Chapter 5: Mixed Strategy Nash Equilibrium

Game Theory:

An Introduction with Step-by-Step Examples

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Introduction

- We have considered games that had at least one NE
  - Prisoner's Dilemma, Battle of the Sexes, and Chicken games
  - But do all games have at least one NE?
- If we restrict players to choose a specific strategy with certainty, some games may not have a NE.
  - This occurs when players do not want to be *predictable* –playing in the same way every time they face the game –such as in board games and professional sports.

Introduction

- Example: Penalty kicks in Soccer

Kicker

		Aim left	*Aim Right*
Goalie	Dive Left	0,0	-10,16
	Dive Right	-10,16	0,0

Matrix 5.1a. Anticoordination Game

- $BR_G(L) = L$, $BR_G(R) = R$ for the Goalie, and
- $BR_K(L) = R$; $BR_K(R) = L$
- Intuitively, the goalie tries to move in the same direction as the kicker, so she prevents the latter from scoring.
- Meanwhile, the kicker seeks to aim to the opposite location of the goalie to score a goal.

Pure Strategy Nash Equilibrium (psNE)

- Definition. Strategy profiles where players use a specific strategy with 100 percent probability are referred to as “**pure-strategy NE**”

		Kicker	
		Aim left	*Aim Right*
Goalie	Dive Left	$\underline{0}$,0	-10,$\underline{16}$
	Dive Right	-10,$\underline{16}$	$\underline{0}$,0

Matrix 5.1b. Anticoordination Game underlining best response payoffs

- In the above game, if we restrict players to use a specific strategy, there will be no mutual best response, and this game will have no NE, i.e., $psNE = \{\emptyset\}$.
- However, if we allow players to *randomize,* such as playing left with probability = ${}^{1}/_{3}$, and right with the remaining probability = ${}^{2}/_{3}$, we can find the NE of the game.
  - (These strategies are called Mixed Strategy NE (msNE), which we discuss in later slides.)

Another Example

- Consider a game where police department chooses where to locate most of its police patrols and, simultaneously, a criminal organization decides where to run its business
- Matrix 5.2 shows that the police seeks to choose the same action as the criminal, while the latter seeks to miscoordinate by selecting the opposite location as the police patrol to avoid being caught.

		Criminal	
		Street *A*	Street *B*
Police	Street *A*	$\underline{10}$,0	-1,$\underline{6}$
	Street *B*	0,$\underline{8}$	$\underline{7}$,-1

Matrix 5.2. Police and Criminal Game

- A similar argument applies to firm monitoring, such as a polluting firm choosing how many emissions to abate and an environmental protection agency deciding the frequency of its inspections.

Mixed Strategy

- Consider an individual i with a binary strategy set $S_i = \{H, L\}$ representing, for instance, a firm choosing between high and low prices.
- Define a player i's mixed strategy (or randomization) as a probability distribution over her pure strategies ($H \text{ and } L$), as follows $\sigma_i = \{\sigma_i(H), \sigma_i(L)\}$, where

$$\sigma_i(H) = p \quad \text{and} \quad \sigma_i(L) = 1 - p$$

  indicating the probability assigned to each strategy s_i.

  - We require that $\sigma_i(H), \sigma_i(L) \geq 0$ and $\sigma_i(H) + \ \sigma_i(L) = 1$.

Mixed Strategy

- Definition. **Mixed Strategy.** Consider a discrete strategy set $S_i = \{s_1, s_2, \dots, s_m\}$ where $m \geq 2$ denotes number of pure strategies. The mixed strategy

$$\sigma_i = \{\sigma_i(s_1), \sigma_i(s_2), .., \sigma_i(s_m)\}$$

is a probability distribution over the pure strategies in S_i, with the property that:

1. $\sigma_i(s_k) \geq 0$ for every pure strategy s_k, and
2. $\sum_{\mathrm{k=1}}^{\mathrm{m}} \mathrm{\sigma_i(s_k)} = 1$

- When mixed strategy concentrates all probability weight on a pure strategy

$$\sigma_i(s_j) = 1 \text{ while } \sigma_i(s_k) = 0 \text{ for all } j \neq k,$$

it is commonly called a “degenerated mixed strategy” because, graphically, it collapses to a pure strategy.

Mixed Strategy

- However, to avoid unnecessary complications, we only use the term "mixed strategy" to probability distributions over at least two pure strategies.
- As a remark, the above definition can be applied to games where players choose their strategies from a continuous strategy space, e.g., an output level so that $s_i > 0$.
- In this context, player $i's$ probability distribution over her pure strategies in S_i can be represented with a cumulative distribution function

$$F_i\colon S_i \ \rightarrow \ [0,1]$$

mapping every strategy $s_i \ \in \ S_i$ into cumulative probability.

- For instance, if s_i denotes firm $i's$ output level, the probability that this firm produces an output level equal or lower than $\bar{s}$ is $F_i(\bar{s})$ and, because $F_i(\bar{s})$ is a probability, it must satisfy $F_i(\bar{s}) \in [0,1]$.

Best Response with Mixed Strategies

- Definition. **Best response with Mixed Strategies.** Player $i's$ mixed strategy σ_i is a best response to her opponents' mixed strategy σ_{-i} if and only if her expected utility from σ_i satisfies

$$EU_i(\sigma_i, \sigma_{-i}) \geq EU_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \neq \sigma_i$$

- Mixed strategy σ_i is player $i's$ best response to her opponents' mixed strategy σ_{-i} if no other randomization σ_i' (potentially including the use of pure strategies) yields a higher expected utility than σ_i does.

- We use expected utility because player i needs to compute her expected payoff from randomizing over at least two of her pure strategies and, potentially, her rivals also randomize.

Mixed Strategy Nash Equilibrium (msNE)

- Definition. **Mixed Strategy Nash Equilibrium (msNE).** A strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a mixed strategy Nash equilibrium if and only if $\sigma_i^* = BR_i(\sigma_{-i}^*)$ for every player i.
  - Therefore, when player i chooses her equilibrium strategy σ_i^*, she is optimally responding to her opponents' strategies, σ_{-i}^*,
    - implying that players are choosing mutual best responses and, thus, have no incentives to unilaterally deviate.
  - This definition is, then, analogous to that of the pure strategy NE in Chapter 3, but using mixed strategy σ_i^* rather than pure strategy s_i^*.

Goalie Example Revisited

			Kicker	
			Prob. q	Prob. 1-q
			Aim left	*Aim Right*
Goalie	Prob. p	Dive Left	0,0	-10,16
	Prob. 1-p	Dive Right	-10,16	0,0

Matrix 5.3. Anticoordination Game –including probabilities

For **mixed strategy**

- The goalie must be indifferent between diving left and right. Her expected utility from diving left must coincide with that of diving right

$$EU_{Goalie}(Left) = EU_{Goalie}(Right)$$

- Let p and $(1-p)$ denote the probability with which the goalie randomizes, and q and $(1-q)$ be the probability with which the kicker randomizes
- Intuitively,
  - $p = 1$ means that the goalie dives left with 100 percent probability, whereas $p = 0$ indicates the opposite
  - $0 < p < 1$ means the goalie randomizes her diving decision
  - Similar argument applies to Kicker’s probability, q

Tool 5.1. How to Find msNEs in a two-player game

1. Focus on player 1. Find her expected utility of choosing one pure strategy. Repeat for each pure strategy.
2. Set these expected utilities equal to each other.
3. Solve for player 2’s equilibrium mixed strategy, σ_2^*.
    a. If players have two pure strategies, step 2 just entails an equality. Solving for σ_2 in this context yields a probability, σ_2^* which should satisfy $0 < \sigma_2^* < 1$.
    b. If players have three pure strategies, step 2 entails several equalities, which gives rise to a system of two equations and two unknowns. The solution to this system of equation is, nonetheless, a player 2’s equilibrium mixed strategy, σ_2^* .
4. Focus now on player 2. Repeat steps 1-3, to obtain player 1’s equilibrium mixed strategy, σ_1^*.

Example 5.1. Finding msNE: Goalie

			Kicker	
			Prob. q	Prob. 1-q
			Aim left	*Aim Right*
Goalie	Prob. p	Dive Left	0,0	-10,16
	Prob. 1-p	Dive Right	-10,16	0,0

Matrix 5.3. Anticoordination Game –including probabilities

- Goalie's expected utility from <u>diving left</u>:

$$EU_{Goalie}(Left) = \underbrace{q \times 0}_{\substack{kicker\ aims\\ left}} + \underbrace{(1-q) \times (-10)}_{\substack{kicker\ aims\\ right}} = 10q - 10$$

- Goalie's expected utility from <u>diving right</u>:

$$EU_{Goalie}(Right) = \underbrace{q \times (-10)}_{\substack{kicker\ aims\\ left}} + \underbrace{(1-q) \times 0 = -10q}_{\substack{kicker\ aims\\ right}}$$

- $EU_{Goalie}(Left) = EU_{Goalie}(Right)$

$$10q - 10 = -10q \Rightarrow q = {}^{1}\!/_{2}$$

Example 5.1. Finding msNE: Kicker

Following the same steps we have,

- Kicker’s expected utility from <u>aiming left</u>:

$$EU_{Kicker}(Left) = \underbrace{p \times 0}_{\substack{goalie\ dives\\ left}} + \underbrace{(1-p) \times 16}_{\substack{goalie\ dives\\ right}} = 16 - 16p$$

- Kicker’s expected utility from <u>aiming right</u>:

$$EU_{Kicker}(Right) = \underbrace{p \times 16}_{\substack{goalie\ dives\\ left}} + \underbrace{(1-p) \times 0 = 16p}_{\substack{goalie\ dives\\ right}}$$

- $EU_{Kicker}(Left) = EU_{Kicker}(Right)$

$$16 - 16p = 16p \Rightarrow p = {}^{1}/_{2}$$

- $msNE = \left\{ \underbrace{\left(\frac{1}{2}\ Dive\ Left, \frac{1}{2}\ Dive\ Right\right)}_{Goalie}; \underbrace{\left(\frac{1}{2} Aim\ Left, \frac{1}{2}\ Aim\ Right\right)}_{Kicker} \right\}$
  - Remember that players do not need to randomize with the same probability. They only did in this case because payoffs are symmetric in Matrix 5.3.

Graphical Visualization: msNE

- How to graphically represent the best response of each player?
- Let's start with the **Goalie**
  - Goalie chooses to dive left if:

$$EU_{Goalie}(Left) > EU_{Goalie}(Right)$$
$$10q\ - 10 > -10q\ \Rightarrow q > \ ^1/_2$$

  - Mathematically, this means that, for all $q > \ ^1/_2$, the goalie chooses to dive left (i.e., $p = 1$).
  - In contrast, for all $q < \ ^1/_2$, the goalie responds by diving right (i.e., $p = 0$).

Graphical Visualization: msNE

We can summarize the BRF of Goalie as:

$$BR_{Goalie}(q)\begin{cases} Left\ if\ q > \frac{1}{2} \\ \{Left, Right\}\ if q = \frac{1}{2}, and \\ Right\ if q < \frac{1}{2} \end{cases}$$

Figure 5.1 depicts this best response function

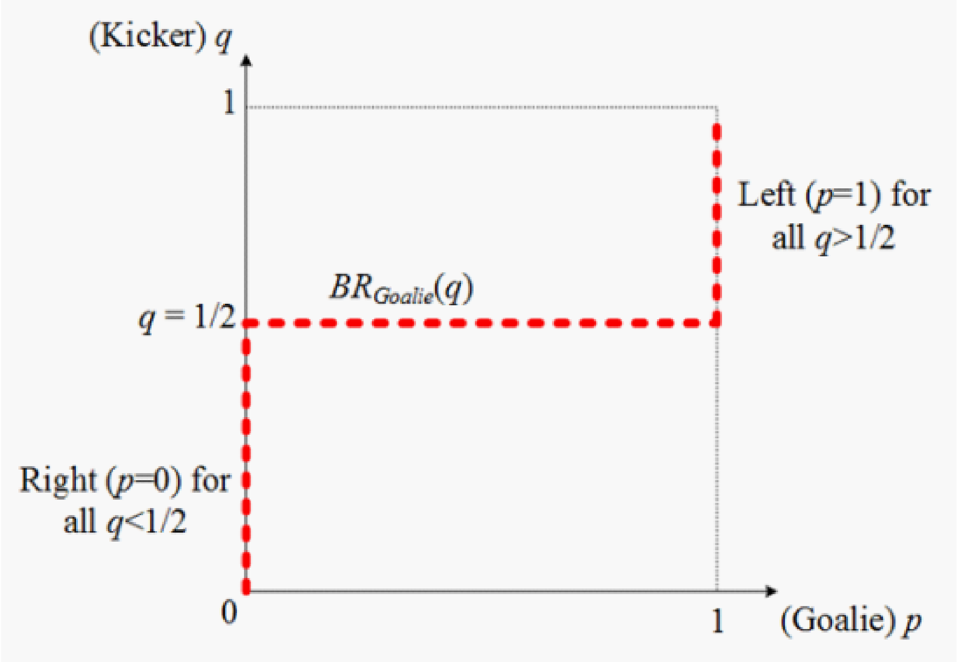


Figure 5.1. The goalie's best responses.

Graphical Visualization: msNE

- For **Kicker**
  - Kicker aims left if:

$$EU_{Kicker}(Left) > EU_{Kicker}(Right)$$
$$16\ - 16p > 16\ \ \Rightarrow p < \ ^1/_2$$

  - Mathematically, this means that, for all $\mathrm{p} < \ ^1/_2$, the kicker chooses to aim left (i.e., $\mathrm{q} = 1$).
  - In contrast, for all p $> \ ^1/_2$, the kicker aims right (i.e., $\mathrm{q} = 0$).

Graphical Visualization: msNE

We can summarize the BRF of Kicker as:

$$BR_{Goalie}(q)\begin{cases} Left\ if\ p<\dfrac{1}{2} \\ \{Left, Right\}\ if p=\dfrac{1}{2}, and \\ Right\ if p>\dfrac{1}{2} \end{cases}$$

Figure 5.2 depicts this best response function

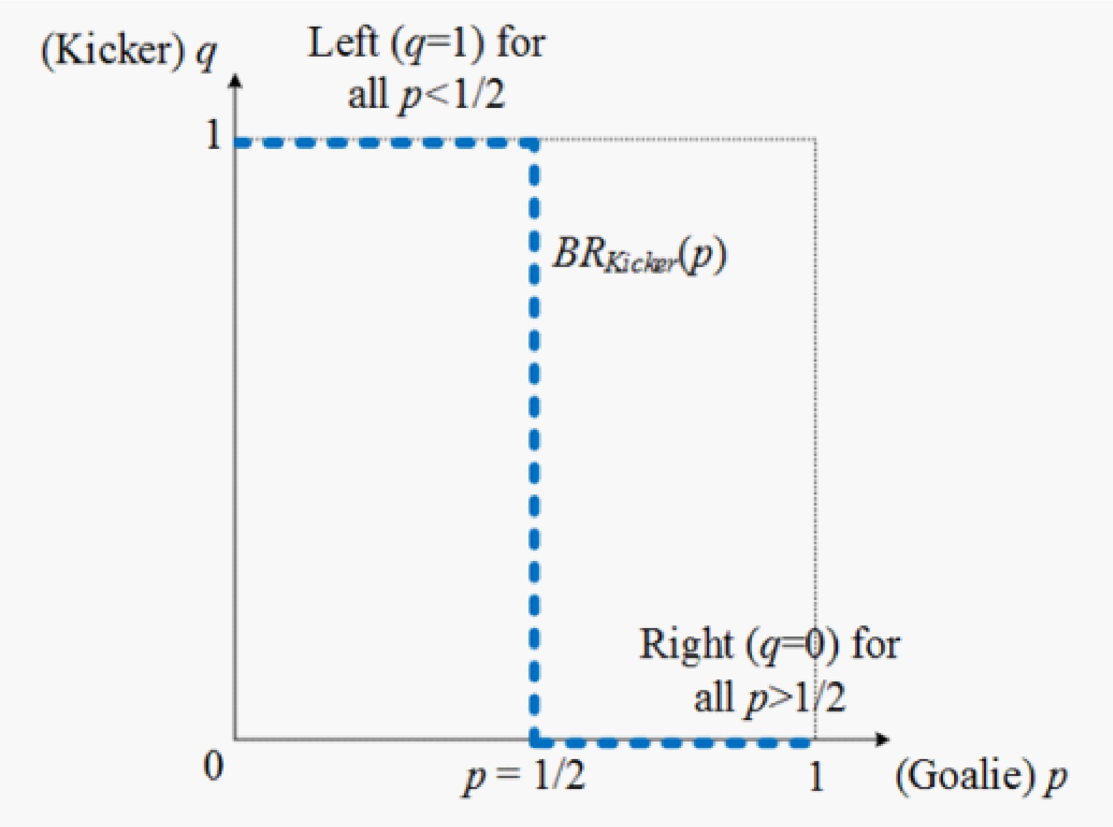


Figure 5.2. Kicker's best responses.

Graphical Visualization: msNE

- Putting together their response
- Figure 5.3 superimposes the goalie’s and the kicker’s best response functions
- The player’s best responses only cross each other at one point, where $p = q = {}^{1}/_{2}$

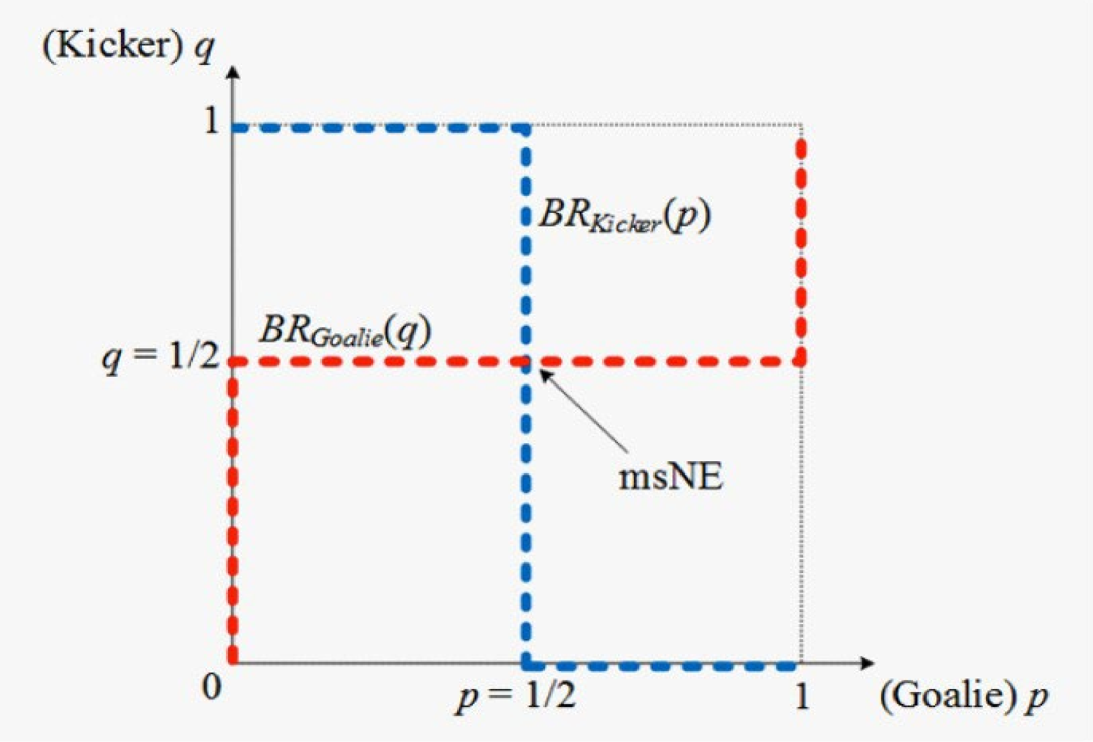


Figure 5.3. Both players’ best responses.

Some Lessons

- Indifference.
  - If it is optimal to randomize over a collection of pure strategies, then a player receives the same expected payoff from each of those pure strategies.
- When analyzing player i, ignore her probability to randomize
  - We make player i indifferent between two (or more) of her pure strategies, we write player i's expected utility from choosing pure strategy, s_i, $EU_i(s_i)$ as a function of her rival's randomization
- Never use strictly dominated strategies
  - If a pure strategy is strictly dominated, then such a pure strategy cannot be part of psNE or a msNE.
- Odd number of equilibria
  - In almost all finite games (games with a finite set of players and available actions), there is a finite and odd number of equilibria

Extensions: Mixed Strategy Equilibria in Games with $k \geq 3$ pure strategies

- Example: Rock-Paper-Scissors game

Player 2

		R	*P*	*S*
Player 1	R	0,0	-1,1	1,-1
	P	1,-1	0,0	-1,1
	S	-1,1	1,-1	0,0

Matrix 5.4. Rock-Paper-Scissors Game

Player 2

		R	*P*	*S*
Player 1	R	0,0	-1,1	1,-1
	P	1,-1	0,0	-1,1
	S	-1,1	1,-1	0,0

Matrix 5.4a. Rock-Paper-Scissors Game – Underlining best response payoffs

- No psNE exists in this game
- Player's payoffs are symmetric entailing that they randomize with the same probabilities: r, p, and $1 - r - p$, where r denotes the probability that every player i chooses Rock, p represents the probability she selects Paper, and

  $1 - r - p$ is the probability she plays Scissors

Example 5.2. Finding msNE when players have three pure strategies

- If player i randomizes, she must be indifferent between all her three pure strategies

$$EU_i(R) = EU_i(P) = EU_i(S)$$

- We separately find player 1's expected utility from each pure strategy
- To compute $EU_1(R)$, focus at the top row of Matrix 5.4a where:
  - If player 2 chooses Rock, which happens with probability r, player 1 earns a payoff of zero,
  - If player 2 chooses Paper, which happens with probability p, player 1 earns a payoff of -1, and
  - If player 2 chooses Scissors, which happens with probability $1 - r - p$, player 1 earns a payoff of 1
  - In summary, player 1's expected utility from Rock is:

$$EU_1(R) = r0 + p(-1) + (1 - r - p)1 = 1 - r - 2p$$

- Working in a similar way:

$$EU_1(P) = r1 + p0 + (1 - r - p)(-1) = 2r + p - 1$$

$$EU_1(S) = r(-1) + p1 + (1 - r - p)0 = p - r$$

Example 5.2. Finding msNE when players have three pure strategies

- We can now set the expected utilities from Rock, Paper, and Scissors equal to each other.
- First, from $EU_1(R) = EU_1(S)$ we find that:

$$1 - r - 2p = p - r \ \Rightarrow p = \frac{1}{3}$$

- Second, $EU_1(P) = EU_1(S)$ we find that:

$$2r + p - 1 = p - r \Rightarrow r = \frac{1}{3}$$

- Therefore $1 - \mathrm{r} - p = \frac{1}{3}$.
- We can then summarize the msNE of this game as that, every player i, randomizes according to the mixed strategy

$$msNE(r^*, p^*, 1 - p^* - r^*) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$$

which means that every player assigns the same probability weight to each of her pure strategies.

Finding Mixed Strategy Equilibria in games with $N \geq 2$ players

“**Snob Effect**” game

- Consider that every player i independently and simultaneously chooses between alternatives X and Y, where X can be interpreted as the snob option, while Y is the conforming option.
- If she is the only player choosing X, she earns a payoff of a (she is the ”cool girl” in the group), but a payoff of b otherwise (if anyone else chooses X too)
- When she chooses Y, she earns a payoff of c regardless of how many other players choose X.
- Payoffs satisfy $a > c > b$.

Finding Mixed Strategy Equilibria in games with $N \geq 2$ players

"**Snob Effect**" game

- Before extending the game to N players, let us consider the two-player version.

			Player 2	
			Prob. p	Prob. 1-p
			X	Y
Player 1	Prob. p	X	b, b	a, c
	Prob. 1-p	Y	c, a	c, c

- Two pure strategy NEs: (*X*,*Y*) and (*Y*,*X*), as in anticoordination games.
- What about msNE?
  - Player 1 is indifferent between X and Y if and only if $pb + (1-p)a = pc + (1-p)c$, which yields $p^* = \frac{a-c}{a-b}$. (Same probability for player 2, since payoffs are symmetric.)
  - Probability p^* is positive and smaller than one since $a > c > b$. Check!
  - p^* is increasing in $a - c$, but decreasing in $a - b$. Intuition.

Finding Mixed Strategy Equilibria in games with $N \geq 2$ players

“**Snob Effect**” game

- We can now extend the game to N players.
- In this setting, we seek to identify a symmetric msNE where every player i chooses X with probability p.
- Therefore, we need to:
  1. Find the expected utility from X and from Y, and
  2. Set these expected utilities equal to each other to obtain the equilibrium probability p^*

Game continued

- **Expected Utility from X.** When player i chooses X, his expected utility is

$$EU_i(X) = \underbrace{(1-p)^{N-1}}_{\text{No other player chooses X}} a + \underbrace{[1-(1-p)^{N-1}]}_{\text{At least someone else chooses X}} b$$

To understand the probabilities of each event, recall that the probability with which every player chooses X is p, so the probability with which she plays Y is $1-p$, implying that all other $N-1$ players (everyone but player i) choose alternative Y with probability

$$\underbrace{(1-p) \times (1-p) \times \cdots \times (1-p)}_{N-1\ times} = (1-p)^{N-1}$$

Game continued

- **Expected Utility from $\boldsymbol{Y}$.** When player i chooses Y, she earns a payoff of c with certainty, entailing that $EU_i(Y) = c$.
- **Indifference condition.** If player i randomizes between X and Y, she must be indifferent, thus earning the same expected payoff from each pure strategy, $EU_i(X) = EU_i(Y)$, which means:

$$(1-p)^{N-1}a + [1- \ (1-p)^{N-1}]b = c$$

Rearranging, yields

$$p^* = 1 - \left(\frac{c-b}{a-b}\right)^{\frac{1}{N-1}}$$

Comparative Statics

- For instance, when $a = 2, b = 0, and\ c = 1$, the probability simplifies to

$$p^* = 1 - \left({}^{1}/_{2}\right)^{\frac{1}{N-1}}$$

which decreases in the number of players, N.

- Intuitively, as the population grows, the probability that someone selects alternative X increases, driving each player to individually decrease their probability of choosing the snob option X, opting instead for conforming option Y, with probability $1 - p^* = \left({}^{1}/_{2}\right)^{\frac{1}{N-1}}$, which increase in N

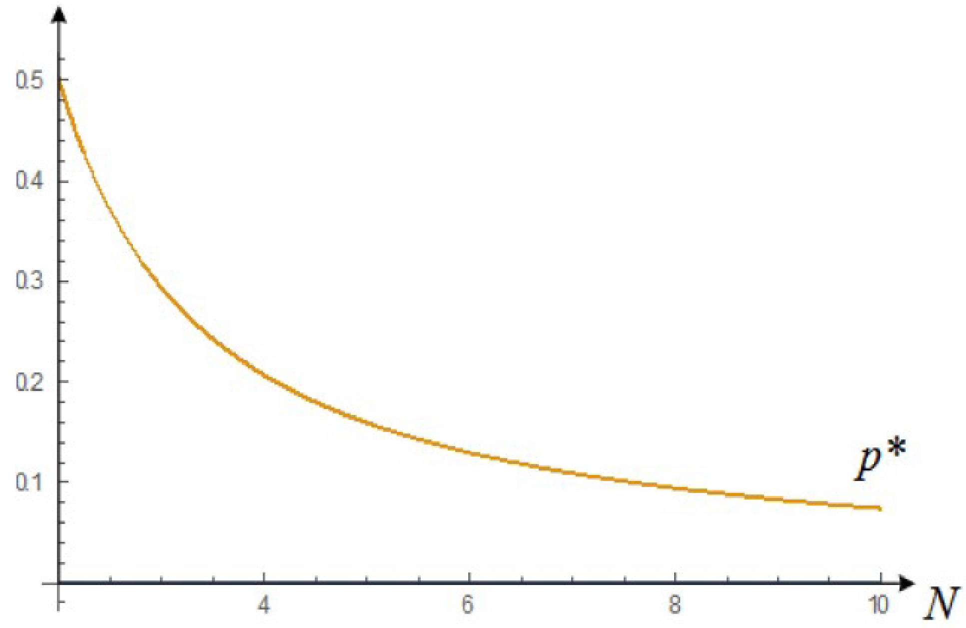


Figure 5.4. Equilibrium probability p^*.

Comparative Statics

$$p^* = 1 - \left(\frac{c-b}{a-b}\right)^{\frac{1}{N-1}}$$

- We can also examine how the above expression p^* changes when:
  - Players are more attracted to the snob option, $a - b$, which captures the payoff gain of snob option relative to the conforming option.
  - An increase in $a - b$ *<u>increases</u>* the probability of choosing X.
- In contrast, an increase in $c - b$, which measures the payoff loss of not being the only individual choosing X, *<u>decreases</u>* the probability of choosing X.

Strictly Competitive Games

- Definition. **Strictly Competitive Games.** A two-player game is strictly competitive if, for every two strategy profiles s and s',
  - if $u_1(s) > u_1(s')$ then $u_2(s) < u_2(s')$; and
  - if $u_1(s) = u_1(s')$, then $u_2(s) = u_2(s')$.

- Intuitively:
  - If player 1 prefers strategy profile s to s', then player 2 has the opposite preference order: preferring s' over s; and
  - if player 1 is indifferent between s and s', player 2 must also be indifferent between these two strategy profiles.
- *Example:* The penalty kicks game is an example of a strictly competitive game where we can test the above definition (next slide).

Strictly Competitive Games

		Kicker	
		Aim left (l)	*Aim Right (r)*
Goalie	Dive Left (L)	0,0	-10,16
	Dive Right (R)	-10,16	0,0

- Comparing (L, l) and (L, r), we see that the goalie prefers the former, since $0 > -10$, while the kicker prefers the latter because $0 < 16$.
- Comparing (L, l) and (R, l), we find that the goalie prefers the former, since $0 > -10$, while the kicker prefers the latter because $0 < 16$.
- Comparing (L, l) and (R, r), we see that the goalie is indifferent, and so is the kicker, both players earning a payoff of zero in both strategy profiles.
- Comparing (R, l) and (L, r), we find that the goalie is indifferent between these two strategy profiles, earning -10 in both of them. A similar argument applies to the kicker, who earns a payoff of 16 in both strategy profiles.
- We can confirm the definition of strictly competitive games (i.e., opposite preferences of players 1 and 2) holds for every two strategy profiles, s and s'.

Games that are not strictly competitive

- A two-player game is *not strictly competitive* if, for at least two strategy profiles, s and s', every player $i's$ utility satisfies $u_i(s) > \ u_i(s')$.

- Example

Criminal

		Street A	Street B
Police	Street A	10,0	-1,6
	Street B	0,8	7,-1

Matrix 5.6. Police and Criminal Game

  - Comparing strategy profiles (A, A) and (B, B), along the main diagonal, we can see that the police prefers (A, A) to (B, B), since her payoff satisfies $10 > 7$.
    - Similar argument applies for the criminal, as her payoff satisfies $0 > -1$.
  - Because we found that players' preferences over strategy profiles are aligned, rather than misaligned, we can already claim that the game is *not* strictly competitive without having to compare other pairs of strategy profiles.

Constant-sum Games

- Definition. **Constant-sum games.** A two-player game is a constant-sum game if, for every strategy profile s, player's payoffs satisfy

$$u_1(s) + u_2(s) = K, \quad \text{where } K > 0 \text{ is a constant.}$$

- Then, players' payoffs must add up to the same constant across all cells in the matrix.
- If, instead, players' payoffs add up to a different number in at least one of the cells, then we can claim that the game is *not* constant sum.
    - It can still be strictly competitive, but not constant sum.

Constant-sum Games

Counterexample:

		Player 2	
		l	r
Player 1	U	10,0	9,3
	D	9,3	10,0

Matrix 5.7. A strictly competitive game that is non constant-sum

- The game is strictly competitive (check as practice).
- It is not a constant-sum game since players payoff in strategy profiles like (U, l) and (D, r) add up to 10, while those strategy profiles (U, r) and (D, l) add up to 12.

Constant-sum Games

- Constant-sum games are always strictly competitive:
  - Condition $u_1(s) + u_2(s) = K$ can be rewritten as $u_1(s) = K - u_2(s)$.
  - Then, if player 1’s payoff increases when moving from s to s', then player 2’s payoff must decrease.

- We now introduce a special class of constant-sum games, those in which *K*=0, called zero-sum games.

Zero-sum Games

- Definition. **Zero-sum games.** A two-player game is a zero-sum game if, for every strategy profile s, player's payoffs satisfy

$$u_1(s) + u_2(s) = 0.$$

- Alternatively, condition $u_1(s) + u_2(s) = 0$ can be expressed as $u_1(s) = -u_2(s)$.
    - Intuitively, every dollar that player 1 earns comes from the same dollar that player 2 loses and vice versa.

Player 2

		Heads	*Tails*
Player 1	*Heads*	1,-1	-1,1
	Tails	-1,1	1,-1

Matrix 5.8. Matching Pennies Game

- Matching pennies game is zero-sum game. Rock-paper-scissors is another example.
- Specifically, in Matrix 5.8, we have that either $1 + (-1) = 0$ or $-1 + 1 = 0$.

Security Strategies

- Definition. **Security Strategies.** In a two-player game, player $i's$ security strategy, i**,** solves

$$\max_{\sigma_i} \min_{\sigma_j} u_i(\sigma_i, \sigma_j)$$

  - Consider the "worst-case scenario" $w_i(\sigma_i) = \min_{\sigma_j} u_i\,(\sigma_i, \sigma_j)$
  - Player i anticipates that player j chooses her strategy σ_j to maximize her own payoff, which entails minimizing i's payoff, $u_i(\sigma_i, \sigma_j)$.
    - This is because players interact in a strictly competitive game.
  - Player i then chooses her strategy σ_i to maximize the payoff across all worst-case scenarios.
  - Intuitively, player i seeks to find the strategy σ_i that provides her with the "best of the worst" payoffs, as represented with the max-min problem.
    - This explains why security strategies are sometimes known as max-min strategies.

Tool 5.2. How to find security strategies in a two-player game

1. Find the expected utility of player 1's randomization, fixing player 2's strategy.
2. Repeat step 1 until you considered all strategies of player 2, fixing one at a time.
3. *"Min" part.* Find the lower envelope of player 1's expected utility. That is, for each strategy σ_1, find the lowest expected utility that player 1 earns.
4. *"Max" part.* Find the highest expected utility of the lower envelope identified in step 3, and the corresponding strategy σ_1. This is player 1's security strategy, σ_1^{sec}.
5. To find the security strategy for player 2, follow a similar process in steps 1-4 above.

Example 5.3. Finding Security Strategies

Example

		Player 2	
		l	r
Player 1	U	10,0	9,3
	D	9,3	10,0

Matrix 5.9. A Strictly Competitive Game that is non constant-sum

To find the security strategy for player 1, we follow the next steps:

1. We find player 1’s expected utility of randomizing between U and D, with associated probabilities p and $1-p$, respectively. First, we fix player 2’s strategy at column l, which yields:

$$EU_1(p|l) = p \times 10 + (1-p) \times 9 = 9 + p$$

Example 5.3. Finding Security Strategies

2. We now find her expected utility of randomizing, but fixing player 2's strategy at column r, as follows:

   $EU_1(p|r) = p \times 9 + (1-p) \times 10 = 10 - p$

3. To find the lower envelope of the previous two expected utilities, we can depict each line as a function of p, as we do in Figure 5.5. The lower envelope is the segment $9 + p$ for all $p \leq \frac{1}{2}$, but segment $10 - p$ otherwise.

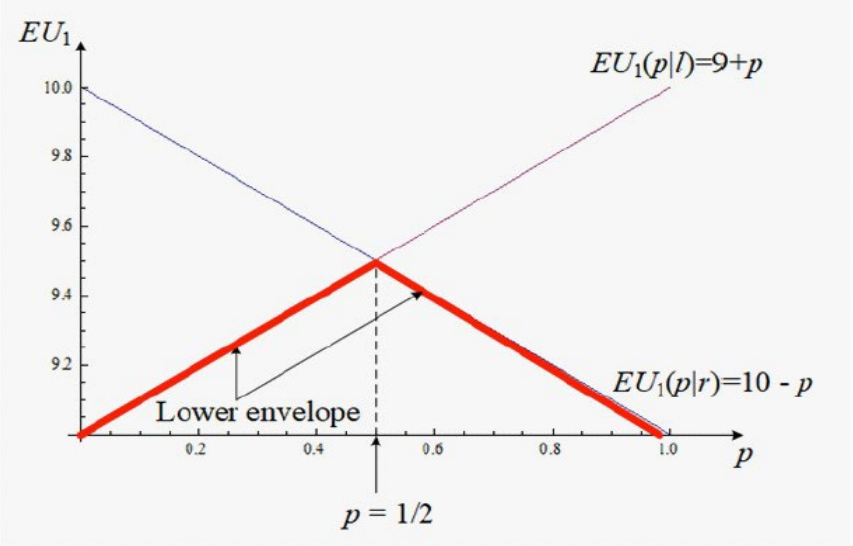


Figure 5.5. Lower envelope and security strategies.

Example 5.3. Finding Security Strategies

4. Among all points in the lower envelope, player 1 enjoys the highest utility at $p = \frac{1}{2}$, which yields an expected payoff $EU_1(p|l) = 9 + \frac{1}{2} = 9.5$, as illustrated in Figure 5.5 by the height of the crossing point between $EU_1(p|l)$ and $EU_1(p|r)$. This is player 1’s security strategy, $p^{Sec} = \frac{1}{2}$.
5. Following the same steps for player 2, we find that, since payoffs are symmetric, her security strategy is $q^{Sec} = \frac{1}{2}$.

Security Strategies and NE

- At this point, you may be wondering about the relationship between security strategies and msNE.
- We obtain the same equilibrium result from both solution concepts, but only for two-player strictly competitive games.
- Consider the previous example from Matrix 5.9:

$$(p^{sec}, q^{sec}) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Let us now confirm that the msNE produces the same result.

Example 5.4. Solved by msNE

- Assuming that player 1 randomizes between U and D with probabilities p and $1-p$, respectively, and player 2 mixes l and r with probabilities q and $1-q$, respectively,
  - We find that player 1’s expected utility from choosing U is:

$$EU_1(U) = q \times 10 + \ (1-q)\ \times\ 9 = 9 + q$$

  - Similarly, player 1’s expected utility from choosing D is:

$$EU_1(D) = q \times 9 + \ (1-q)\ \times\ 10 = 10 - q$$

  - Therefore, player 1 randomizes between U and D when she is indifferent between these two pure strategies $EU_1(U) = EU_1(D)$, which entails

$$9 + q = 10 - q \ \Rightarrow q = \frac{1}{2}$$

Example 5.4. Solved by msNE

- Player 2’s expected utilities
  - $EU_2(l) = p \times 0 + (1-p) \times 3 = 3 - 3p$, and
  - $EU_2(r) = p \times 3 + (1-p) \times 0 = 3p$
  - $EU_2(l) = EU_2(r) \Rightarrow 3 - 3p = 3p \Rightarrow p = \frac{1}{2}$

- Summarizing, we can claim that the msNE of this game is $(p, q) = \left(\frac{1}{2}, \frac{1}{2}\right)$, which coincides with the security strategies we found in example 5.3.

Example 5.5. Security strategies and msNE yield different equilibrium outcomes

Player 2

		l	r
Player 1	U	3,5	-1,1
	D	2,6	1,2

Matrix 5.10. A Game that is not strictly competitive

- The above game is not strictly competitive. We can find strategy profile where players' interests are aligned; both players prefer, for instance, (U, l) to (D, r).
- Since the game is not strictly competitive, we can expect that security strategies may produce a different equilibrium prediction than msNE.

Example 5.5. Security strategies and msNE yield different equilibrium outcomes

- For player 1:
  - When player 2 chooses l, player 1's expected payoff from randomizing between U and D with probabilities p and $1-p$ respectively,
  $$EU_1(p|l) = p \times 3 + (1-p) \times 2 = 2 + p$$
  - When player 2 chooses r, player 1's expected utility is
  $$EU_1(p|r) = p \times (-1) + (1-p) \times 1 = 1 - 2p$$

Example 5.5. Security strategies and msNE yield different equilibrium outcomes

- $EU_1(p|l)$ lies above $EU_1(p|r)$ for all $p \in [0,1]$.
- This means that the lower envelope coincides with $EU_1(p|r) = 1 - 2p$ for all values of p.
- The highest point of this lower envelope occurs at $p = 0$, so player 1 assigns no probability weight to U or, alternatively, that she plays D in pure strategies.
- This means that D is player 1's security strategy.

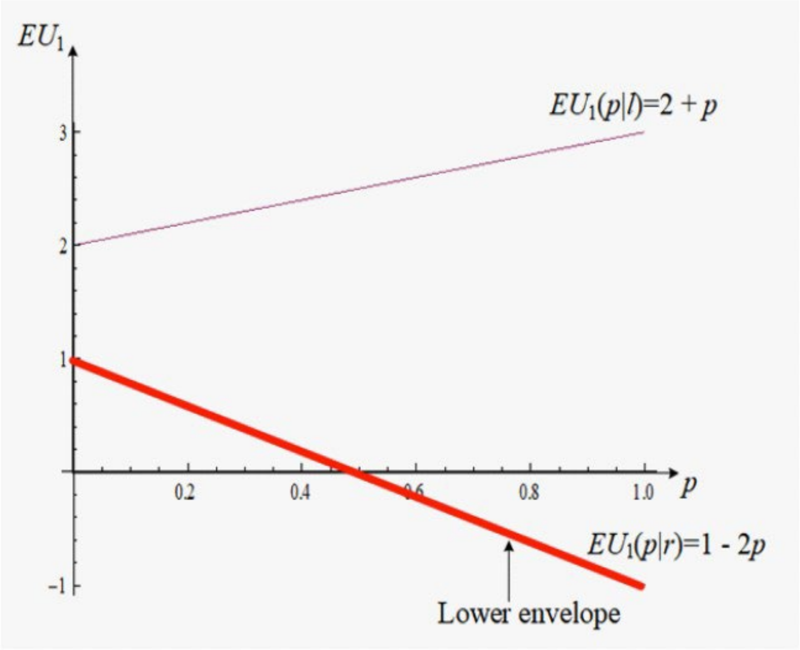


Figure 5.6. Lower envelope and security strategies - Corner solution.

Security strategies and msNE yield different equilibrium outcomes

Similarly, for player 2

- $EU_2(q|U) = q \times 5 + (1-q) \times 1 = 1 + 4q,\ \text{and}$
- $EU_2(q|D) = q \times 6 + (1-q) \times 2 = 2 + 4q.$

- Since $EU_2(q|U)$ < $EU_2(q|D)$ for all values of q,we can claim that U is the lower envelope.
- We can, then, notice that the highest point of $1 + 4q$ occurs at $q = 1$, meaning that player 1 puts full probability weight on l, which becomes his security strategy.

In summary, the security strategy profile in this game is (D, l).

Example contd. & solving by msNE

- For **msNE**
  - We can facilitate our analysis by noticing that strategy l strictly dominates r since it yields a strictly higher payoff than r regardless of the row that player 1 chooses $(5 > 1 \text{ and } 6 > 2)$.
  - We know players put no probability weight in strictly dominated strategies, so we can delete column r from the matrix and obtain:

		Player 2
		l
Player 1	U	3,5
	D	2,6

Matrix 5.10. A Game that is not strictly competitive – After deleting column r

  - Turning now to player 1, we do not need to consider his randomization since, at this point, he has a clear best response to l, U. Therefore, the psNE (no msNE) is (U, l).
  - This equilibrium outcome does not coincide with the security strategy profile (D, l).

Correlated Equilibrium

- Example

		Player 2	
		$Swerve$	$Stay$
Player 1	$Swerve$	6,6	$\underline{2,7}$
	$Stay$	$\underline{7,2}$	0,0

Matrix 5.11. Modified Chicken game

- By underlining best response payoffs, we can find two psNEs: $(Swerve, Stay)$ and $(Stay, Swerve)$.
- The game also has msNE, where player 1 chooses $Swerve$ with probability $p = \frac{2}{3}$, and the same probability applies to player 2, q= $\frac{2}{3}$ (since payoffs are symmetric).
- In this msNE, Player 1's expected utility in equilibrium is

$$EU_1(\sigma^*) = \underbrace{\frac{2}{3}\left(\frac{2}{3}6 + \frac{1}{3}2\right)}_{Player\ 1\ chooses\ Swerve} + \underbrace{\frac{1}{3}\left(\frac{2}{3}7 + \frac{1}{3}0\right)}_{Player\ 1\ chooses\ Stay} = \frac{28}{9} + \frac{14}{9} = \frac{14}{3} \cong 4.67$$

And a similar expected payoff accrues to player 2.

Correlated Equilibrium

- *Natural question*: Can players reach a higher expected payoff if, instead, they rely on a probability distribution, such as a coin toss, that each player privately observes before playing the game, and that informs the player about which action to choose?
- Intuitively, the probability distribution can be interpreted as an external "recommender" who:
  - First, draws one strategy profile, $s = (s_i, s_{-i})$, such as one cell in Matrix 5.11.
  - Second, the recommender makes recommendation s_i to player i, without informing her of the recommendation s_{-i} that her rivals receive.
- Definition. **Correlated Equilibrium**. A probability distribution over strategy profiles is a correlated equilibrium if every player i follows his recommendation, s_i.
  - Intuitively, a probability distribution over strategy profiles is stable in the sense that every player i has no incentives to unilaterally deviate from the recommendation, s_i.
- For simplicity, we first examine public signals, then privately observed signals.

Example 5.6. Correlated Equilibrium with Public Signals

- Consider the game in Matrix 5.11 and assume that players observe a public signal that assigns probability:
  - α to one of the psNEs in this game, $(Stay, Swerve)$, and
  - $1-\alpha$ to the other psnE, $(Swerve, Stay)$.

- A public signal could be:
  - A traffic light,
  - Coin toss,
  - Dice toss,

or any other stochastic mechanism that players agree on before starting the game, that yields this probability distribution (summarized in Matrix 5.12).

		Player 2	
		$Swerve$	$Stay$
Player 1	$Swerve$	6,6	$\underline{2,7}$
	$Stay$	$\underline{7,2}$	0,0

Matrix 5.11. Modified Chicken game

		Player 2	
		$Swerve$	$Stay$
Player 1	$Swerve$	0	$1-\alpha$
	$Stay$	α	0

Matrix 5.12. Correlated equilibrium with public signals – Probability of each strategy profile

Example 5.6. Correlated Equilibrium with Public Signals

		Player 2	
		$Swerve$	$Stay$
Player 1	$Swerve$	6,6	$\underline{2,7}$
	$Stay$	$\underline{7,2}$	0,0

Matrix 5.11. Modified Chicken game

		Player 2	
		$Swerve$	$Stay$
Player 1	$Swerve$	0	$1-\alpha$
	$Stay$	α	0

Matrix 5.12. Correlated equilibrium with public signals – Probability of each strategy profile

- In this context, player 1 does not have incentives to deviate.
- Upon observing $(Stay, Swerve)$, payoff from following the recommendation is 7, and that of unilaterally deviating to $Swerv$e decreases to 6.
- Similarly, upon observing $(Swerve, Stay)$, payoff is 2, but decreases to 0 if she unilaterally deviates.
- By symmetry, the same argument applies to player 2.
- As a consequence, a continuum of correlated equilibria can be sustained, where players alternate between the two psNEs of the game with probabilities α and $1-\alpha$, respectively.

Example 5.7. Correlated Equilibrium with Private Signals

- Consider the game in Matrix 5.11 again, where the recommendations assign the same probability weight to $(Swerve, Swerve)$, $(Swerve, Stay)$, and $(Stay, Swerve)$, as summarized in Matrix 5.13:

		Player 2	
		$Swerve$	$Stay$
Player 1	$Swerve$	$^1/_3$	$^1/_3$
	$Stay$	$^1/_3$	0

Matrix 5.13. Correlated equilibrium with private signals – Probability of each strategy profile

- **Bottom row.** Intuitively, if player 1 receives the recommendation of $Stay$, she knows that the only strategy profile recommended by the public signal is $(Stay, Swerve)$.
- **Top row.** However, if she receives the recommendation of $Swerve$, she knows that player 2 may have received:
  - The same recommendation, $Swerve$, with probability $\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}}={}^1/_2$, or
  - The opposite recommendation, $Stay$, with probability $\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}}={}^1/_2$.
- A similar interpretation applies to the recommendations that player 2 receives.

Example 5.7. Correlated Equilibrium with Private Signals

We can now show that player 1 does not have incentives to deviate from this recommendation profile:

- If he receives the recommendation of $Stay$, his payoff is 7 at $(Stay, Swerve)$, which she cannot improve by unilaterally deviating to $Swerve$ (earning only 6).
- If instead, she receives the recommendation of $Swerve$, her expected payoff is

$$\frac{1}{2}6 + \frac{1}{2}2 = 4,$$

which he cannot improve by unilaterally deviating to $Stay$, as that only yields

$$\frac{1}{2}7 + \frac{1}{2}0 = 3.5.$$

- Since payoffs are symmetric, a similar argument applies to player 2, making the above recommendation stable.
- We can then say that the recommendation profile can be sustained as a correlated equilibrium, with expected payoff:

$$\frac{1}{3}6 + \frac{1}{3}2 + \frac{1}{3}7 = 5$$

which exceeds that in the msNE of the game, 4.67.

Existence of Correlated equilibrium

- Finally, note that every psNE, $s^* = (s_i^*, s_{-i}^*)$ can be defined as a (trivial) correlated equilibrium:
  - where the probability distribution recommends player i to choose s_i^* with probability 1.
  - That is, every player is recommended to play as she would under the psNE.
- A similar argument applies to msNE $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$, where the probability distribution recommends:
  - Player i to randomize according to the same mixed strategy that she uses in the msNE of the game.
- In other words, if σ^* is a NE, it must also be a correlated equilibrium, but the converse is not necessarily true.

$$\sigma^* \text{ is a NE} \underset{\nLeftarrow}{\Rightarrow} \sigma^* \text{ is a correlated equilibrium}$$

- Examples in the Chicken game:
  - (Stay, Swerve) is a NE, and it can be a correlated equilibrium.
  - Any of the public or private randomizations we saw before are correlated equilibria, but they aren't NE.

Equilibrium Refinements in Strategic-form games (Technical)

- Mixed strategies can help us discard NEs which seem fragile to small strategic mistakes, as if a player's hands could "tremble" when choosing her strategy.

Player 2

		l	r
Player 1	U	$\underline{1},\underline{1}$	$\underline{0},0$
	D	$0,\underline{0}$	$\underline{0},\underline{0}$

- The above game has two psNE: (U,l) and (D,r).
- The second one seems more fragile to trembles:
  - if player 1 deviates from D to U, even if U only occurs with a small probability, player 2's BR would change from r to l.
- A similar argument applies if player 2 deviates from r to l, by a small prob.
- The issue, of course, is that in (D,r) players use weakly dominated strategies.
- We next seek to rule out psNEs that aren't robust to trembles.

Equilibrium Refinements in Strategic-form games (Technical)

- Definition. **Totally mixed strategy.**
- Player $i's$ mixed strategy, σ_i, is "totally mixed" if it assigns a strictly positive probability weight on every pure strategy, that is $\sigma_i(s_i) > 0$ for all s_i.

- Therefore, all pure strategies happen, even with small probability.
- This allows for trembles, where D could occur with 0.001 probability or less.

Equilibrium Refinements in Strategic-form games (Technical)

- Definition. **Trembling-Hand Perfect equilibrium.**
- A mixed strategy profile $\sigma_i = (\sigma_i, \sigma_{-i})$ is a Trembling-Hand Perfect Equilibrium (THPE) if:

1. There exists a sequence of totally mixed strategies for each player i, $\{\sigma_i^k\}_{k=1}^{\infty}$, that converges to σ_i, and
2. for which $\sigma_i \in BR_i(\sigma_{-i}^k)$ for every k.

- Informally, these two requirements say that:
  1. Every player *i*'s totally mixed strategy (which allows for trembles) must converge to σ_i; and
  2. Strategy σ_i is player *i*'s BR to her rivals' strategy profile σ_{-i}^k at every point of the sequence (i.e., for all *k*).
- Second requirement is a bit trickier to show. (Example in a moment.)

Properties of THPE

1. Every THPE must be a NE.
2. Every strategic-form game with finite strategies for each player has a THPE.
3. Every THPE assigns zero probability weight on weakly dominated strategies.

Intuitively, points (1) and (2) show that THPEs are a subset of the set of all NEs in a strategic-form game.

$$\sigma \text{ is a THPE} \underset{\nLeftarrow}{\Rightarrow} \sigma \text{ is a NE}$$

And point (3) helps us rule out strategies D for player 1 and r for player 2 in the 2x2 game we used as a motivation. Therefore, (D,r) is a NE but cannot be supported as a THPE.

Example 5.9. Trembling-hand Perfect Equilibrium

		Player 2	
		l	r
Player 1	U	$\underline{1},\underline{1}$	$\underline{0},0$
	D	$0,\underline{0}$	$\underline{0},\underline{0}$

Matrix 5.14. A Game with two psNEs, but only (U, l) is THPE

- Consider the following sequence of totally mixed strategies

$$\sigma_i^k = \left(1 - \frac{\varepsilon_k}{2}, \frac{\varepsilon_k}{2}\right) \text{ for every player } i \text{, where } \varepsilon_k = \frac{1}{2^k}.$$

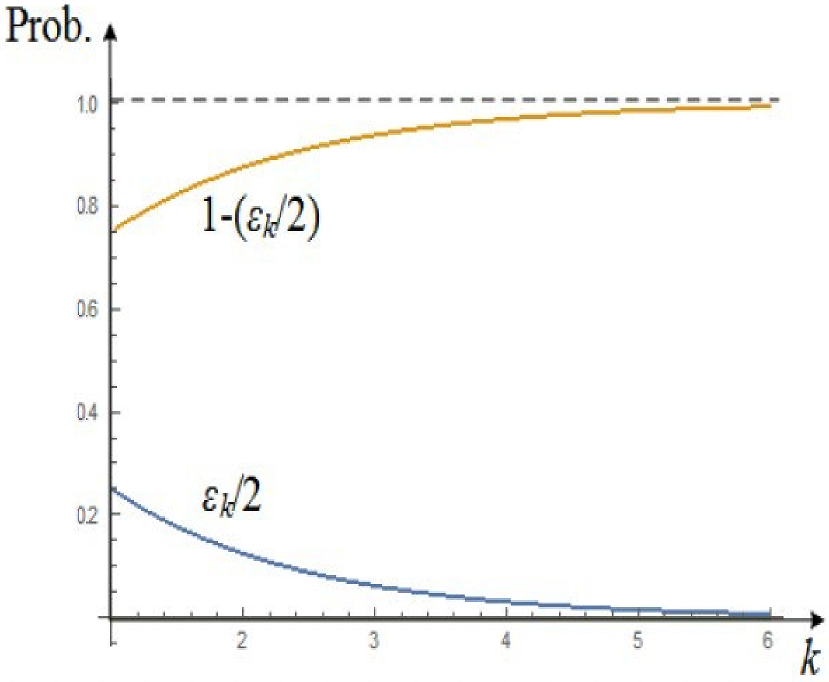
- *Example*:
  - When $k = 1, \varepsilon_1 = \frac{1}{2}$,and σ_i^k becomes $\sigma_i^1 = \left(\frac{3}{4}, \frac{1}{4}\right)$, indicating that every player i makes mistakes with $\frac{1}{4}$ probability.
  - When $k = 2, \varepsilon_2 = \frac{1}{4}$,and σ_i^k becomes $\sigma_i^2 = \left(\frac{7}{8}, \frac{1}{8}\right)$, representing that mistakes are now less likely.
- In the limit, we find that (see figure in next slide)

$$\lim_{k \to +\infty} \sigma_i^k = (1,0) \text{ since } \lim_{k \to +\infty} \varepsilon_k = \lim_{k \to +\infty} \frac{1}{2^k} = 0$$

- which implies that player 1 (2) chooses U (l, respectively) in pure strategies, yielding strategy profile (U, l).

Properties

- Generally, as k increase, mistakes become less likely, and the above totally mixed strategy converges to the psNE (U, l).
- This leads to the following figure:



Example 5.9. Trembling-hand Perfect Equilibrium

- Therefore, the NE (U, l) can be supported as a THPE because:

1. The totally mixed strategy $\sigma_1^k\left(\sigma_2^k\right)$ converges to U (l) ; and
2. U (l) is the best response of player 1 (2) to her rival's totally mixed strategy, $\sigma_2^k\left(\sigma_1^k, respectively\right)$ for all *k*.

- To see point (2), note that:
  - When k=1, σ_2^k becomes $\sigma_2^1 = \left(\frac{3}{4}, \frac{1}{4}\right)$, where U is player 1's best response because $EU_1(U|\sigma_2^1) = \frac{3}{4}1 + \frac{1}{4}0 = \frac{3}{4}$ and $EU_1(D|\sigma_2^1) = \frac{3}{4}0 + \frac{1}{4}0 = 0.$
  - When k=2, σ_2^k becomes $\sigma_2^2 = \left(\frac{7}{8}, \frac{1}{8}\right)$, and U is still player 1's best response because $EU_1(U|\sigma_2^2) = \frac{7}{8}1 + \frac{1}{8}0 = \frac{7}{8}$ and $EU_1(D|\sigma_2^1) = \frac{7}{8}0 + \frac{1}{8}0 = 0.$
  - Same argument applies to every *k* since $EU_1\left(U|\sigma_2^k\right) = \left(1 - \frac{\varepsilon_k}{2}\right)1 + \frac{\varepsilon_k}{2}0 = 1 - \frac{\varepsilon_k}{2} > 0$ and $EU_1\left(D|\sigma_2^k\right) = \left(1 - \frac{\varepsilon_k}{2}\right)0 + \frac{\varepsilon_k}{2}0 = 0.$
  - Same argument applies to player 2's best response to σ_1^k being *l* for every *k*. (Check as a practice.)

Example 5.9. Trembling-hand Perfect Equilibrium

- In contrast, (D,r) cannot be sustained as THPE.
  - While we can find converging sequences of totally mixed strategies (first requirement)…
  - Choosing D (r) is *not* player 1’s (2’s) best response to her rival’s totally mixed strategy for every *k* (second requirement).
- To see this point, consider this totally mixed strategy:

$$\sigma_i^k = \left(\frac{\varepsilon_k}{2}, 1 - \frac{\varepsilon_k}{2}\right) \text{ for every player } i\text{, where } \varepsilon_k = \frac{1}{2^k}.$$

- which assigns the opposite probability weights than that converging to (U,l).
- It converges to psNE (D,r). Check!

Example 5.9. Trembling-hand Perfect Equilibrium

- However, U is player 1’s BR to σ_2^k for every k.
- To see this point, consider that:
  - When k=1, σ_2^k becomes $\sigma_2^1 = \left(\frac{1}{4}, \frac{3}{4}\right)$, and U is player 1’s best response.
  - When k=2, σ_2^k becomes $\sigma_2^2 = \left(\frac{1}{8}, \frac{7}{8}\right)$, and U is still player 1’s best response.
  - Same argument applies for every k.
  - Recall that finding that U is player 1’s BR, instead of D, for *at least one value of k* and for *at least one player* would have been enough to show that (D,r) cannot be sustained as THPE.

ε —Proper Equilibrium

- THPE helps us rule out NEs that aren’t robust to trembles.
- But, which trembles do we allow?
- Myerson (1978) suggested that a rational player, while making mistakes, should put:
  - Higher probability weight on strategies yielding higher payoffs.
  - Lower probability weight on strategies yielding lower payoffs.
- Alternatively, players are less likely to make costly mistakes.

ε —Proper Equilibrium

- Definition. **$\varepsilon -$ proper equilibrium**. For any $\varepsilon > 0$, a totally mixed strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is the $\varepsilon -$ proper equilibrium if, for every player i, and for every two pure strategies s_i,and $s_i' \neq s_i$ such that

$$u_i(s_i, \sigma_{-i}) > u_i(s_i', \sigma_{-i}),$$

  - we must have that probabilities of playing s_i and s_i', $\sigma_i(s_i)$ and $\sigma_i(s_i')$ satisfy

$$\varepsilon \times \sigma_i(s_i) \geq \sigma_i(s_i')$$

- Intuitively, if player $i's$ expected payoff from choosing s_i is higher than that from s_i', then…
  - The probability of playing s_i must be at least "ε times higher" than the probability of playing s_i'.

Example 5.10. ε —Proper Equilibrium

Player 2

		l	r
Player 1	U	$\underline{1},\underline{1}$	$\underline{0},0$
	D	$0,\underline{0}$	$\underline{0},\underline{0}$

Matrix 5.14. A Game with two psNEs, but only (U, l) is THPE

- Consider $\sigma_i = \left(1-\frac{\varepsilon}{a}, \frac{\varepsilon}{a}\right)$ for every player i, where $a \geq 2$ and $0 < \varepsilon < 1$.
- This mixed strategy is an $\varepsilon -$ proper equilibrium because: (1) it is a totally mixed strategy, assigning a positive probability weight to all players' strategies; and (2) for pure strategies U and D, their expected utilities satisfy

$$u_1(U,\sigma_2) = \underbrace{1\left(1-\frac{\varepsilon}{a}\right)}_{\substack{Player\ 2 \\ chooses\ l}} + \underbrace{0\left(\frac{\varepsilon}{a}\right)}_{\substack{Player\ 2 \\ chooses\ r}} = 1-\frac{\varepsilon}{a} > 0 = u_1(D,\sigma_2)$$

Example 5.10. ε —Proper Equilibrium Example

And the probabilities of applying U and D are

$$\varepsilon \times \sigma_1(U) = \varepsilon\left(1 - \frac{\varepsilon}{a}\right) = \frac{\varepsilon(a-\varepsilon)}{a} \text{ and}$$

$$\sigma_1(D) = \frac{\varepsilon}{a}$$

which satisfy

$$\varepsilon \times \sigma_1(U) = \frac{\varepsilon(a-\varepsilon)}{a} \geq \frac{\varepsilon}{a} = \sigma_1(D)$$

since, after rearranging, this inequality simplifies to $a \geq \varepsilon$, which holds given that $a \geq 2$ and $0 < \varepsilon < 1$ by assumption.

(Since the game is symmetric, a similar argument applies to player 2's utility from choosing l and r, and its associated probabilities.

Proper Equilibrium

- Definition. **Proper Equilibrium.** A mixed strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is a proper equilibrium if there exists:
  1. A sequence $\{\sigma_i^k\}_{k=1}^{\infty}$ that converges to σ_i for every player i
  2. A sequence $\{\varepsilon_i^k\}_{k=1}^{\infty}$ where $\varepsilon^k > 0$ for all k, that converges to zero
  3. $\{\sigma_i^k\}_{k=1}^{\infty}$ is an ε_k $-$proper equilibrium for every k

- Proper equilibrium are also THPE, but the converse in not necessarily true.
- In other words:
  - If σ is a proper equilibrium, it must be robust to a sequence of decreasing trembles where costly mistakes are less likely to occur;
  - while σ being THPE only requires that it is robust to any sequence of decreasing trembles.

Example 5.11. Proper Equilibrium

		Player 2	
		l	r
Player 1	U	$\underline{1},\underline{1}$	$\underline{0},0$
	D	$0,\underline{0}$	$\underline{0},\underline{0}$

Matrix 5.14. A Game with two psNEs, but only (U, l) is THPE

- The sequence of totally mixed strategies from example 5.9

$$\sigma_i^k = \left(1 - \frac{\varepsilon_k}{2}, \frac{\varepsilon_k}{2}\right) \text{ for every player } i\text{, where } \varepsilon_k = \frac{1}{2^k},$$

is a proper equilibrium if it satisfies the three requirements in the above definition:

1. A sequence σ_i^k converges to (U, l)
2. ε^kconverges to zero
3. σ_i^k is an ε_k-proper equilibrium for every k (as shown in Example 5.10).

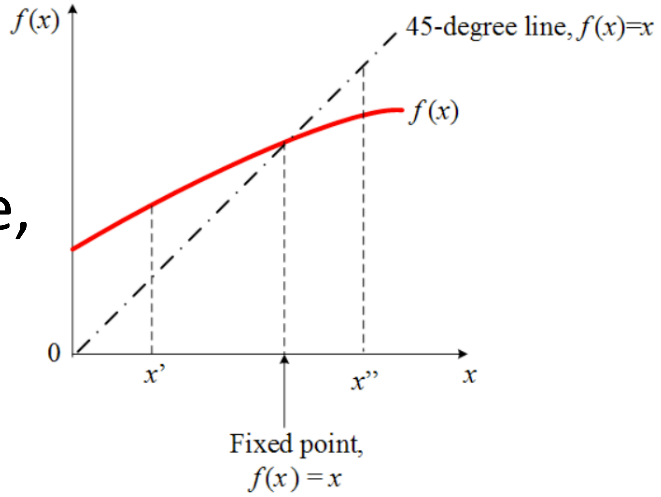
Appendix

Fixed-point theorems, an Introduction

- Consider a function $f\colon X \to X$, mapping elements from X into X, where $X \subset \mathbb{R}^N$.
- We then say that a point $x \in X$ is a “fixed point” if $x \in f(x)$. For instance, if $X \subset \mathbb{R}$, we can define the distance function

$$g(x) = f(x) - x,$$

- which graphically measures the distance from $f(x)$ to the 45-degree line, as illustrated in Figure 5.8a and 5.8b.



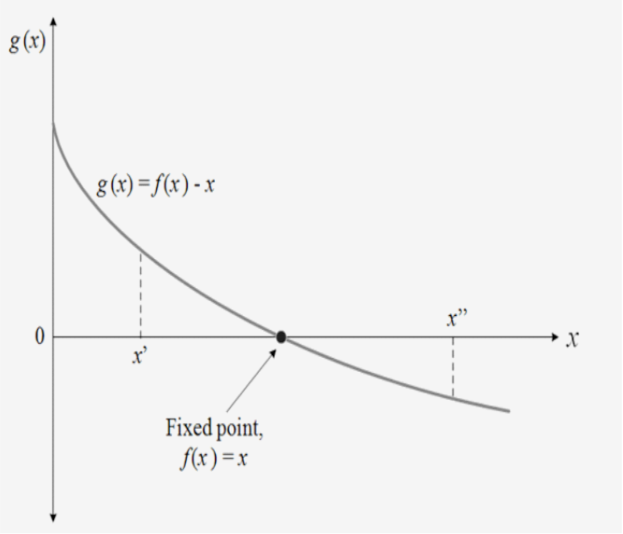
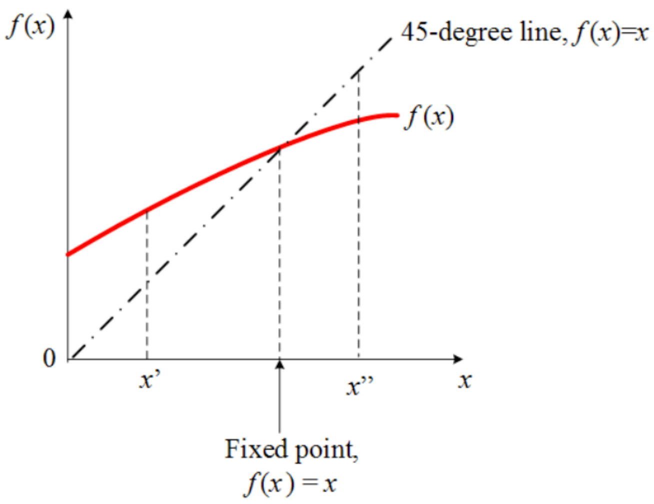


Figure 5.8a. Function $f(x)$ against the 45-degree line.

Figure 5.8b. Distance function $g(x)$.

Fixed-point theorems, an Introduction

- At points such as $x'$where $g(x') > 0$, we have that $f(x') > \ x'$, meaning that $f(x')$ lies above the 45-degree line.
- In contrast, at points $x'' > x'$ where $g(x'') < 0$, we have that $f(x'') < x''$, entailing that $f(x'')$ lies below the 45-degree line.



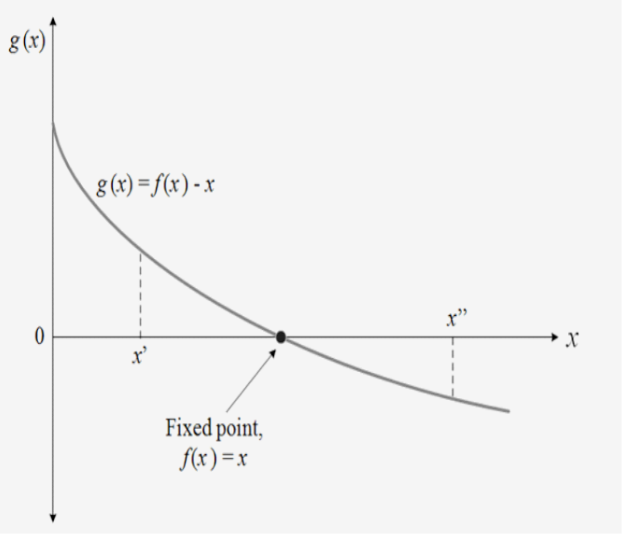


Figure 5.8a. Function $f(x)$ against the 45-degree line.

Figure 5.8b. Distance function $g(x)$.

Fixed-point theorems, an Introduction

- Since x''> x', if the distance function is continuous, we can invoke the intermediate value theorem to say that:
  - There must be an intermediate value, $\hat{x}$, between x' and x'' (or more than one) where $g(\hat{x}) = 0$, implying that
  - $g(\hat{x}) = f(\hat{x}) \ - \hat{x} = 0$, or $f(\hat{x})$ = $\hat{x}$, as required for a fixed point to exist.
- Note that if $f(x)$ was not continuous, then:
  - $g(x)$ would not be continuous either…
  - Allowing for $g(x') > 0$ and $g(x'') < 0$ to occur,
  - yet we could not guarantee the existence of an intermediate point $\hat{x}$ between x' and x'' for which $g(\hat{x}) = 0$.
- Brouwer’s fixed-point theorem formalizes this result (next slide).

Brouwer’s fixed-point theorem

Definition. **Brouwer’s fixed-point theorem.**

If $f: X \to X$ is a continuous function, where $X \subset \mathbb{R}^N$, then it has at least one fixed point, that is, a point $x \in X$ where $f(x) = x$.

- While Brouwer’s fixed-point theorem is useful when dealing with best response functions, it does not apply to best response correspondences.
    - where player i is, for instance, indifferent between two or more of her pure strategies when her opponent chooses strategy s_j.
- The following theorem generalizes Brouwer’s fixed-point theorem to correspondences.
- For a more detailed presentation on fixed-point theorems, see Border (1985).

Kakutani’s fixed-point theorem

Definition. **Kakutani’s fixed-point theorem.**

A correspondence $F: X \rightarrow X$, where $X \subset \mathbb{R}^N$, has a fixed point, that is, a point $x \in X$ where $F(x) = x$, if these conditions hold:

1. X is a compact, convex, and non-empty set.
2. $F(x)$ is non-empty.
3. $F(x)$ is convex.
4. $F(x)$ has a closed graph.

Nash Existence theorem

- First, define player $i's$ pure strategy set, S_i, to be finite, i.e., a discrete list of pure strategies, and denote a mixed strategy for this player as σ_i, where $\sigma_i \in \sum_i$, meaning that player i chooses her randomization among all possible mixed strategies available to her. Therefore, the Cartesian product

$$\sum_i \times \sum_{-i} = \sum$$

denotes the set of all possible mixed strategy profiles in the game, so that every strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ satisfies $\sigma \in \sum$.

- Second, let us define player $i's$ best response correspondence to her rivals choosing σ_{-i} as $\sigma_i \in BR_i(\sigma_{-i})$.

Nash Existence Theorem

- We now define the joint response correspondence $BR(\sigma)$, as the product of $BR_i(\sigma_{-i})$ and $BR_{-i}(\sigma_i)$, that is,

$$BR(\sigma) \equiv BR_i(\sigma_{-i}) \times BR_{-i}(\sigma_{-i})$$

- Importantly, if BR has a fixed point, then, a NE exists.
- Therefore, we next check if BR satisfies the four conditions on Kakutani's fixed-point theorem, as that would guarantee the existence of a NE.

- Before doing that, we identify:
  - X in Kakutani's fixed-point theorem with the set of all possible mixed strategy profiles, $\sum$, and
  - correspondence F with BR.

Nash Existence Theorem

1. $\sum$ is a non-empty, compact, and convex set.

a. The set $\sum$ is non-empty as long as players have some strategies, so we can identify pure or mixed strategy profiles.

b. Recall that if a set is closed and bounded, it is compact. The set of all possible mixed strategy profiles is closed and bounded, thus satisfying compactness.

c. Convexity is satisfied since:

- For any two strategy profiles, σ and σ',
- their linear combination $\lambda\sigma + (1-\lambda)\sigma'$ where $\lambda \in [0,1]$,
- is also a mixed strategy profile, thus being part of $\sum$.

Nash Existence Theorem

2. $BR(\sigma)$ is nonempty.

Since every player $i's$ payoff, $u_i(\sigma_i, \sigma_{-i})$, is linear in both σ_i and σ_{-i} (expected utility is linear in the probabilities)…

She must find a maximum (a best response to her rivals choosing σ_{-i}) among her available strategies, $\sum_i$, which we know it is a compact set from point 1b.

Because $\sigma_i \in BR_i(\sigma_{-i})$ and $\sigma_{-i} \in BR_i(\sigma_i)$ are non-empty (best response exists), then their product, $BR(\sigma)$, must also be non-empty.

3\. $BR(\sigma)$ is convex.

To prove this point, consider two strategies for player i, σ_i and σ_i', that are best responses to her rivals choosing σ_{-i}, that is, $\sigma_i, \sigma_i' \in BR_i(\sigma_{-i})$.

Because both σ_i and σ_i' are best responses, they must both yield the same expected payoff; otherwise, one of them cannot be a best response.

Therefore, a linear combination of σ_i and σ_i', $\lambda\sigma_i + (1-\lambda)\sigma_i'$ where $\lambda \in [0,1]$, must yield the same expected payoff as σ_i and σ_i', thus being a best response as well, that is, $\lambda\sigma_i + (1-\lambda)\sigma_i' \in BR_i(\sigma_{-i})$.

Nash Existence Theorem

4. $BR(\sigma)$ has a closed graph.

This property means that the set $\{(\sigma_i, \sigma_{-i}) \mid \sigma_i \epsilon BR_i(\sigma_{-i})\}$ is "closed," meaning that every player $i's$ best response correspondence has no discontinuities.

The best responses depicted in this chapter, for instance, showed no discontinuities. Because every player $i's$ payoff, $u_i(\sigma_i, \sigma_{-i})$, is continuous and compact, the set $\{(\sigma_i, \sigma_{-i}) \mid \sigma_i \epsilon BR_i(\sigma_{-i})\}$ is closed.

Nash Existence Theorem

- The previous 4 properties guarantee that a NE exists when players face finite strategy spaces (i.e., a list of pure strategies).

- What if they choose their strategies from a continuous strategy space, as when firm set their prices or output levels?
  - Glicksberg (1952) extended the above result to setting with continuous strategy spaces, where $S_i \subset \mathbb{R}^N$, showing that, if:
    1. Every player $i's$ strategy space, S_i , is compact, and
    2. Her utility function, $u_i(.)$, is continuous,
  - Then, a NE exists, in pure or mixed strategies.
  - (For generalization of this result to non-continuous utility functions, see Dasgupta and Maskin (1986).)