firms are active. In this case, all firms exhibit relatively symmetric costs.
  - When c is relatively high, $c_A \leq c < 1$, no firm is active.

(e) *Symmetric firms.* Evaluate your results in part (d) in the case that all firms are symmetric, that is, $c = c'$.

- Evaluating the condition for cutoff c_A at $c = c'$, we obtain

$$c < 1 - \frac{(N-1)(1-c)}{N} = \frac{1+(N-1)c}{N}$$

which, rearranging, yields $c < 1$ that holds by definition. Therefore, $c < c_A$ for all parameter values.

- Similarly, evaluating the condition where we obtained cutoff c_B at $c = c'$, we find that

$$c > 2c - 1$$

which holds given that $c < 1$ by definition.

- Combining the above results, when firms are symmetric in costs, they behave as in the case where $c_B \leq c < c_A$, and all firms produce a positive output.

2. **Exercise 4.14 - A common pool resource with more general cost externalities.** Consider the setting in exercise 4.13, but assume now that every firm i's cost function is

$$C_i(q_i, q_j) = \frac{q_i(q_i + \theta q_j)}{S},$$

where $\theta \geq 0$ denotes the severity of the cost externality. When $\theta = 0$, firm i's costs are unaffected by its rival's appropriation q_j; but when $\theta > 0$, firm i's costs are affected by q_j. Therefore, the setting in exercise 4.12 can be interpreted as a special case of this (more general) model, where $\theta = 1$.

(a) Find every firm i's best response function, $q_i(q_j)$.

- Each firm chooses its appropriation level q_i to solve

$$\max_{q_i \geq 0} \ \pi_i = q_i - \frac{q_i(q_i + \theta q_j)}{S}$$

Following previous exercises, we normalize the price of the good to \$1. Differentiating with respect to q_i, we obtain

$$1 - \frac{2q_i + \theta q_j}{S} = 0.$$

To find the best response function, we first rearrange the equation to $S = 2q_i + \theta q_j$, and solving for q_i we find

$$q_i(q_j) = \frac{S}{2} - \frac{\theta}{2} q_j.$$

This indicates that appropriation levels are strategic substitutes, that is, an increase in firm j's appropriation induces a reduction in firm i's.

(b) How is $q_i(q_j)$ affected by parameter θ? Interpret.

- Differentiating $q_i(q_j)$ with respect to θ, we obtain that

$$\frac{\partial q_i(q_j)}{\partial \theta} = -\frac{q_j}{2}$$

Therefore, when firm j appropriates one more unit, it affects firm i's cost more significantly as the severity of the cost externality, θ, increases. This intuition goes in line with our interpretation in part (a).

(c) Find firms' appropriation in the NE of this game, q_i^*.

- In a symmetric equilibrium, appropriation levels satisfy $q_i = q_j = q_i^*$. Every firm i's best response function is

$$q_i^* = \frac{S}{2} - \frac{\theta}{2} q_i^*$$

Rearranging yields $(1+\theta/2)q_i^* = \frac{S}{2}$, or, multiplying both sides by 2, $(2+\theta)q_i^* = S$. Finally, solving for q_i^*, we obtain the equilibrium appropriation

$$q_i^* = \frac{S}{2+\theta}.$$

Therefore, as the stock becomes more abundant (higher S), equilibrium appropriation increases at a rate of $\frac{1}{2+\theta}$.

(d) How is q_i^* affected by parameter θ? Interpret.

- An increase in θ decreases the equilibrium appropriation. We can find the exact amount by which it decreases by taking a derivative:

$$\frac{\partial q_i^*}{\partial \theta} = -\frac{S}{(2+\theta)^2} < 0.$$

A higher θ increases the negative externality from the other firm's exploitation of the resource, which increases the marginal cost for firm i. The increase in marginal cost (absent a change in the price, or marginal revenue, of the good) reduces the equilibrium appropriation from the firm.

3. **Exercise 5.7 - Finding msNE in the salesman's game.** Consider a salesman who, after returning from a business trip, can be honest about his expenses (H) or lie about them (L) claiming more expenses than he actually had. His boss can either not check the salesman (NC) or check whether his claimed expenses are legit (C). If the salesman is honest, he is reimbursed for his expenses during the trip, entailing a payoff of zero for him, and a payoff of zero for his boss when he does not check on the salesman but $-c$ otherwise (we can interpret c as the cost that the boss incurs when checking). If the salesman lies, he earns $a > 0$ when his boss does not check on him, but when he does, the salesman's payoff is $-b$ from his boss detects his cheating (which occurs with probability p) and a when his boss does not detect his cheating (which happens with probability $1-p$). If the boss checks when the salesman lies, the boss' payoff is $-c + [p0 + (1-p)(-\beta)] = -c - (1-p)\beta$, which embodies his (certain) cost from checking and his expected cost of not detecting the lie, which occurs with probability $1-p$.

   (a) Depict the game's payoff matrix.

   - The following payoff matrix represents the salesman in rows and his boss in columns.

Boss

		C	NC
Salesman	H	$0, -c$	$0, 0$
	L	$p(-b) + (1-p)a, -c - \beta(1-p)$	$a, -\beta$

   (b) Find the best responses of each player.

   - *The salesman.*
     - When the boss chooses C (in the left column), the salesman's best response is $BR_S(C) = H$ when his payoffs satisfy $0 > p(-b) + (1-p)a$ that simplifies to $p > \frac{a}{a+b}$, which happens when there is a high chance that his cheating would be detected by the boss.
     - When the boss chooses NC (in the right column), the salesman's best response is $BR_S(NC) = L$ because $a > 0$ by assumption.
   - *The boss.*
     - When the salesman chooses H (in the top row), the boss' best response is $BR_B(H) = NC$ since her payoffs satisfy $-c < 0$ by assumption.
     - When the salesman chooses L (in the bottom row), the boss' best response is $BR_B(L) = C$ when her payoffs satisfy $-c - \beta(1-p) > -\beta$ that simplifies to $p > \frac{c}{\beta}$, which happens when there is a high chance that she can detect the salesman's cheating.

   (c) Find the psNEs of this game.

   - When $p > \max\left\{\frac{a}{a+b}, \frac{c}{\beta}\right\}$, the following payoff matrix underlines the best response payoffs for each player, where we have no psNE of this game.

Boss

		C	NC
Salesman	H	$\underline{0}, -c$	$0, \underline{0}$
	L	$p(-b) + (1-p)a, \underline{-c - \beta(1-p)}$	$\underline{a}, -\beta$

- When $\frac{a}{a+b} \geq p \geq \frac{c}{\beta}$, the following payoff matrix underlines the best response payoffs for each player, where we have (L, C) as the psNE of this game.

		Boss	
		C	NC
Salesman	H	$0, -c$	$0, \underline{0}$
	L	$\underline{p(-b) + (1-p)a}, \underline{-c - \beta(1-p)}$	$\underline{a}, -\beta$

- When $\frac{c}{\beta} \geq p \geq \frac{a}{a+b}$, the following payoff matrix underlines the best response payoffs for each player, where we have (L, NC) as the psNE of this game.

		Boss	
		C	NC
Salesman	H	$\underline{0}, -c$	$0, \underline{0}$
	L	$p(-b) + (1-p)a, -c - \beta(1-p)$	$\underline{a}, \underline{-\beta}$

- When $p < \min\left\{\frac{a}{a+b}, \frac{c}{\beta}\right\}$, the following payoff matrix underlines the best response payoffs for each player, where we have (L, NC) as the psNE of this game.

		Boss	
		C	NC
Salesman	H	$0, -c$	$0, \underline{0}$
	L	$\underline{p(-b) + (1-p)a}, -c - \beta(1-p)$	$\underline{a}, \underline{-\beta}$

(d) Find the msNE of this game.

- Let the salesman (boss) assign probability r (q) to H (C), then

$$\begin{aligned} EU_S(H) &= 0q + 0(1-q) = 0 \\ EU_S(L) &= [p(-b) + (1-p)a]\, q + a(1-q) = a - (a+b)\, pq \\ EU_B(C) &= -cr - [c + \beta(1-p)](1-r) = -c - \beta(1-p)(1-r) \\ EU_B(NC) &= 0r - \beta(1-r) = -\beta(1-r) \end{aligned}$$

- For the salesman to randomize, we need $EU_S(H) = EU_S(L)$, solving

$$0 = a - (a+b)\, pq$$

which we simplify to

$$q = \frac{a}{(a+b)\, p}$$

which is decreasing in p since

$$\frac{\partial q}{\partial p} = -\frac{a}{(a+b)\, p^2} < 0.$$

Intuitively, when his cheating has a higher chance to be detected, the boss can check on the salesman less frequently.

- For the boss to randomize, we need $EU_B(C) = EU_B(NC)$, solving

$$-c - \beta(1-p)(1-r) = -\beta(1-r)$$

which we simplify to

$$r = \frac{\beta p - c}{\beta p}$$

which is increasing in p since

$$\frac{\partial r}{\partial p} = \frac{c}{\beta p^2} > 0.$$

Intuitively, when his cheating has a higher chance to be detected, he is more likely to be honest in claiming expenses.

- Therefore, the msNE of this game is

$$\left(\left(\frac{\beta p - c}{\beta p}H, \frac{c}{\beta p}L\right), \left(\frac{a}{(a+b)p}C, \frac{bp - a(1-p)}{(a+b)p}NC\right)\right).$$

- Note that when $p < \frac{a}{a+b}$, $BR_S(C) = L$ so that lying becomes the strictly dominant strategy of the salesman whether the boss checks on him or not. In this context, the salesman assigns full probability weight to L and the boss will still randomize between C and NC if and only if $p = \frac{c}{\beta}$. Otherwise, when $p > \frac{c}{\beta}$ ($p < \frac{c}{\beta}$), $BR_B(L) = C$ ($BR_B(L) = NC$) and the boss will assign full probability weight on (not) checking.

4. **Exercise 5.18 - Proper equilibrium - Example.** Consider the Hawk-Dove game in the matrix below (an example of an anticoordination game), where player 1 chooses between Hawk and Dove in rows (H or D), and player 2 chooses between these two pure strategies in columns (h and d).

		Player 2 h	d
Player 1	H	$6,6$	$0,12$
	D	$12,0$	$0,0$

(a) Find the pure strategy NEs in this game.

- The following matrix underlines best response payoffs, identifying three pure strategy NEs, (D,h), (D,d), and (H,d).

		Player 2 h	d
Player 1	H	$6,6$	$\underline{0},\underline{12}$
	D	$\underline{12},\underline{0}$	$\underline{0},\underline{0}$

(b) Check with of the NEs found in part (a) are proper.

- Let us first consider the NE (D, h) and the totally mixed strategy $\sigma_1^k = (\varepsilon^k, 1-\varepsilon^k)$. While σ_1^k converges to $(0,1)$, implying that player 1 chooses D in the limit, h is not player 2's best response to σ_1^k. Indeed, his expected payoff from responding with h to σ_1^k is

$$6\varepsilon^k + 0(1-\varepsilon^k) = 6\varepsilon^k,$$

whereas his expected payoff from responding with d is

$$12\varepsilon^k + 0(1-\varepsilon^k) = 12\varepsilon^k.$$

Therefore, (D, h) is not a proper equilibrium.

- Second, let us consider the opposite strategy profile, (H, d). Consider the totally mixed strategy $\sigma_2^k = (\varepsilon^k, 1-\varepsilon^k)$ for player 2. In this case, σ_2^k converges to $(0,1)$, implying that player 2 chooses d in the limit. In this context, D is player 1's best response to σ_2^k since his expected payoff from responding with D,

$$12(1-\varepsilon^k) + 0\varepsilon^k = 12(1-\varepsilon^k),$$

is higher than that from responding with H,

$$6(1-\varepsilon^k) + 0\varepsilon^k = 6(1-\varepsilon^k).$$

Therefore, (H, d) is not a proper equilibrium either.

- Finally, consider the strategy profile (D, d) and the totally mixed strategy $\sigma_i^k = (\varepsilon^k, 1-\varepsilon^k)$, where σ_i^k converges to $(0,1)$, for every player $i = \{1, 2\}$. In this context, the expected payoff of player 1 responding with D to σ_2^k is

$$12\sigma_i^k + 0\left(1-\sigma_i^k\right)$$

that exceeds his payoff from responding with H, $6\sigma_i^k + 0\left(1-\sigma_i^k\right)$. Similarly, the expected payoff of player 2 responding with d to σ_1^k is

$$12\sigma_i^k + 0\left(1-\sigma_i^k\right),$$

which exceeds his payoff from responding with h, $6\sigma_i^k + 0\left(1-\sigma_i^k\right)$, so that (D, d) is a proper equilibrium.

5. **Exercise 6.22 - Stackelberg with one leader and two followers.** Consider an industry with three firms, facing inverse demand function $p(Q) = 1 - Q$, and constant marginal cost c, where $c \in [0, 1]$. Firm 1 is the industry leader, choosing its output q_1 in the first stage. In the second stage, firms 2 and 3 observe q_1 and simultaneously and independently respond choosing their output levels q_2 and q_3.

   (a) Find firm 2 and 3's best response functions. [*Hint*: They should be a function of q_1.]

- Every follower $i = \{2,3\}$ chooses q_i to solve

$$\max_{q_i \geq 0} \ \pi_i(q_i) = (1 - q_1 - q_i - q_j)q_i - cq_i$$

where $j \neq i$.
Differentiating with respect to q_i, and assuming interior solutions, we have

$$1 - q_1 - 2q_i - q_j - c = 0$$

Rearranging, the best response of firm i is

$$q_i(q_1, q_j) = \frac{1-c}{2} - \frac{1}{2}(q_1 + q_j)$$

- Invoking symmetry, we have that both followers produce the same output level, that is, $q_i = q_j$, yielding a best response function

$$q_i(q_1) = \frac{1-c}{3} - \frac{1}{3}q_1$$

for every follower $i = \{2,3\}$. This best response function originates at $\frac{1-c}{3}$ when the leader is inactive (which indicates that every follower produces the same output as in a standard setting of Cournot competition), but decreases in the leader's output at a rate of $\frac{1}{3}$.

(b) Find the leader's equilibrium output, q_1.

- In the first period, the leader anticipates both followers' best response functions, $q_2(q_1)$ and $q_3(q_1)$, and chooses q_1 to solve

$$\max_{q_1 \geq 0} \ \pi_1(q_1) = [1 - q_1 - q_2(q_1) - q_3(q_1)]\, q_1 - cq_1$$

Substituting $q_2(q_1) = q_3(q_1) = \frac{1-c}{3} - \frac{q_1}{3}$ into the above expression, yields

$$\max_{q_1 \geq 0} \pi_1(q_1) = \frac{(1 - q_1 - c)\, q_1}{3}$$

Differentiating with respect to q_1, and assuming interior solutions, we have

$$1 - 2q_1 - c = 0$$

Rearranging, the equilibrium output of firm 1 is

$$q_1 = \frac{1-c}{2}.$$

(c) Describe the SPE of the game and the output triplet (q_1, q_2, q_3) that arises in equilibrium.

- Inserting the leader's output, $q_1 = \frac{1-c}{2}$, into every follower i's best response function, $q_i(q_1) = \frac{1-c}{3} - \frac{1}{3}q_1$, yields

$$q_i = \frac{1-c}{3} - \frac{1}{3}\underbrace{\frac{1-c}{2}}_{q_1} = \frac{1-c}{6}.$$

Therefore, the SPE is

$$(q_1, q_2, q_3) = \left(\frac{1-c}{2}, \frac{1-c}{6}, \frac{1-c}{6}\right)$$

for firms 1, 2, and 3, respectively.

(d) Compare your results in part (c) against those of a more standard Stackelberg game where each firm acts in a different stage (firm 1 chooses q_1, followed by firm 2, and then followed by firm 3). Interpret your findings.

- In the third stage, firm 3 chooses q_3 to solve

$$\max_{q_3 \geq 0} \pi_3(q_3) = (1 - q_1 - q_2 - q_3)q_3 - cq_3$$

Differentiating with respect to q_3, and assuming interior solutions, we have

$$1 - q_1 - q_2 - 2q_3 - c = 0$$

which we rearrange to yield the best response function

$$q_3(q_1, q_2) = \frac{1-c}{2} - \frac{1}{2}(q_1 + q_2).$$

- In the second stage, firm 2 anticipates firm 3's best response function, $q_3(q_1, q_2)$, and chooses q_2 to solve

$$\max_{q_2 \geq 0} \pi_2(q_2) = \left[1 - q_1 - q_2 - \underbrace{\left(\frac{1-c}{2} - \frac{1}{2}(q_1 + q_2)\right)}_{q_3(q_1, q_2)}\right] q_2 - cq_2$$

Differentiating with respect to q_2, and assuming interior solutions, we have

$$1 - q_1 - 2q_2 - c = 0$$

which we rearrange to yield the best response function

$$q_2(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1.$$

Inserting this information in firm 3's best response function, we obtain

$$\begin{aligned} q_3(q_1) &= \frac{1-c}{2} - \frac{1}{2}\left(q_1 + \underbrace{\frac{1-c}{2} - \frac{1}{2}q_1}_{q_2(q_1)}\right) \\ &= \frac{1-c}{4} - \frac{1}{4}q_1 \end{aligned}$$

- In the first stage, firm 1 anticipates firm 2's and 3's best response functions and chooses q_1 to solve

$$\max_{q_1 \geq 0} \pi_1(q_1) = \left[1 - q_1 - \left(\frac{1-c}{2} - \frac{1}{2}q_1\right) - \left(\frac{1-c}{4} - \frac{1}{4}q_1\right)\right] q_1 - cq_1$$

Differentiating with respect to q_1, and assuming interior solutions, we have

$$1 - 2q_1 - c = 0$$

so that the equilibrium output of firm 1 is

$$q_1 = \frac{1-c}{2}$$

- Substituting $q_1 = \frac{1-c}{2}$ into firm 2's best response function, $q_2(q_1)$, yields

$$q_2(q_1) = \frac{1-c}{2} - \frac{1}{2}\frac{1-c}{2} = \frac{1-c}{4}$$

Substituting $q_1 = \frac{1-c}{2}$ into firm 3's best response function, $q_3(q_1)$, yields

$$q_3(q_1) = \frac{1-c}{4} - \frac{1}{4}\frac{1-c}{2} = \frac{1-c}{8}$$

- Summarizing, the SPE is

$$(q_1, q_2, q_3) = \left(\frac{1-c}{2}, \frac{1-c}{4}, \frac{1-c}{8}\right)$$

for firms 1, 2, and 3, respectively, where firm 1 produces the same output as when firms 2 and 3 act simultaneously in the second stage. However, firm 2 (3) produces more (less) output than that in part (c), because firm 2 enjoys a first-mover advantage when setting its output relative to firm 3.

(e) Consider now that firms 2 and 3 can coordinate their output levels q_2 and q_3 in order to maximize their joint profits in the second stage of the game. Find their output decisions (still as a function of q_1), and the leader's output in equilibrium. How are your results in part (c) affected?

- The followers choose their aggregate output, $q = q_2 + q_3$, to solve

$$\max_{q \geq 0} \pi(q) = (1 - q_1 - q)q - cq$$

Differentiating with respect to q, and assuming interior solutions, we have

$$1 - q_1 - 2q - c = 0$$

which we rearrange to yield

$$q(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$$

which decreases in the leader's output at a rate of $\frac{1}{2}$

- In the first period, the leader anticipates that the followers will produce $q(q_1)$ in the subsequent stage and chooses q_1 to solve

$$\max_{q_1 \geq 0} \ \pi_1(q_1) = [1 - q_1 - q(q_1)]\, q_1 - cq_1$$

Substituting $q(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$ into the above expression, yields

$$\max_{q_1 \geq 0} \pi_1(q_1) = \frac{(1 - q_1 - c)\, q_1}{2}$$

Differentiating with respect to q_1, and assuming interior solutions, we have

$$1 - 2q_1 - c = 0$$

Rearranging, the equilibrium output of firm 1 is

$$q_1 = \frac{1-c}{2}.$$

- Substituting the leader's output, $q_1 = \frac{1-c}{2}$, into $q(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$, yields

$$q = \frac{1-c}{2} - \frac{1}{2}\underbrace{\frac{1-c}{2}}_{q_1} = \frac{1-c}{4}$$

meaning that each follower produces half of this amount, that is, $\frac{q}{2} = \frac{1-c}{8}$. Therefore, the SPE is

$$(q_1, q_2, q_3) = \left(\frac{1-c}{2}, \frac{1-c}{8}, \frac{1-c}{8}\right)$$

for firms 1, 2, and 3, respectively, where firm 1 produces the same output as in part (c), but firms 2 and 3 coordinate to reduce their output level from $\frac{1-c}{6}$ to $\frac{1-c}{8}$ units.

6. **Exercise 6.31 - Cost-reducing investment followed by Cournot competition-I.** Consider a duopoly market with two firms selling a homogeneous product, facing inverse demand curve $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output, and facing marginal cost c, where $1 > c \geq 0$. In the first stage of the game, every firm i chooses its investment in cost-reducing technology, z_i; and, in the second stage, observing the profile of investment levels (z_i, z_j), firms compete à la Cournot.

   Investment z_i reduces firm i's marginal cost, from c to $c - \frac{1}{4}z_i$; and the cost of investing z_i in the first stage is $\frac{1}{2}z_i^2$. For simplicity, assume no discounting of future payoffs.

   (a) *Second stage.* Operating by backward induction, find firm i's best response function $q_i(q_j)$ in the second stage. How is it affected by a marginal increase in z_i? And by a marginal increase in z_j?

- Firm i's profit maximization problem in second stage is:

$$\max_{q_i \geq 0} \ \pi_{i2} = (1 - q_i - q_j)q_i - \left(c - \frac{1}{4}z_i\right) q_i$$

Differentiating with respect to q_i, we obtain

$$1 - 2q_i - q_j - c + \frac{1}{4}z_i = 0$$

Solving for q_i gives firm i's best response function:

$$q_i(q_j) = \frac{1 - \left(c - \frac{1}{4}z_i\right)}{2} - \frac{1}{2}q_j$$

which originates at $\frac{1-\left(c-\frac{1}{4}z_i\right)}{2}$ and decreases in q_j at a rate of $\frac{1}{2}$. Note that when $z_i = 0$, the vertical intercept simplifies to $\frac{1-c}{2}$, as in standard Cournot models of quantity competition.

- The slope of the best response function is unaffected by z_i, but the vertical intercept increases in z_i since

$$\frac{\partial}{\partial z_i}\left(\frac{1 - \left(c - \frac{1}{4}z_i\right)}{2}\right) = \frac{1}{8} > 0$$

Therefore, as firm i's investment in the cost-reducing technology increases, its net cost decreases, and the best response function shifts upward. However, firm i's best response function is unaffected by its rival's investment in the cost-reducing technology, z_j.

(b) Find equilibrium output and profits in the second stage as a function of investment levels (z_i, z_j). Are they increasing in z_i? Are they increasing in z_j? Interpret.

- Inserting $q_j(q_i)$ into firm i's best response $q_i(q_j)$, we obtain

$$q_i = \frac{1 - \left(c - \frac{1}{4}z_i\right)}{2} - \frac{1}{2}\left(\underbrace{\frac{1 - \left(c - \frac{1}{4}z_j\right)}{2} - \frac{1}{2}q_i}_{q_j(q_i)}\right)$$

which simplifies to

$$q_i(z_i, z_j) = \frac{1-c}{3} + \frac{2z_i - z_j}{12}.$$

When $z_i = z_j = 0$, firm i produces the same output as under standard Cournot competition, $q_i = \frac{1-c}{3}$; when $z_i < \frac{z_j}{2}$, firm i produces fewer units; and when $z_i > \frac{z_j}{2}$, firm i produces more units.

- Inserting $q_i(z_i, z_j)$ into firm j's best response function, we find a symmetric output for firm j, as follows,

$$q_j(z_i, z_j) = \frac{1-c}{3} + \frac{2z_j - z_i}{12}.$$

Differentiating with respect to z_i and z_j, we obtain

$$\frac{\partial q_i(z_i, z_j)}{\partial z_i} = \frac{\partial q_j(z_i, z_j)}{\partial z_j} = \frac{1}{6} > 0 \text{ and}$$

$$\frac{\partial q_i(z_i, z_j)}{\partial z_j} = \frac{\partial q_j(z_i, z_j)}{\partial z_i} = -\frac{1}{12} < 0$$

Therefore, as firm i's (j's) investment in the cost-reducing technology z_i (z_j) increases, firm i produces more (fewer, respectively) units.

- Substituting the equilibrium outputs into firm i's second-period profit function, we obtain that

$$\begin{aligned}
\pi_{i2} &= \left[1 - \left(\frac{1-c}{3} + \frac{2z_i - z_j}{12}\right) - \left(\frac{1-c}{3} + \frac{2z_j - z_i}{12}\right)\right]\left(\frac{1-c}{3} + \frac{2z_i - z_j}{12}\right) \\
&\quad - \left(c - \frac{1}{4}z_i\right)\left(\frac{1-c}{3} + \frac{2z_i - z_j}{12}\right) \\
&= \left(\frac{1-c}{3} + \frac{1}{4}z_i - \frac{z_i + z_j}{12}\right)\left(\frac{1-c}{3} + \frac{2z_i - z_j}{12}\right) \\
&= \frac{[4(1-c) + 2z_i - z_j]^2}{144}
\end{aligned}$$

Differentiating profits with respect to z_i and z_j, we obtain

$$\frac{\partial \pi_{i2}(z_i, z_j)}{\partial z_i} = \frac{2z_i - z_j}{36} + \frac{1-c}{9}, \text{ and}$$

$$\frac{\partial \pi_{i2}(z_i, z_j)}{\partial z_j} = -\frac{2z_i - z_j}{72} - \frac{1-c}{18}$$

where the first derivative is unambiguously positive, thus indicating that firm i's profits increase in its own investment in the cost-reducing technology, z_i. Similarly, the second derivative is unambiguously negative, thus indicating that firm i's profits decrease in its rival's investment in the cost-reducing technology, z_j.

(c) *First stage.* Find the equilibrium investment levels that firms choose in the first stage, z_i^* and z_j^*.

- Firm i's profit maximization problem in first stage is:

$$\max_{z_i \geq 0} \pi_{i1} = \underbrace{\frac{[4(1-c) + 2z_i - z_j]^2}{144}}_{\pi_{i2}\text{, found in part (b)}} - \frac{1}{2}z_i^2$$

Differentiating with respect to z_i, we obtain

$$\frac{2z_i - z_j}{36} + \frac{1-c}{9} - z_i = 0$$

Solving for z_i, we have

$$z_i(z_j) = \frac{2(1-c)}{17} - \frac{1}{34}z_j$$

which originates at $\frac{2(1-c)}{17}$ and decreases in z_j at a rate of $\frac{1}{34}$. Intuitively, an increase in firm j's investment, z_j, induces firm i to respond decreasing its own investment, z_i.

- In a symmetric equilibrium, both firms invest the same amount in cost-reducing technologies, $z_i = z_j = z$. Inserting this property into the above first-order condition, yields

$$\frac{z}{36} + \frac{1-c}{9} - z = 0$$

Solving for z, we obtain

$$z^* = \frac{4(1-c)}{35}.$$

(d) *Joint venture.* If, in the first stage, firms could coordinate their investment levels (z_i, z_j) to maximize their joint profits, what would their investment levels be? This investment decision resembles a "joint venture," where firms coordinate their R&D activities, or any other decision, and then compete in a subsequent stage (in this case, à la Cournot). Compare your results with those in part (c).

- Firms' joint profit maximization problem in first stage is:

$$\begin{aligned}\max_{z_i, z_j \geq 0} \pi_1 &= \pi_{i1} + \pi_{j1} \\ &= \left[\frac{[4(1-c)+2z_i - z_j]^2}{144} - \frac{1}{2}z_i^2\right] + \left[\frac{[4(1-c)+2z_j - z_i]^2}{144} - \frac{1}{2}z_j^2\right]\end{aligned}$$

Differentiating with respect to z_i and z_j, we obtain

$$\frac{1-c}{18} - \frac{67z_i + 4z_j}{72} = 0$$

$$\frac{1-c}{18} - \frac{67z_j + 4z_i}{72} = 0$$

In a symmetric equilibrium, both firms invest the same amount in cost-reducing technologies, $z_i = z_j = z$. Inserting this property into the above first-order conditions, yields

$$\frac{1-c}{18} - \frac{71z}{72} = 0$$

Solving for z, we get

$$z^{JV} = \frac{4(1-c)}{71}$$

where the superscript JV denotes "joint venture."

- Comparing firms' investment in part (c) and (d) by taking the difference between z^* and z^{JP}, we have

$$\begin{aligned} z^* - z^{JV} &= \frac{4(1-c)}{35} - \frac{4(1-c)}{71} \\ &= \frac{144(1-c)}{2485} > 0 \end{aligned}$$

Intuitively, when firms internalize their investment in cost-reducing technologies (in the joint venture), they invest less than in equilibrium. In other words, when every firm i independently chooses its investment in cost-reducing technologies, z_i, it ignores the externality that its investment imposes on its rival (namely, expanding the cost differential between the two firms).