Chapter 9: Auction Theory

Game Theory:

An Introduction with Step-by-Step Examples

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Introduction

- Auctions have been historically reported as early as Babylon around 500 BC and in the Roman Empire in 193 AD.
  - They remain popular nowadays through online auctions such as eBay, eBid, QuiBids, AuctionMaxx, DealDash, or LiveAuctioneers, etc.
- Understanding optimal bidding behavior in different auction formats, then, is not only an interesting application of BNEs but also a useful tool for online shopping.
- We will consider different auction formats:
  - First-price auction (FPA). The bidder submitting the highest bid wins the auction. She pays the bid she submitted.
  - Second-price auction (SPA). The bidder submitting the highest bid wins the auction. But does not pay her bid; she pays, instead, the second-highest bid.
  - Common value auctions.

Auctions as allocation mechanisms

Two basic criteria to characterize auctions: the assignment and payment rules.

a. **Assignment Rule**

- Informally, the assignment (or allocation) rule in an auction just answers the question “who gets the object?”
  - In the first-price auction, for instance, the individuals submitting the highest bid receives the object for sale.
  - A similar assignment rule applies to other auction formats, such as:
    - the second-price, third-price, k^{th}-price, or all-pay auctions (APA), where the object still goes to the bidder submitting the highest bid.
  - In settings where the seller offers $m \geq 2$ units of the same good (multi-unit auctions), the assignment rule in first-price auctions, for instance, determines that:
    - the first unit of the object goes to the individual submitting the highest bid,
    - the second unit to the individual submitting the second-highest bid,
    - and similarly for the remaining units.

Auctions as allocation mechanisms

a. Assignment Rule

- In lottery auctions, however, bidder i may not receive the object even when she submits the highest bid. In particular, bidder i's probability of winning the auction is

$$\Pr(win) = \frac{b_i}{b_i + B_{-i}}$$

where $B_{-i} = \sum_{j \neq i} b_j$ denotes the aggregate bids of player i's rivals. Therefore, bidder i's probability of winning:

i. Increases in her own bid, b_i;
ii. Satisfies $Pr(win) < 1$ if at least one of her rivals submits a positive bid, $B_{-i} > 0$; and
iii. Decreases in her rivals' bids.

- *Example*: bidder i submits $b_i = \$100$, $N = 10$, and $b_j = \$5$ for every $j \neq i$, then $\Pr(win) = \frac{100}{100+(10\times5)} = \frac{2}{3}$.

Auctions as allocation mechanisms

b. Payment Rule

- The criterion answers the question "how much each bidder pays," which allows for the winner and losers to pay a monetary amount to the seller.
- In the first-price auction, the winner is the only bidder paying to the seller, and she must pay the bid she submitted.
- In the second-price auction, the winner is again the only player to the seller, but in this case she pays the second-highest bid (not the bid that she submitted).
- A similar argument applies to k^{th}-price auctions, where only the winning bidder pays to the seller, specifically the k^{th}-highest bid.
- In the all-pay auction, all bidders (the winner and the $N-1$ losers) pay the bid that each of them submitted.

Setting

- All auctions we consider in this chapter will share these ingredients:
- A seller offers an object to $N \geq 2$ bidders.
- Every bidder i privately observes her valuation for the object, $v_i \in [0,1]$, but does not observe bidder j's valuation, v_j, for every bidder $j \neq i$.
- It is common knowledge, however, that valuations v_i and v_j are independent and identically distributed (i.i.d.)
- v_j is drawn from a cumulative distribution function $F_j(v) = \Pr\{v \leq v_j\}$, with positive density in all its support, i.e., $f_j(v) = F_j'(v) > 0$ for all $v \in [0,1]$.
- After observing her valuation v_i, every bidder i simultaneously and independently submits her bid $b_i \geq 0$ for the object.
- The seller observes the profile of submitted bids, $b = (b_1, b_2, \ldots, b_N)$, and:
  - According to the assignment and payment rules of the auction, the seller declares:
    - a winning bidder,
    - the price that the winning bidder must pay for the object,
    - and potentially the prices that losing bidders must pay.

Second-price auctions (SPA)

- Assignment rule in SPA prescribes that the object goes to the individual who submitted the highest bid.
- Payment rule says that the winner must pay the second-highest bid while all other bidders pay zero.
- *Tie-breaking rule*: If two or more bidders submit the highest bid, then the object is randomly assigned among them with equal probabilities.

Second-price auctions (SPA)

- We seek to show that submitting a bid equal to her valuation,

$$b_i(v_i) = v_i$$

is a ***weakly dominant strategy*** for every player, i.e., it is a weakly dominant BNE of this game.

- This means that:
  - Regardless of the valuation bidder i assigns to the object, and
  - Independently of her opponents' bids,
- submitting a bid equal to her valuation, $b_i(v_i) = v_i$, yields an expected profit equal to or higher than deviating to:
  - A bid lower than her valuation, $b_i(v_i) < v_i$, or
  - A bid higher than her valuation, $b_i(v_i) > v_i$.

Case 1: Bid equal to his valuation

1.1 Bidder i wins.

- Define the highest competing bid among all bidder i's rivals, h_i, as follows
$$h_i = \max\{b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_N\}$$
- If the highest competing bid, h_i, lies below bidder i's bid, $h_i < b_i$:
  - bidder i wins the auction,
  - earning a net payoff of $v_i - h_i$ because in a SPA, the winning bidder does not pay the bid she submitted, but rather the second highest bid, h_i.

1.2 Bidder i loses.

- If, instead, the highest competing bid lies above her bid, $h_i > b_i$, then:
- bidder i loses the auction, earning zero payoff.

1.3 Tie in bids.

- If a tie occurs, where $b_i = h_i$, the object is randomly assigned, and bidder i's expected payoff becomes $\frac{1}{2}(v_i - h_i)$, where $\frac{1}{2}$ is the probability that bidder i receives the object.
- However, because $v_i = h_i$, in this case, the bidder earns a zero expected payoff.

Case 2: Downward deviations, bidding below her valuation

2.1 If the highest competing bid h_i lies below her bid (i.e., $h_i < b_i < v_i$):

- bidder i still wins the auction,
- earning a payoff $v_i - h_i$, as when she does not shade her bid.
- This indicates that bidder i's payoff does not increase relative to case 1.1, thus not having incentives to shade her bid.

2.2 If the highest competing bid, h_i, lies between b_i and v_i (i.e., $b_i < h_i \leq v_i$):

- bidder i loses,
- earning a payoff of zero.
- As a consequence, bidder i's payoff is the same (zero) as when she does not shade her bid, as shown in case 1.2.

2.3 If the highest competing bid h_i is higher than v_i (i.e., $b_i < v_i < h_i$):

- bidder i loses the auction,
- yielding the same outcome as when she submits a bid, $b_i = v_i$.
- In this case, bidder $i's$ payoff exactly coincides when she bids according to her valuation (case 1.3) and when she shades her bid.

Case 2: Downward deviations, bidding below her valuation

- Overall, when bidder i shades her bid:
  - she earns the same or lower payoff than when she submits a bid that coincides with her valuation for the object.
- In other words, she does not have strict incentives to shade her bid:
  - her payoff would not strictly improve from doing so,
  - regardless of the exact position of the highest competing bid.

Case 3: Upward deviations, bidding above her valuation

3.1 If the highest competing bid h_i lies below bidder i's valuation, v_i:

- she wins,
- earning a payoff of $v_i - h_i$.
- Her payoff in this case coincides with that when submitting a bid equal to her valuation, $b_i = v_i$, as in case 1.1, implying that she has no strict incentives to bid above her valuation.

3.2 If the highest competing bid h_i lies between v_i and b_i $(v_i < h_i < b_i)$:

- bidder i wins the object,
- but earns a negative payoff because $v_i - h_i < 0$.
- If, instead, bidder i submits a bid equal to her valuation, $b_i = v_i$, she would have lost the object, earning a zero payoff.
- In other words, bidder i would be better off submitting a bid $b_i = v_i$, and losing, than submitting a bid $b_i > v_i$, and winning but earning a negative payoff.

3.3 If the highest competing bid h_i lies above b_i $(v_i < b_i < h_i)$:

- bidder i loses the auction,
- earning a zero payoff.
- When she submits a bid equal to her valuation, $b_i = v_i$, in case 1.3, a tie occurs, but her expected payoff $\frac{1}{2}(v_i - h_i)$, is zero given that $v_i = h_i$ in that case. Therefore, bidder i has no strict incentives to deviate from bidding $b_i = v_i$.

Case 3: Upward deviations, bidding above her valuation

- Overall, bidder i can earn the same payoff as in cases 1.1-1.3, or a lower payoff,
  - but cannot strictly improve her payoff.
- In short, bidding according to her valuation $b_i = v_i$, is a weakly dominant strategy for every bidder i in the SPA.

Discussion

- **Bidding BNE vs. Dominant Strategies.**
  - Every bidder i finds that bidding according to her valuation is:
    - not only a BNE of the SPA,
    - it is the BNE where every player uses weakly dominant strategies.
  - If bidder i finds that her bidding function $b_i^* = b_i(v_i)$ is a BNE, this means that

$$EU_i(b_i^*, b_{-i}^* | v_i) \geq EU_i(b_i, b_{-i}^* | v_i)$$

    for every bid $b_i \neq b_i^*$ and every valuation v_i.
    - Intuitively, equilibrium bid b_i^* provides bidder i with a higher expected payoff than any other bid $b_i \neq b_i^*$, conditional on her rivals selecting equilibrium bids b_{-i}^* .
  - But we showed something stronger than that! (Next slide.)

Discussion

- **Bidding BNE vs. Dominant Strategies.**
  - When we say that bidder i finds that her equilibrium bid b_i^* is a *weakly dominant strategy*, we mean that
$$EU_i(b_i^*, b_{-i}|v_i) \geq EU_i(b_i, b_{-i}|v_i)$$
    for:
    - every bid $b_i \neq b_i^*$
    - every valuation v_i, and
    - every bidding profile her rivals use b_{-i} → *NEW.*
  - This inequality entails that bidder i's expected payoff from submitting bid b_i^* is higher than from any other bid $b_i \neq b_i^*$, *regardless* of the specific bidding profile that her rivals use, b_{-i}, that is:
    - both when they submit equilibrium bids, $b_{-i} = b_{-i}^*$,
    - and when they do not, $b_{-i} \neq b_{-i}^*$.
  - This is a strong property in the bidding strategy in SPAs, saying that bidder i, when submitting $b_i(v_i) = v_i$, can essentially, ignore her opponents' bids:
    - both when they submit equilibrium bids and when they do not.

Discussion

- **No Bid shading.**
  - Intuitively, by shading her bid, $b_i(v_i) < v_i$, bidder i:
    - lowers the chance that she wins the auction, but…
    - does not lower the price that she pays upon winning.
    - No tradeoff!
  - In the FPA, in contrast, bid shading gives rise to a trade-off:
    - A lower chance of winning the auction,
    - But paying a lower price upon winning the object.

First-price auctions (FPA)

- Assignment rule coincides with that in the SPA:
  - the winner is the bidder submitting the highest bid.
- Payment rule in the FPA, however, differs:
  - the winning bidder must pay the highest bid.
- This seemingly small difference between both auction formats give rise to bid shading in the FPA
  - a result that we could not sustain in equilibrium when bidders face a SPA.

First-price auctions (FPA)

Step 1. Writing bidder i's maximization problem.

- Bidder i's maximization problem is as follows:

$$\max_{b_i \geq 0} \quad Pr(win) \times \underbrace{(v_i - b_i)}_{\substack{net\ payoff\ that \\ bidder\ i\ earns \\ when\ she\ wins}} - Pr(lose) \times \underbrace{0}_{\substack{payoff \\ from\ losing}}$$

- At this point, we need to write the probability of winning, $Pr(win)$, as a function of bidder i's bid, b_i.
- To do this, note that every bidder i uses a symmetric bidding function $b_i\colon [0,1] \to \mathbb{R}_+$, a function mapping her valuation $v_i \in [0,1]$ into a positive dollar amount (her bid).

First-price auctions (FPA)

- If bidding functions are symmetric across players and monotonic, bidder i wins when her bid satisfies $b_j \leq b_i$,
  - which must indicate that her valuation satisfies $v_j \leq v_i$.
  - This ranking between valuations v_i and v_j occurs if $\Pr(v_j \leq v_i) = F(v_i)$.
- Therefore, when bidder i faces $N-1$ rivals, her probability of winning the auction is:
  - the probability that her valuation exceeds that of all other $N-1$ rivals.
- Since valuations are i.i.d., we can write this probability as the product

$$Pr(v_j \leq v_i) \times Pr(v_k \leq v_i) \times \ldots \times Pr(v_l \leq v_i)$$

$$= \underbrace{F(v_i) \times F(v_i) \times \ldots \times F(v_i)}_{N-1\ times} = F(v_i)^{N-1}$$

where bidders $j \neq k \neq l$ represent $i's$ rivals.

First-price auctions (FPA)

As a result, we can express the above expected utility maximization problem as follows:

$$\max_{b_i \geq 0} \quad \underbrace{F(v_i)^{N-1}}_{Pr(win)} (v_i - b_i)$$

First-price auctions (FPA)

- Using the above bidding function, we can write
  - $b_i(v_i) = x_i$, where $x_i \in \mathbb{R}_+$ is the bidder i's bid when her valuation is v_i.
  - Or $v_i = b_i^{-1}(x_i)$ by applying the inverse of $b_i(.)$ on both sides.
- So, the program becomes

$$\max_{x_i \geq 0} \quad F\left(b_i^{-1}(x_i)\right)^{N-1}(v_i - x_i)$$

First-price auctions (FPA)

Step 2. Finding equilibrium bids.

- Differentiating with respect to x_i, yields
$$-\left[F\left(b_i^{-1}(x_i)\right)^{N-1}\right]+(N-1)F\left(b_i^{-1}(x_i)\right)^{N-2}f\left(b_i^{-1}(x_i)\right)\frac{\partial b_i^{-1}(x_i)}{\partial x_i}(v_i-x_i)=0$$
- Since $b_i^{-1}(x_i)=v_i$, we can use the inverse function theorem to obtain $\frac{\partial b_i^{-1}(x_i)}{\partial x_i}=\frac{1}{b'\left(b_i^{-1}(x_i)\right)}$. So simplifying and rearranging,

$$F(v_i)^{N-1}b'(v_i)+\ (N-1)F(v_i)^{N-2}f(v_i)x_i=(N-1)\ F(v_i)^{N-2}f(v_i)v_i$$

- The left-hand side is $\frac{\partial\left[F(v_i)^{N-1}b_i(v_i)\right]}{\partial v_i}$, which let us write the above expression more compactly as
$$\frac{\partial[F(v_i)^{N-1}b_i(v_i)]}{\partial v_i}=(N-1)F(v_i)^{N-2}f(v_i)v_i$$

First-price auctions (FPA)

Step 2. Finding equilibrium bids.

- Integrating both sides of $\frac{\partial\left[F(v_i)^{N-1}b_i(v_i)\right]}{\partial v_i} = (N-1)F(v_i)^{N-2}f(v_i)v_i$, yields

$$F(v_i)^{N-1}b_i(v_i) = \int_0^{v_i}(N-1)\,F(v_i)^{N-2}f(v_i)v_i dv_i$$

- We could now solve for $b_i(v_i)$ and obtain an equilibrium bidding function

$$b_i(v_i) = \frac{N-1}{F(v_i)^{N-1}}\int_0^{v_i} F(v_i)^{N-2}f(v_i)v_i dv_i$$

- But this presentation does not help us see the role of bid shading.
  - For that, we need to represent the bid as a function of bidder *i*'s valuation, v_i, minus one term capturing bid shading.
  - For that, we need to apply integration by parts on the RHS. (Next slide.)

First-price auctions (FPA)

Step 3. Applying integration by parts.

- Recall that

$$g(x)h(x) = \int g'(x)h(x)dx + \int g(x)h'(x)dx$$

- Reordering this expression, we find

$$\int g'(x)h(x)dx = g(x)h(x) - \int g(x)h'(x)dx$$

- Applying this in the above expression (RHS), yields:

$$\int_0^{v_i} \underbrace{[(N-1)F(x)^{N-2}f(x)]}_{g'(x)} \underbrace{x}_{h(x)} \; dx = \underbrace{F(v_i)^{N-1}}_{g(x)} \underbrace{v_i}_{h(x)} - \int_0^{v_i} \underbrace{F(x)^{N-1}}_{g(x)} \underbrace{1}_{h'(x)}$$

First-price auctions (FPA)

Step 3. Applying integration by parts.

- Inserting this results in the right-hand side of first-order condition, yields

$$F(v_i)^{N-1} b_i(v_i) = F(v_i)^{N-1}\, v_i - \int_0^{v_i} F(v_i)^{N-1} dv_i$$

- Solving for $b_i(v_i)$, we obtain the equilibrium bid in the FPA:

$$b_i(v_i) = v_i - \underbrace{\frac{\int_0^{v_i} F(v_i)^{N-1} dv_i}{F(v_i)^{N-1}}}_{Bid\ shading}$$

- Intuitively, bidder i submits a bid:
  - equal to her valuation for the object, v_i,
  - *less* an amount captured by the second term, which we refer as her “bid shading.”

First-price auctions (FPA)

Step 4. Checking Monotonicity

- We finally check that the above bidding function $b_i(v_i)$ is monotonically increasing in bidder $i's$ valuation, v_i. A marginal increase in v_i produces the following effect in bidder $i's$ equilibrium bidding function:

$$\frac{\partial b_i(v_i)}{\partial v_i} = 1 - \frac{F(v_i)^{N-1}F(v_i)^{N-1} - (N-1)F(v_i)^{N-2}\int_0^{v_i} F(v_i)^{N-1} dv_i}{[F(v_i)^{N-1}]^2}$$

$$= \frac{(N-1)F(v_i)^{N-2} f(v_i) \int_0^{v_i} F(v_i)^{N-1} dv_i}{[F(v_i)^{N-1}]^2}$$

which is positive since $F(v_i) \in [0,1]$, $f(v_i) > 0$ for all v_i and $N \geq 2$ by definition.

But this effect is less than proportional (See next slide).

First-price auctions (FPA)

Step 4. Checking Monotonicity

- Note also that $\frac{\partial b_i(v_i)}{\partial v_i} = \frac{(N-1)F(v_i)^{N-2}f(v_i)\int_0^{v_i} F(v_i)^{N-1}dv_i}{[F(v_i)^{N-1}]^2} < 1$:

$$\Rightarrow \frac{(N-1)f(v_i)\int_0^{v_i} F(v_i)^{N-1}dv_i}{F(v_i)^N} = (N-1)\ \frac{f(v_i)}{F(v_i)} \times \frac{\int_0^{v_i} F(v_i)^{N-1}dv_i}{F(v_i)^{N-1}}$$

$$\Rightarrow (N-1) \times \frac{dlogF(v_i)}{dv_i} \times \frac{\int_0^{v_i} F(v_i)^{N-1}dv_i}{F(v_i)^{N-1}}$$

- Let $\log F(v_i) = g(v_i)$, then we have:

$$\Rightarrow (N-1) \times \frac{dg(v_i)}{dv_i} \times \frac{\int_0^{v_i} e^{(N-1)g(v_i)}dv_i}{e^{(N-1)g(v_i)}} = (N-1)\frac{\int_0^{v_i} e^{(N-1)g(v_i)}dg(v_i)}{e^{(N-1)g(v_i)}}$$

$$\Rightarrow \left.\frac{\left(\frac{1}{N-1}\right)e^{(N-1)g(v_i)}}{e^{(N-1)g(v_i)}}\right|_0^{v_i} \times (N-1) = \frac{e^{(N-1)g(v_i)} - e^{(N-1)g(0)}}{e^{(N-1)g(v_i)}} < \frac{e^{(N-1)g(v_i)}}{e^{(N-1)g(v_i)}} = 1.$$

Meaning that an increase in bidder i's valuations leads her to increase her bid, but less than proportionally.

Example 9.1. FPA with uniformly distributed valuations

- Consider, for instance, when individual valuations are uniformly distributed, i.e. $F(v_i) = v_i$.
- In this setting, we obtain $F(v_i)^{N-1} = v_i^{N-1}$ and $\int_0^{v_i} F(v_i)^{N-1} dv_i = \frac{1}{N} v_i^N$, producing a bidding function of

$$b_i(v_i) = v_i - \frac{\frac{1}{N}\ v_i^N}{v_i^{N-1}} = v_i - \underbrace{\frac{v_i}{N}}_{\substack{Bid\\shading}} = v_i \left(\frac{N-1}{N}\right)$$

- In this context, every bidder shades her bid by $\frac{v_i}{N}$, which increases in the number of competing bidders.
- In addition, the equilibrium bidding function $b_i(v_i) = v_i \left(\frac{N-1}{N}\right)$ is monotonically increase in the valuation that bidder i assigns to the object, v_i, as required.

Example 9.1. FPA with uniformly distributed valuations

- When only 2 bidders compete for the object, $N = 2$, this bidding function simplifies to $b_i(v_i) = \frac{v_i}{2}$ as depicted in Figure 9.1.
- When $N = 3$, equilibrium bids increase to $b_i(v_i) = \frac{2v_i}{3}$.
- When $N = 4$, equilibrium bids increase to $b_i(v_i) = \frac{3v_i}{4}$.

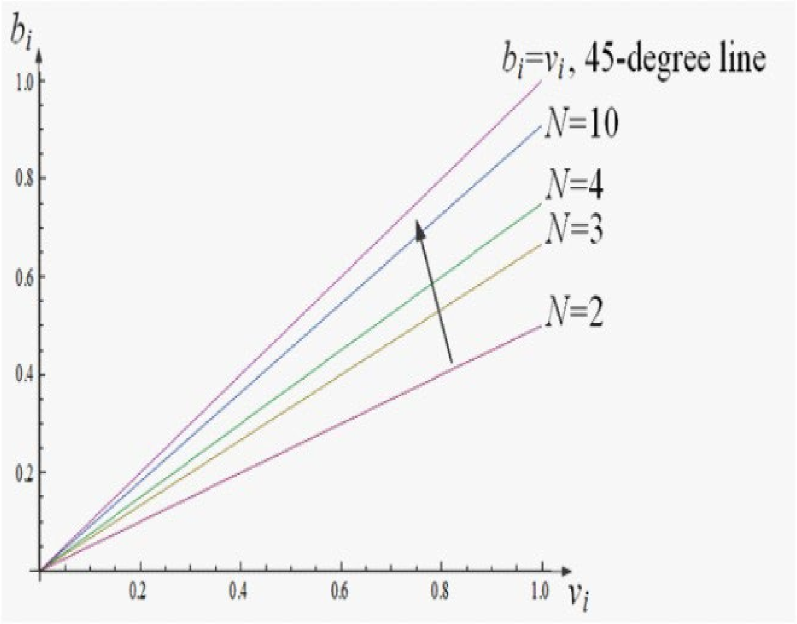


Figure 9.1. Equilibrium bids in the first-price auction with uniformly distributed valuations.

Example 9.1. FPA with uniformly distributed valuations

- Informally, as more bidders participate in the auction, every bidder i submits more aggressive bids since:
  - she faces a higher probability than another bidder j has a higher valuation for the object.
  - And, given symmetric bidding functions, bidder j submits a higher bid than she does, leading to bidder i to lose the auction.
- In the extreme case that $N \to \infty$, the bidding function converges to $b_i(v_i) = v_i$.

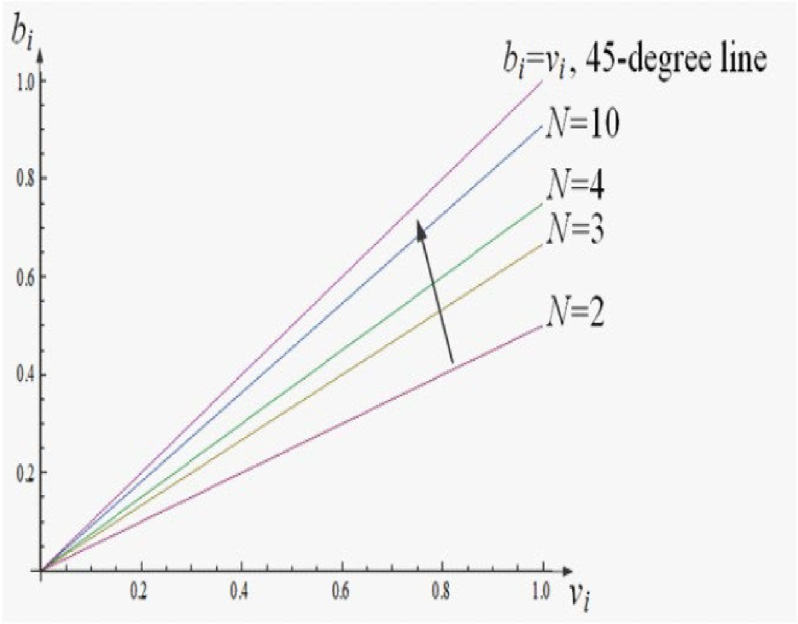


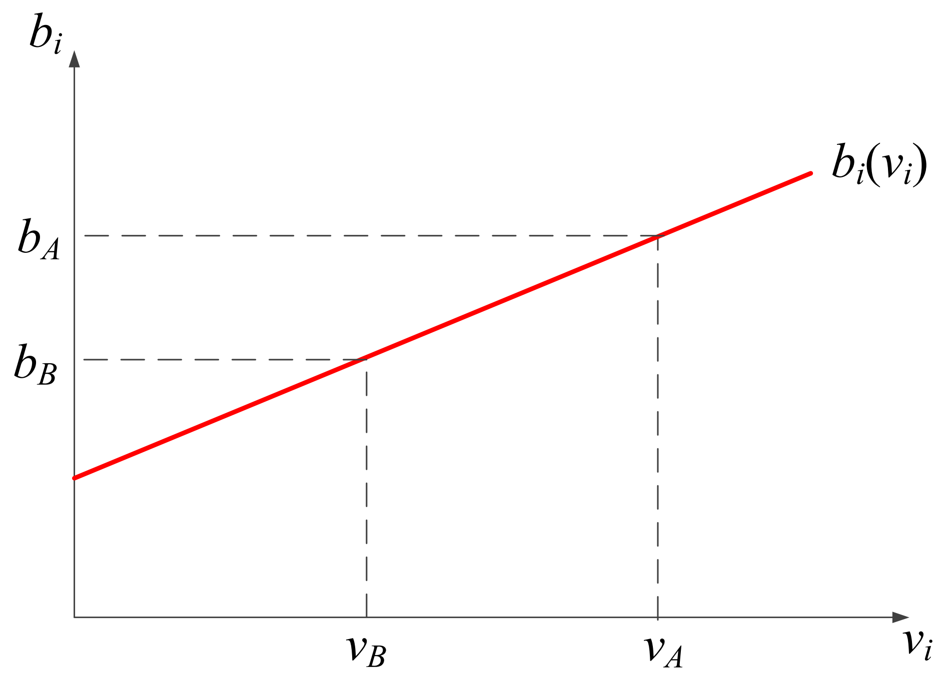
Figure 9.1. Equilibrium bids in the first-price auction with uniformly distributed valuations.

Efficiency in auctions

- An auction is deemed "efficient" when it assign the object to the individual with the highest valuation.
  - That is, when the assignment rule allocates the object to bidder i if only if her valuation, v_i, satisfies $v_i > v_j$ for all $j \neq i$.
- Otherwise, if bidder j receives the object despite having a lower valuation than bidder i:
  - these two bidders could negotiate at the end of the auction,
  - with bidder i paying a price p that satisfies $v_i > p > v_j$,
  - making both bidders better off.
- The assignment rule that allocates the object to bidder j is Pareto inefficient.
  - We can find an alternative allocation that improves the payoff of at least one individual without making any other individual worse off.
- If bidders use a *symmetric, strictly increasing, bidding function* in equilibrium:
  - the winner of the auction must be the individual with the highest valuation,
  - making the auction efficient.

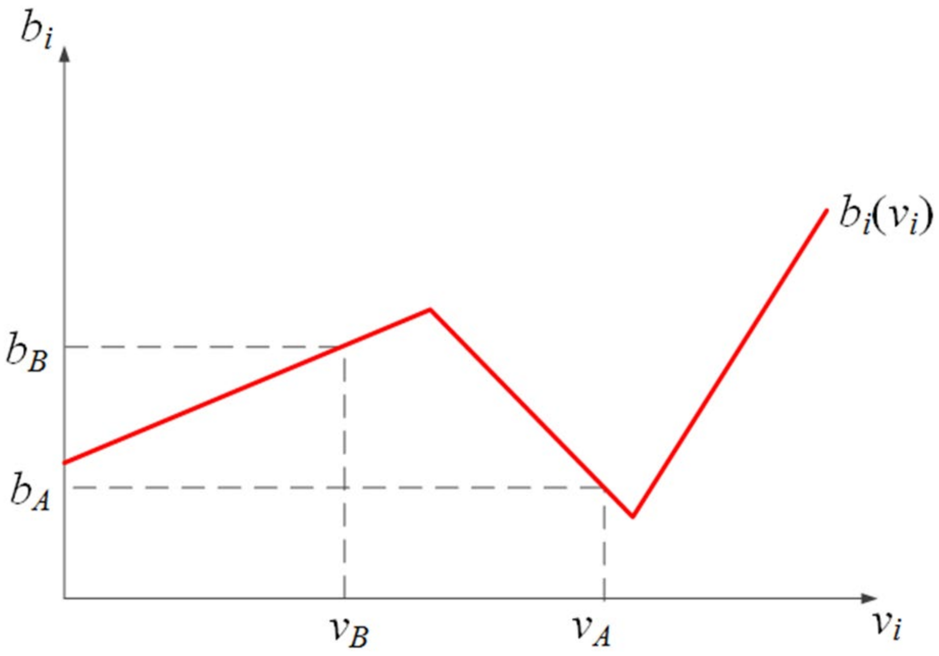
Efficiency in auctions

- *Symmetric* and *strictly increasing* bidding function:
  - Because bidder A has a higher valuation than B does, he submits a higher bid and wins the object.



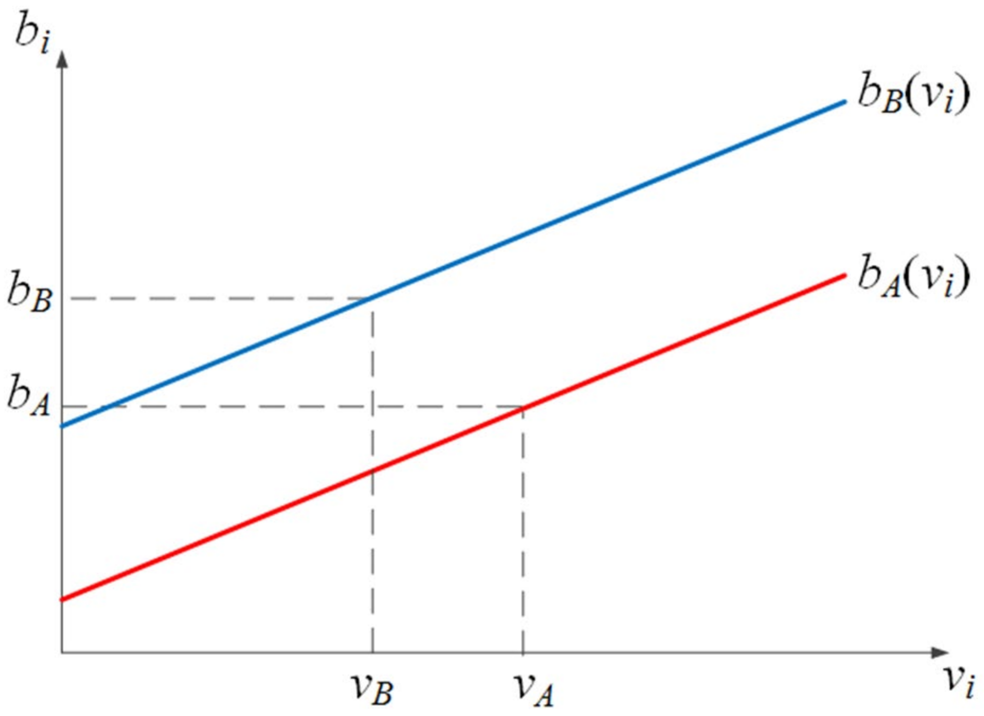
Efficiency in auctions

- *Symmetric* but *Not strictly increasing* bidding function:
  - Bidder A has a higher valuation than B does, but he may submit a lower bid and lose the object.



Efficiency in auctions

- *Asymmetric* but *strictly increasing* bidding functions:
  - Bidder A has a higher valuation than B does, but he may submit a lower bid and lose the object.

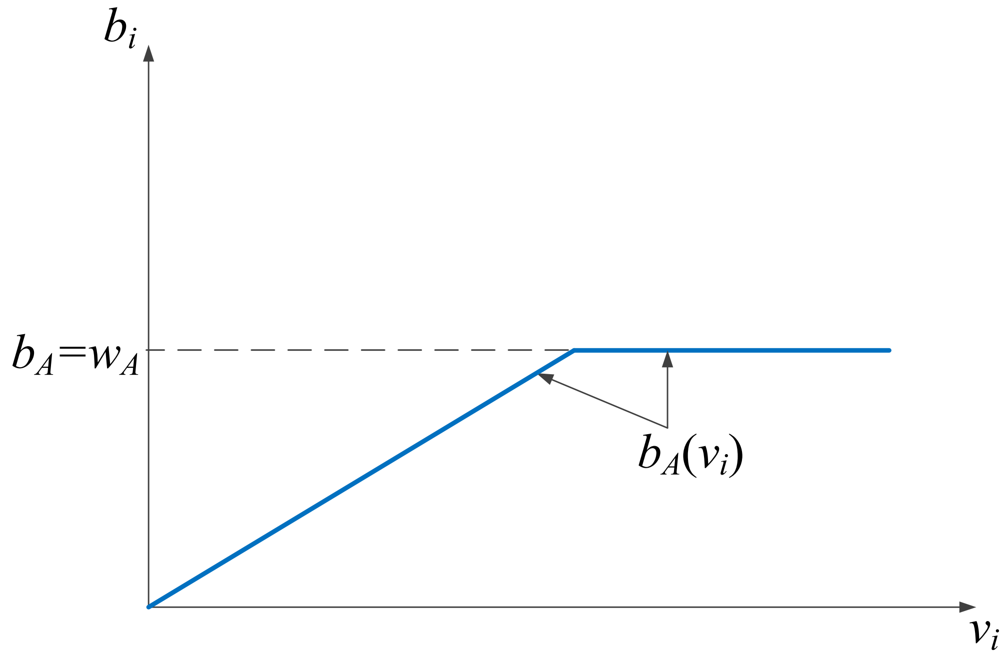


Efficiency in auctions

- Two typical features that “break” efficiency in auctions:
  - Budget constraints, either symmetric or asymmetric.
  - Asymmetric risk averse bidders.

Budget constraints

- When bidder i faces a budget constraint w_i,
  - her bidding function is increasing in v_i for all $b_i < w_i$ (affordable bids).
  - But becomes flat at the height of w_i for all valuations for which $b_i > w_i$ (unaffordable bids)

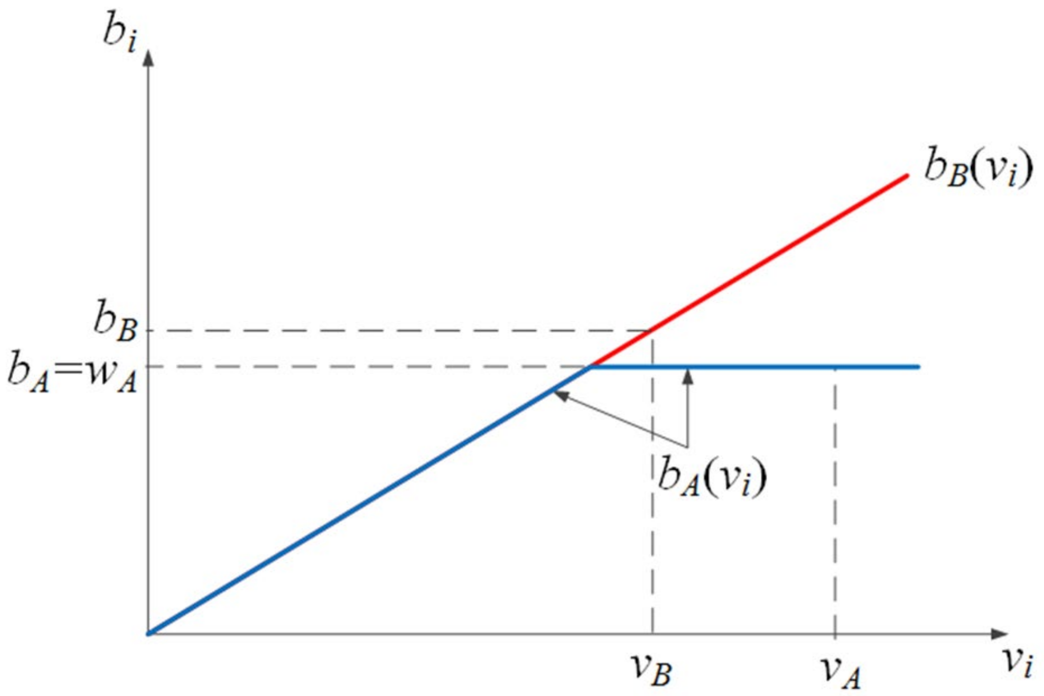


Budget constraints

- Budget constraints imply that bidding functions are ***weakly*** increasing in v_i, not ***strictly*** increasing,
  - so we cannot guarantee that the auction is efficient.
  - Recall that we need bidders to use "*symmetric, strictly increasing, bidding functions* in equilibrium"
- Figure to illustrate this point (next slide).

Budget constraints

- Bidder A values the object more than B does…
  - yet he may submit a lower bid, losing the object.



Budget constraints

- Budget constraints imply that bidding functions are <u>*weakly*</u> increasing in v_i, not <u>*strictly*</u> increasing,
  - so we cannot guarantee that the auction is efficient.
  - Recall that we need bidders to use "*symmetric, strictly increasing, bidding functions* in equilibrium"
- This occurs even if:
  - only one individual suffers from budget constraints,
  - or if all bidders face the same budget constraint $w_i = w$,
  - making their bidding functions symmetric (but not strictly increasing).

Asymmetric risk averse bidders

- Another context where efficiency is not satisfied is that where players exhibit different degrees of risk aversion.
- Bidders competing in a FPA submit more aggressive bids when they become more risk-averse.
  - If both bidders are symmetric in their risk aversion, the above condition hold (i.e. bidders use a symmetric, strictly increasing, bidding function).
- When bidders are asymmetric in their risk preferences, however, the FPA is not efficient.

Asymmetric risk averse bidders

- Consider FPA with two bidders, A and B, where bidder A exhibits more (less) risk aversion.
- Scenarios:
  - Bidder A can be risk averse, while bidder B is risk neutral or risk-loving,
  - both bidders can be risk-averse but A is more risk averse than B,
  - or both are risk lovers but A is less so than B.
- In any of these settings, bidder A submits a more aggressive bid than B does, $b_A > b_B$, implying that A wins the auction.
- If bidder A values the object more than B does, $v_A > v_B$, the outcome of the auction is still efficient; but otherwise the outcome is inefficient.
- Generally, then, we cannot guarantee that the object goes to the individual who values the object the most, especially if bidders are relatively asymmetric in their risk aversion, entailing that the FPA is not efficient when players are risk averse.

Seller's expected revenue: Expected revenue in the FPA

Step 1. *Finding each bidder's payment*

- The seller receives a payment from bidder i if she wins the auction. In other words, bidder i's payment is
$$m(v_i) = Pr(win)\ \times\ b_i\,(v_i)$$
- We know that $Pr(win) =\ [F(v_i)]^{N-1}$, so

$$m(v_i) = [F(v_i)]^{N-1}\ \times\ b_i(v_i) = G(v_i) \times\ b_i(v_i)$$, where $G(v_i) = [F(v_i)]^{N-1}$

- Recall that the equilibrium bidding function in the FPA is

$$b_i(v_i) = v_i - \frac{\int_0^{v_i} F\,(x)^{N-1} dx}{F(v_i)^{N-1}} = v_i - \frac{\int_0^{v_i} G(x)\,dx}{G(v_i)} = \frac{G(v_i)v_i - \int_0^{v_i} G(x)\,dx}{G(v_i)}$$

Seller's expected revenue: Expected revenue in the FPA

Step 1. *Finding each bidder's payment*

- Applying integration by parts in the numerator, we obtain

$$G(v_i)v_i - \int_0^{v_i} G(x)\,dx = \int_0^{v_i} xg(x)\,dx$$

- So we can rewrite the equilibrium bidding function:

$$b_i(v_i) = \frac{\int_0^{v_i} xg(x)\,dx}{G(v_i)}$$

- Inserting this equilibrium bidding function in bidder i's expected payment, yields

$$m(v_i) = G(v_i) \times \overbrace{\left(\frac{\int_0^{v_i} xg(x)dx}{G(v_i)}\right)}^{Bid,\ b_i(v_i)} = \int_0^{v_i} xg(x)dx$$

Seller's expected revenue: Expected revenue in the FPA

Step 2. *Finding the expected payment.*

- Since the seller cannot observe bidder $i's$ value for the object, she needs to take expectations over all possible values to find the expected payment from this bidder, $E[m(v_i)]$, as follows:

$$E[m(v_i)] = \int_0^1 m(x)f(x)dx = \int_0^1 \underbrace{\left[\int_0^{v_i} xg(x)dx\right]}_{m(v_i)} f(x)dx$$

Step 3. *Sum across all bidders.*

- The seller sums across all N bidders participating in the auction, which yields the expression of her revenue in the FPA.

$$R^{FPA} = \sum_{i=1}^{N} E[\pi_i(v_i)] = N \times E[m(v_i)] = N \int_0^1 \left[\int_0^{v_i} xg(x)dx\right] f(x)dx$$

Example 9.2. Expected Revenue in FPA with Uniformly Distributed Valuations

- When valuations are uniformly distributed, $F(v_i) = v_i$, we obtain that:
  - $f(v_i) = F'(v_i) = 1$
  - $G(v_i) = [F(v_i)]^{N-1} = v_i^{N-1}$, which implies that:
  - $g(v_i) = G'(v_i) = (N-1)v_i^{N-2}$,
  - $v_i g(v_i) = v_i\,(N-1)v_i^{N-2} = (N-1)v_i^{N-1}$,
- Therefore, bidder i's equilibrium bid is

$$b_i(v_i) = v_i\left(\frac{N-1}{N}\right)$$

- **Step 1**. The expected payment from bidder i is:

$$m(v_i) = \int_0^{v_i} xg(x)dx = \int_0^{v_i} \underbrace{(N-1)x^{N-1}}_{xg(x)} dx = \frac{N-1}{N}[x^N]_0^{v_i} = \frac{N-1}{N}\left(v_i^N - 0\right) = \frac{N-1}{N}v_i^N$$

Example 9.2. Expected Revenue in FPA with Uniformly Distributed Valuations

- **Step 2.** Therefore, bidder i's expected payment is:

$$E[m(v_i)] = \int_0^1 m(v_i)f(v_i)dv_i = \int_0^1 \underbrace{\frac{N-1}{N}v_i^N}_{m(v_i)} \underbrace{1}_{f(v_i)} dv_i = \frac{N-1}{N}\int_0^1 v_i^N dv_i = \frac{N-1}{N}\left[\frac{v_i^{N+1}}{N+1}\right]_0^1 = \frac{N-1}{N(N+1)}$$

- **Step 3.** Finally, the seller sums across all N bidders to obtain the expected revenue from the FPA, R^{FPA}, as follows,

$$R^{FPA} = \sum_{i=1}^{N} E[m(v_i)] = N \times E[m(v_i)] = N\frac{N-1}{N(N+1)} = \frac{N-1}{N+1}$$

Example 9.2. Expected Revenue in FPA with Uniformly Distributed Valuations

- Therefore,

$$R^{FPA} = \frac{N-1}{N+1}$$

- The expected revenue is:
  - increasing in the number of bidders, N,
  - but at a decreasing rate.
  - Approaches 1 when $N \to \infty$.
- This result goes in line with that in Example 9.1:
  - as more bidders compete in the auction, they submit more aggressive bids, i.e., $b_i(v_i)$ increases in N,
  - increasing as a result the expected winning bid that the seller earns.

Expected Revenue in the SPA

- In this auction format, the seller anticipates that every bidder i submits a bid $b_i(v_i) = v_i$.
- The winning bidder pays the second-highest bid.
- The second-highest bid coincides with the second-highest valuation for the object, $v_1^{[2]}$. That is,

$$R^{SPA} = E\left[v_1^{[2]}\right] = \int_0^1 x f^{[2]}(x) dx$$

Expected Revenue in the SPA

- We first identify the cumulative distribution function $F^2(x) = Pr(v^2 \leq x)$ which happens when two events occur:
  1. The valuations of all N bidders are below x, or formally, $v_i \leq x$ for every bidder i. This event happens with probability
  $$Pr\{v_1 \leq x\} \times \ldots \times Pr\{v_N \leq x\} = \underbrace{F(x) \times \ldots \times F(x)}_{N\ times} = [F(x)]^N$$
  2. The valuations of $N - 1$ bidders are below x, $v_i \leq x$, but that of only one bidder j is above x, $v_i > x$. This even can occur in N different ways:
     - $v_1 > x$ for bidder 1 but $v_i \leq x$ for every bidder $i \neq 1$;
     - $v_2 > x$ for bidder 2 but $v_i \leq x$ for every bidder $i \neq 2$;
     - …
     - $v_N > x$ for bidder N but $v_i \leq x$ for every bidder $i \neq N$.

Expected Revenue in the SPA

- Each of these N cases happens with probability

$$\underbrace{[1 - F(x)]}_{v_i > x \; for \; i} \times \underbrace{[F(x)]^{N-1}}_{v_j \leq x \; for \; every \; j \neq i}$$

where $[1 - F(x)]$ denotes the probability that $v_i > x$ for a given bidder i, while $[F(x)]^{N-1}$ represents the probability that $v_j \leq x$ for all other bidders $j \neq i$.

- Summing over the above N cases, we find that event (2) happens with probability

$$\sum_{i=1}^{N}[1 - F(x)][F(x)]^{N-1} = N(1 - F(x)) \, [F(x)]^{N-1}$$

- Summarizing, the cumulative distribution function of the second-highest valuation $F^{[2]}(x)$, is

$$F^{[2]}(x) = [F(x)]^N + N(1 - F(x))[F(x)]^{N-1}$$

Expected Revenue in the SPA

- Rearranging,

$$F^{[2]}(x) = N[F(x)]^{N-1} - (N-1)[F(x)]^{N}$$

- Differentiating with respect to x:

$$f^{[2]}(x) = N(N-1)F(x)^{N-2}[1-F(x)]f(x)$$

- Inserting density function into seller's expected revenue

$$R^{SPA} = \int x f^{[2]}(x)dx = \int x \underbrace{N(N-1)F(x)^{N-2}[1-F(x)]f(x)}_{f^{[2]}(x)}dx$$

Example 9.3. Expected Revenue in SPA with uniformly distributed valuations

- When valuations are uniformly distributed, $F(x) = x$ and $f(x) = 1$, the seller's expected revenue becomes

$$
\begin{aligned}
R^{SPA} &= \int_0^1 xN(N-1)x^{N-2}[1-x]dx \\
&= (N-1)\int_0^1 Nx^{N-1}dx - N(N-1)\int_0^1 x^N dx \\
&= (N-1)x^N\Big|_0^1 - \frac{N(N-1)x^{N+1}}{N+1}\Big|_0^1 \\
&= (N-1) - \frac{N(N-1)}{N+1} \\
&= \frac{N-1}{N+1}
\end{aligned}
$$

which coincides with that in the FPA, R^{SPA}, found in Example 9.2., thus being increasing and concave in the number of bidders, N.

Revenue Equivalence Principle

- When bidders' valuations are uniformly distributed, $R^{FPA} = R^{SPA}$, implying that the seller can expect to earn the same revenue from both auction formats.
- This "revenue equivalence" result extends to several other auction formats:
  - yielding the same expected revenue as the FPA and SPA,
  - and to settings where bidders' valuations are non-uniformly distributed.
- We can identify the two main requirements that two auction formats must satisfy to yield the same expected revenue for the seller:
  1. Same allocation rule in both auction formats, e.g., the bidder submitting the highest bid receives the object, as in the FPA and SPA.
  2. Same expected utility of the bidder who has the lowest valuation for the object, e.g., zero in most auction formats since this bidder loses the auction, not receiving the object.

Revenue Equivalence Principle

- It is straightforward to note that the comparison of FPA and SPA satisfies conditions (1) and (2), thus generating the same expected revenue.
- But the comparison of FPA and APA does not satisfy condition (2):
  - because the bidder with the lowest valuations earns a zero payoff in the FPA but a negative payoff in the APA after paying here bid.
- Therefore, the FPA and APA do not necessarily generate the same expected revenue.
- Similarly, the comparison of the FPA and the lottery auction does not satisfy condition (1),
  - since the lottery auction does not necessarily assign the object to the individual submitting the highest bid in the FPA.
- As a consequence, the FPA and the lottery auction do not yield the same expected revenue for the seller.

Common value auctions and the winner's curse

- Every bidder i shares a common value for the object, v, but privately observes a noisy signal s_i about the object's value drawn from $F(s_i)$, where $s_i \in [0,1]$.
- Based on this signal, every bidder i submits her bid, b_i.
- This setting is known as "common value" auctions
- Bidders participate in a first-price, sealed-bid auction:
  - if bidder i wins, her realized payoff becomes $v - b_i$,
  - and if she loses her payoff is zero.
- Experimentally tested with a jar of nickels (Explain).
- For simplicity, we consider a setting with only two bidders, and that the true value is equal to the average of bidders' signals, so that $v = \frac{s_i + s_j}{2}$.

Bid shading is a must!

- Note that bidder i falls prey of the winner's curse if her bid exceeds the object's true value (which no bidder observes), $b_i > v$,
  - thus earning a negative payoff, $v - b_i < 0$, from winning the auction.
- In particular, this occurs if

$$b_i > \frac{s_i + s_j}{2}$$

- When bidder $i's$ bid is a function of her privately observed signal, $b_i = \alpha s_i$, where $\alpha \in [0,1]$, this inequality becomes

$$\alpha s_i > \frac{s_i + s_j}{2} \Rightarrow s_i > \frac{s_j}{2\alpha - 1}$$

- When bidder i submits a bid equal to the signal she received, the winner's curse occurs if $s_i > s_j$.

Bid shading is a must!

- Intuition:
  - If every bidder submits a bid that coincides with her privately observed signal, the bidder who received the highest signal ends up submitting the highest bid,
  - She wins the auction
  - But suffers form the *winner's curse*.
- In other words, the fact that she won the auction means that she received an overestimated signal of the object.
- If instead, bidder i submits a bid equal to $\frac{3}{4}$ of the signal she received, $\alpha = \frac{3}{4}$, the winner's curse only emerges if $s_i > 2s_j$;
  - that is, when bidder i's signal is larger than the double of bidder j's.
- More generally, as bidder i shades her bid more severely (decreasing α),
  - Ratio $\frac{s_j}{2\alpha - 1}$ increases,
  - and the winner's curse is less likely to occur.

Equilibrium Bidding in Common Value Auctions

Step 1. *Finding the expected utility.*

- Bidder $i's$ expected payoff from participating in the auction is

$$Pr\left(b_i > b_j\right) \times \left\{E\left[v|s_i, b_i > b_j\right] - b_i\right\}$$

where $E\left[v|s_i, b_i > b_j\right]$ is bidder $i's$ expected valuation, conditional on her signal s_i, and on knowing that she submitted the highest bid, i.e., $b_i > b_j$.

- Since bidder j uses bidding function $b_j = \alpha s_j$, as this expression becomes

$$Pr\left(b_i > \alpha s_j\right) \times \left\{E\left[v|s_i, b_i > \alpha s_j\right] - b_i\right\}$$

- Solving for s_j in the probability (first term) and in the inequality inside the expectation operator (second term), we obtain that

$$Pr\left(\frac{b_i}{\alpha} > s_j\right) \times \left\{E\left[v|s_i, \frac{b_i}{\alpha} > s_j\right] - b_i\right\}$$

Equilibrium Bidding in Common Value Auctions

- Step 1. *Finding the expected utility.*
  - Next, inserting $v = \frac{s_i+s_j}{2}$ in the expectation operator, yields
$$Pr\left(\frac{b_i}{\alpha} > s_j\right) \times \left\{E\left[\frac{s_i+s_j}{2}\middle|\frac{b_i}{\alpha} > s_j\right] - b_i\right\}$$
  - Recall that bidder i observes her signal, s_i, so that $E[s_i] = s_i$, but does not know her rival's, s_j, entailing that
$$Pr\left(\frac{b_i}{\alpha} > s_j\right) \times \left\{\frac{s_i}{2} + \frac{1}{2}E\left[s_j\middle|\frac{b_i}{\alpha} > s_j\right] - b_i\right\}$$
  - Using the uniform distribution on s_j, we have
$$Pr\left(\frac{b_i}{\alpha} > s_j\right) = \frac{b_i}{\alpha} \quad \text{and} \quad E\left[s_j\middle|\frac{b_i}{\alpha} > s_j\right] = \frac{b_i}{2\alpha}$$

  because s_j, which is a uniformly distributed random variable, falls into the range $\left[0, \frac{b_i}{\alpha}\right]$, yielding an expected value of $\frac{b_i}{2\alpha}$.
  - Inserting these results into bidder $i's$ expected payoff, we obtain that
$$\frac{b_i}{\alpha}\left[\frac{s_i}{2} + \frac{1}{2}\left(\frac{b_i}{2\alpha}\right) - b_i\right]$$

Equilibrium Bidding in Common Value Auctions

Step 2. *Taking first-order conditions.*

- Every bidder i chooses her bid b_i to maximize her expected utility, solving the following problem

$$\max_{b_i \geq 0} \quad \frac{b_i}{\alpha}\left[\frac{s_i}{2} + \frac{1}{2}\left(\frac{b_i}{2\alpha}\right) - b_i\right]$$

Taking the first-order conditions with respect to b_i, we obtain

$$\frac{1}{\alpha}\left[\frac{s_i}{2} + \frac{1}{2}\left(\frac{b_i}{2\alpha}\right) - b_i\right] + \frac{b_i}{\alpha}\left(\frac{1}{4\alpha} - 1\right) = 0$$

Simplifying, yields

$$\frac{1 - 4\alpha}{2\alpha^2} b_i = -\frac{s_i}{2\alpha} \quad \Rightarrow \quad b_i = \frac{\alpha}{4\alpha - 1} s_i$$

Equilibrium Bidding in Common Value Auctions

Step 3. *Finding the equilibrium bidding function.*

- Recall that we considered a symmetric bidding function $b_i = \alpha s_i$ for a generic value of α.
- The above expression, $b_i = \frac{\alpha}{4\alpha-1} s_i$, is indeed linear in signal s_i, so we can write

$$\alpha s_i = \frac{\alpha}{4\alpha-1} s_i \quad \text{or}$$

$$\alpha = \frac{\alpha}{4\alpha - 1} \Rightarrow 4\alpha - 1 = 1 \Rightarrow \alpha = \frac{1}{2}$$

- In summary, a symmetric BNE has every bidder i using the bidding function

$$b_i(s_i) = \frac{1}{2} s_i$$

That is, she submits a bid equal to half of her privately observed signal, s_i.

Equilibrium Bidding in Common Value Auctions

- **Extension to N bidders**:

$$b_i(s_i) = \frac{(N+2)(N-1)}{2N^2} s_i$$

where $(N+2)(N-1) < 2N^2$ simplifies to $N^2 - N + 2 > 0$, which holds since $N \geq 2$ by definition.

- In addition, $\frac{\partial b_i(s_i)}{\partial n} = \frac{(4-N)}{2N^3} s_i$, which is positive for all $n < 4$, but negative otherwise.

Equilibrium Bidding in Common Value Auctions

- A natural question at this point is whether this bidding function helps bidders avoid the winner's curse.
  - Exercise 9.16 helps you confirm that it does.
  - *Hint*: We only need to evaluate the winning bidder's utility at her equilibrium bid. No need to compute expected utility (we know she won).