Chapter 2: Equilibrium Dominance

Game Theory:

An Introduction with Step-by-Step Examples

by Ana Espinola-Arredondo and Felix Muñoz-Garcia

Introduction

- In most solution concepts, every player “searches” for the strategy providing her with the highest payoff, given the time structure and given her available information.
- Strict dominance, instead, will just “rule out” strategies that a rational player would never choose.
- This occurs when a strategy profiles her with a strictly lower payoff than other available strategies *regardless* of her opponents’ behavior.

Introduction

- We apply that “ruling out” strategy iteratively, moving from one player to another, until we cannot rule out any other strategies.
- The result will be the strategy profiles surviving IDSDS (Iterative Deletion of Strictly Dominated Strategies).
- We will apply IDSDS to typical games in economics and social sciences.
- We will finish the chapter talking about two related solution concepts: IDWDS and SDE.

Introduction

Definition: Strictly Dominated Strategy

- Player i finds that a strategy s_i strictly dominates strategy s_i' if and only if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for every strategy profile $s_{-i} \in S_{-i}$ of player i's rivals, and where $s_i, s_i' \in S_i$, and $s_i' \neq s_i$.

- Intuition:
  - Strategy s_i that provides a player with a strictly lower payoff than another available strategy, s_i', regardless of her opponents' strategy.
  - A rational player should, then, never choose s_i, independently on how she believes her rivals behave.
  - We can delete strategy s_i from player *i*'s strategy set.

Tool 2.1. How to Find Strictly Dominated Strategies

1. Fix your attention on one strategy of the column player, s_2 (i.e., one specific column). Find a strategy s_1' that yields a strictly lower payoff than some other strategy s_1 for the row player, that is $u_1(s_1, s_2) > u_1(s_1', s_2)$ for a given strategy of player 2.
2. Repeat step 1, but now fix your attention on a different column. That is, find if the above payoff inequality also holds, $u_1(s_1, s_2') > u_1(s_1', s_2')$, which is now evaluated at a different strategy of the column player, s_2'.
3. If, after repeating step 1 for all possible strategies of player 2 (all columns), you find that strategy s_1' yields a strictly lower payoff for player 1 than strategy s_1, you can claim that strategy s_1' is strictly dominated by s_1.

Otherwise, player 1 does not have a strictly dominated strategy.

Analogous argument for player 2, switching subscripts and switching “row” for “column” everywhere.

Example 2.1: Finding Strictly Dominated Strategies

		Firm 2		
		h	*m*	*l*
	H	2,2	3,1	5,0
Firm 1	M	1,3	2,2	2.5,2.5
	L	0,5	2.5,2.5	3,3

Matrix 2.1 Strictly Dominated Strategies in the Output game

Example: Finding Strictly Dominated Strategies

Firm 2

		h	m	l
	H	2,2	3,1	5,0
Firm 1	M	1,3	2,2	2.5,2.5
	L	0,5	2.5,2.5	3,3

Firm 1 finds row L to be strictly dominated by H since:

- When firm 2 chooses h, 2>0.
- When firm 2 chooses m, 3>2.5.
- When firm 2 chooses l, 5>3.

A similar argument applies to row M, which is strictly dominated by H since:

- When firm 2 chooses h, 2>0.
- When firm 2 chooses m, 3>2.
- When firm 2 chooses l, 5>2.5.

A similar argument applies to firm 2: columns m and l are both strictly dominated by h. Check as a practice.

Tool 2.2. Applying IDSDS

We next seek to delete as many strictly dominated strategies as possible for each player. We use IDSDS, following the next steps:

1. From the definition of rationality, we know that a player would never use strictly dominated strategies,
   - So we can delete them from her original strategy set, S_i,
   - Obtaining S_i' (which is technically a subset of S_i).
2. We can then proceed by also using common knowledge of rationality.
   - In this context, this entails that every player j can put in her opponent's shoes,
   - Identify all strictly dominated strategies for her opponent, and
   - Delete them from j's strategy set S_j, obtaining S_j'.

Applying IDSDS

3. Continue again: Player i considers now her rival's reduced strategy set S_j' and finds whether some of his own strategies in $S_i' \times S_j'$ now become strictly dominated.
   - At the end of this step, we obtain a further reduced strategy set S_i''.
   - In some games, this set may coincide with the original set in the third step, $S_i'' = S_i'$, if we could not identify more strictly dominated strategies for player i; while in other games, strategy set S_i'' is a strict subset of S_i'.

Applying IDSDS

4. Starting with strategy sets S_i'' and S_j'', we repeat the above process, seeking to find more strictly dominated strategies for either player.
5. The process continues until no player can identify more strictly dominated strategies to delete.
   - The remaining strategies are referred to as the “strategy profile(s) surviving IDSDS.”

Example 2.2: Applying IDSDS

Firm 2

		h	l
	H	4,4	0,2
Firm 1	M	1,4	2,0
	L	0,2	0,0

Matrix 2.2a. When IDSDS yields a unique equilibrium

Starting with firm 1:

Row L is strictly dominated by M. Check!

Example: IDSDS

		Firm 2	
		h	l
Firm 1	H	4,4	0,2
	M	1,4	2,0

Matrix 2.2b. IDSDS yields a unique equilibrium – After one round of IDSDS

Moving to firm 2:

Column l is strictly dominated by h. Check!

Example: IDSDS

		Firm 2
		h
Firm 1	H	4,4
	M	1,4

Matrix 2.2c. IDSDS yields a unique equilibrium – After two rounds of IDSDS

		Firm 2
		h
Firm 1	H	4,4

Matrix 2.2d. IDSDS yields a unique equilibrium – After three rounds of IDSDS

Then, the unique strategy profile surviving IDSDS is (H, h), yielding an equilibrium payoff of 4 to each firm.

IDSDS Properties

1. Does the order of deletion matter?
   - The order of deletion does not matter when applying IDSDS.
   - We would predict different equilibrium results if we started deleting strictly dominated strategies for player 1 or 2.
   - See Example 2.3 in the book, where we redo Example 2.2 but starting with firm 2.

2. Deleting more than one strategy at a time
   - We can delete all strictly dominated strategies we found for that player, so we do not need to delete on strictly dominated strategy at a time, saving us valuable time.
   - We did this in Example 2.3.

3. Multiple equilibrium predictions
   - In Examples 2.2 and 2.3 we found unique equilibrium predictions.
   - In this case, we say the game is "dominance solvable."
   - That's not necessarily the case for all games, as we see next.

Example 2.4. Multiple Equilibria

Firm 1 finds H to be strictly dominated by L.

After deleting row H, we obtain the matrix at the bottom of the slide.

Firm 2

Firm 1		h	m	l
	H	2,3	1,4	3,2
	M	5,3	2,1	1,2
	L	3,6	4,7	5,4

Matrix 2.3a. When IDSDS yields more than one equilibrium-I

Firm 2 finds l to be strictly dominated by h.

After deleting column l, we obtain the matrix in the next slide

Firm 2

Firm 1		h	m	l
	M	5,3	2,1	1,2
	L	3,6	4,7	5,4

Matrix 2.3b. When IDSDS yields more than one equilibrium-II

Example 2.4. Multiple Equilibria

		Firm 2	
		h	m
Firm 1	M	5,3	2,1
	L	3,6	4,7

Matrix 2.3c. When IDSDS yields more than one equilibrium-III

At this point, we find no other strictly dominated strategies for firm 1, or for firm 2. Check!

Then,

$$IDSDS = \{(M, h), (M, m), (L, h), (L, m)\},$$

meaning that IDSDS doesn't provide us a very precise equilibrium prediction.

IDSDS: Prisoner’s Dilemma

		Player 2	
		Confess	Not Confess
Player 1	Confess	-4,-4	0,-8
	Not Confess	-8,0	-2,-2

Matrix 2.4a. The Prisoner’s Dilemma game

Starting with player 1, he finds that NC is strictly dominated by C since:

- $-8 < -4$ when player 2 confesses, in the left column.
- $-2 < 0$ when player 2 doesn’t confess, in the right column.

IDSDS: Prisoner's Dilemma

		Player 2	
		Confess	Not Confess
Player 1	Confess	-4,-4	0,-8

Matrix 2.4b. The Prisoner's Dilemma game after one round of IDSDS

Player 2 finds NC to be strictly dominated by C since $-8 < -4$.

IDSDS: Prisoner's Dilemma

Deleting the NC column, we are left with just one cell:

		Player 2
		Confess
Player 1	Confess	-4,-4

Matrix 2.4c. The Prisoner's Dilemma game after two rounds of IDSDS

IDSDS then predicts a unique strategy profile, (C, C), with an equilibrium payoff of -4 to every player.

Notice that the outcome is not Pareto optimal, yet no player has *unilateral incentives to deviate*.

Other Prisoner’s Dilemma Games

- Similar conflicts between individual and group incentives are common in economics and social sciences:
  - Price wars between firms
  - Tariff wars between countries
  - Use of negative campaigning in politics.
- Some movies with Prisoner’s Dilemma game scenes:
  - The dark knight: https://www.youtube.com/watch?v=K4GAQtGtd_0.
  - Murder by numbers, https://www.youtube.com/watch?v=UAR7WDrL3Ec.
  - The hunger games: https://www.youtube.com/watch?v=rCWL-pRX_hE.

Prisoner's Dilemma Game, General form

		Player 2	
		Confess	Not Confess
Player 1	Confess	a, a	b, c
	Not Confess	c, b	d, d

Matrix 2.4d. The Prisoner's Dilemma game – General Form

- For players to find Confess to be strictly dominant, we need that:
  - $a > c$, in the left column and
  - $b > d$, in the right column.
- We aren't ranking payoffs *a* and *d*, but:
  - If $a < d$ we can say that the strategy profile surviving IDSDS, (C, C), is not Pareto optimal.
  - Otherwise, the equilibrium of this game according to IDSDS, (C, C), is also Pareto optimal.

Coordination Games – Battle of the Sexes

		Wife: Football, F	Wife: Opera, O
Husband	Football, F	10,8	6,6
	Opera, O	4,4	8,10

Matrix 2.5a. The Battle of the Sexes game

- Husband has no strictly dominated strategy. Check.
- Same argument for wife.
- Then, the game as a whole is our equilibrium prediction after applying IDSDS (i.e., four strategy profiles).

Coordination Games, Other examples

- Interaction of two or more firms investing in different technologies:
  - Firms can share files, or parts, only if they both invest in the same technology.
- Friends buying game consoles that run on different operating systems.
- Friends joining social media platforms.
- Choosing the side of the road to drive (driving conventions).
- More generally, games where players benefit from *positive network externalities* when choosing the same strategy as their rivals.

Coordination Games, General form

		Wife: Football, F	Wife: Opera, O
Husband	Football, F	a_H, a_W	b_H, c_W
	Opera, O	c_H, b_W	d_H, d_W

Matrix 2.5b. Coordination games – General Form

For this to be a coordination game (no strictly dominated strategy for any player), we need payoffs to satisfy:

- $a_i > c_i$
- $d_i > c_i$

for every player $i = \{H, W\}$.

This general form allows for symmetric payoffs when attending the same event, or asymmetric payoffs.

Pareto Coordination Games – Stag Hunt

Pareto Coordination Game – Stag Hunt

		Player 2	
		Stag	Hare
Player 1	Stag	6,6	1,4
	Hare	4,1	2,2

Matrix 2.6a. The stag hunt game

- Player 1 has no strictly dominated strategies.
- Player 2 doesn't have any strictly dominated strategy either.
- Hence, IDSDS predicts the original four strategy profiles.
  - In these cases, we say that "IDSDS has no bite."

General Form and Pareto Coordination Games

The general form of Pareto coordination games

		Player 2	
		Stag	Hare
Player 1	Stag	a, a	c, b
	Hare	b, c	d, d

Matrix 2.6b. Pareto coordination games – General Form

Payoffs need to satisfy $a > b \geq d > c$.

So, when player 2 selects:

- Stag, player 1 is better off choosing Stag as well since $a > b$
- Hare, player 1 is better off choosing Hare as well since $d > c$.

Anti Coordination Game – The Game of Chicken

Player 2

Player 1		Swerve	Stay
	Swerve	-1, -1	-8,10
	Stay	10, -8	-30, -30

Matrix 2.7a. Anticoordination game

- Player 1 has no strictly dominated strategies.
  - When player 2 chooses to $Swerve$, player 1 is better off choosing $Stay$ because $10 > -1$, but if player 2 chooses to $Stay$, player 1 is better off choosing $Swerve$ because $-8 > -30$.
- Player 2 doesn't have any strictly dominated strategy either.
- Hence, IDSDS predicts the original four strategy profiles.
  - In these cases, we say that "IDSDS has no bite."

Anti Coordination Game – The Game of Chicken

		Player 2	
		Swerve	Stay
Player 1	Swerve	a, a	c, b
	Stay	b, c	d, d

Matrix 2.7b. Anticoordination game – General form

For players to exhibit incentives to anticoordinate, payoffs need to satisfy $b > a > c > d$.

So, when player 2 selects:

- Swerve, player 1 is better off choosing Stay since $b > a$
- Stay, player 1 is better off choosing Swerve since $c > d$.

Anti Coordination Game – The Game of Chicken

		Player 2	
		Swerve	Stay
Player 1	Swerve	-1, -1	-8,10
	Stay	10, -8	-30, -30

Matrix 2.7a. Anticoordination game

Movies with Anti-coordination games scenes

- Rebel without a Cause: https://www.youtube.com/watch?v=BGtEp7zFdrc
- Footloose: https://www.youtube.com/watch?v=ZL57muBck0w.
- Stand by Me: https://www.youtube.com/watch?v=6L-IbkWHsOE.
- The hunt for the Red October: https://www.youtube.com/watch?v=mh599VtMB1c.
- A beautiful mind: https://www.youtube.com/watch?v=LJS7Igvk6ZM

Symmetric Games

- **Definition:** A two-player game is symmetric if both players' strategy sets coincide, $S_A = S_B$, and payoffs are unaffected by the identity of the player choosing each strategy, that is:

$$u_A(s_A, s_B) = u_B(s_A, s_B)$$

for every strategy profile (s_A, s_B).

- This property is also known as that the payoff function satisfies *anonymity*:
  - Starting from a setting where player A chooses s_A and player B chooses s_B, …
  - if we were to switch the identities of players A and B, so that player A becomes B and player B becomes A,…
  - every player's payoff would be unaffected.

Symmetric Games – Visual interpretation

Visually, the matrix of a symmetric game must satisfy the following properties:

1. Same number of rows and columns, so both players have the same number of available strategies.
2. Same action labels in rows and columns, so player A and B face the same strategy sets.
3. Payoffs along the main diagonal must coincide for every player (i.e. both players earn the same payoff when they choose the same strategy).
    1. Formally, this property entails that, if both players choose the same strategy, $s_A = s_B = s$, their payoffs must also coincide, $u_A(s, s) = u_B(s, s)$.
    2. Otherwise, an identity "switch" (formally known as a permutation) would affect players' payoffs.
4. Cells above the main diagonal must be mirror images of those below the main diagonal.
    1. This implies that when $s_A \neq s_B$, we can satisfy the condition $u_A(s_A, s_B) = u_B(s_A, s_B)$.
    2. Otherwise, an identity "switch" would affect players' payoffs.

Examples: Prisoner's Dilemma and Game of Chicken are symmetric. Battle of the Sexes is not.

Symmetric Games - Example

		Player 2	
		Confess	Not Confess
Player 1	Confess	-4,-4	0,-8
	Not Confess	-8,0	-2,-2

Matrix 2.4a. The Prisoner’s Dilemma game

- This game satisfies the above 4 properties.
  - Same number of rows and columns.
  - Same action labels.
  - Payoffs along the main diagonal coincide across players.
  - Payoffs away from the main diagonal are mirror images of each other. (next slide explains this point).

Symmetric Games - Example

		Player 2	
		Confess	Not Confess
Player 1	Confess	-4,-4	0,-8
	Not Confess	-8,0	-2,-2

Matrix 2.4a. The Prisoner's Dilemma game

- Consider the strategy profile $(s_1, s_2) = (C, NC)$, where player 1 confesses but player 2 does not.
- In this setting, player 1's payoff is $u_1(C, NC) = 0$, while player 2's is $u_2(C, NC) = -8$.
- If we switch their identities, so that player 1 becomes player 2 (and does not confess), and player 2 becomes player 1 (and confesses), their payoffs are $u_2(NC, C) = 0$ for player 2 (who was originally player 1), and $u_1(NC, C) = -8$ for player 1 (who was originally player 2), thus being unaffected by the identity switch.
- A similar argument applies to other strategy profiles in this game.

Asymmetric Games

- **Definition:** A two-player game is asymmetric if:
  - Players face different strategy sets, $S_A \neq S_B$, or
  - Despite facing the same strategy sets, $S_A = S_B$, their payoffs satisfy

$$u_A(s_A, s_B) \neq u_B(s_A, s_B)$$

    for at least one strategy profile (s_A, s_B).

- In other words, a game is asymmetric if it violates at least one of the 4 properties required for a game to be symmetric.

Asymmetric Games - Example

		Wife	
		Football, F	Opera, O
Husband	Football, F	10,8	6,6
	Opera, O	4,4	8,10

Matrix 2.5a. The Battle of the Sexes game

- Consider the strategy profile $(s_H, s_W) = (F, F)$, where both husband and wife go to the football game.
- In this context, the husband's payoff is $u_H(F, F) = 10$, while his wife's is $u_W(F, F) = 8$.
- If we switch their identities, the wife's payoff becomes $u_w(F, F) = 10$ and $u_H(F, F) = 8$, implying that their payoffs are affected.
- Graphically, this could be seen in Matrix 2.5a:
  - While satisfying Properties #1 and #2 for the game to be symmetric.
  - It violates Property #3 (players' payoffs don't coincide along the main diagonal), and
  - It also violates Property #4 (players' payoffs above and below the main diagonal are not mirror images).

Randomization to bring IDSDS Further

- Sometimes IDSDS has little or no “bite” in the game.
  - Consider Battle of the Sexes game, as an example of Coordination game; or the Game of Chicken, as an examples of anticoordination game).
  - You can also consider the following 3x3 matrix.

Firm 2

Firm 1		h	m	l
	H	0,10	4,6	4,6
	M	4,6	0,10	6,4
	L	10,0	6,4	4,6

Matrix 2.8. Applying IDSDS and allowing for randomizations- I

- Firm 1 has no strictly dominated strategies we can delete:
  - While row H yields strictly lower payoffs than L when firm 2 chooses h or m ($0 < 10$ and $4 < 6$, respectively), it yields the same payoff when firm 2 chooses l (4).
- A similar argument applies to Firm 2 which does not have strictly dominated strategy either.
- What can we do?

Randomization to bring IDSDS Further

Don’t despair! When we allow players to randomize between two or more strategies, IDSDS might still have some “bite.”

			Firm 2		
			h	m	l
Firm 1		H	0,10	4,6	4,6
	Prob. $(1-q) \rightarrow$	M	4,6	0,10	6,4
	Prob. $q \rightarrow$	L	10,0	6,4	4,6

Matrix 2.8. Applying IDSDS and allowing for randomizations- I

Randomization to bring IDSDS Further

			Firm 2		
			h	m	l
		H	0,10	4,6	4,6
Firm 1	Prob. $(1-q) \rightarrow$	M	4,6	0,10	6,4
	Prob. q $\rightarrow$	L	10,0	6,4	4,6

Matrix 2.8. Applying IDSDS and allowing for randomizations- I

- However, strategy H for firm 1 seems to produce a lower payoff than a combination of rows M and L.
- Specifically, M and L yield weakly higher payoffs than H does when firm 2 picks columns h or l, but yields extreme payoffs when firm 2 chooses column m (which are clearly above or below that of H).
- This indicates that, if we could create a linear combination between the payoffs in row M and L, player 1 would receive an expected utility that lies strictly above that of strategy H, ultimately helping us to delete this top row from the matrix.

Randomization to bring IDSDS Further

			Firm 2		
			h	m	l
		H	0,10	4,6	4,6
Firm 1	Prob. $(1-q) \rightarrow$	M	4,6	0,10	6,4
	Prob. $q \rightarrow$	L	10,0	6,4	4,6

Matrix 2.8. Applying IDSDS and allowing for randomizations- I

- Let us first compute the expected utility from assigning:
  - A probability weight $q \in (0,1)$ to strategy L, and
  - The remaining probability weight $1-q$ to strategy M.
- That is, we create the mixed strategy $\sigma = qL + (1-q)M$.
- Intuitively, a randomization could be flipping a coin, $q = 1/2$, but that's too specific. At this point, we don't know which probability weight we should assign to L and M to guarantee that their expected payoff exceeds the certain payoff from playing H.

Randomization to bring IDSDS Further

			Firm 2		
			h	m	l
		H	0,10	4,6	4,6
Firm 1	Prob. $(1-q) \rightarrow$	M	4,6	0,10	6,4
	Prob. $q \rightarrow$	L	10,0	6,4	4,6

Matrix 2.8. Applying IDSDS and allowing for randomizations- I

- Mixed strategy $\sigma = qL + (1-q)M$ yields the following expected utilities:

$$EU(\sigma|h) = q10 + 4(1-q) = 4 + 6q$$
$$EU(\sigma|h) = q6 + 0(1-q) = 6q$$
$$EU(\sigma|h) = q4 + 6(1-q) = 6 - 2q$$

Randomization

Inserting these expected payoffs into Matrix 2.8 yields

		Firm 2		
		h	m	l
Firm 1	H	0	4	4
	$qL + (1-q)M$	$4+6q$	$6q$	$6-2q$

Matrix 2.9. Applying IDSDS and allowing for randomization –II

Randomization

We can then say that, for the randomization $\sigma = qL + (1-q)M$ to strictly dominate strategy H, we need that Firm 1's expected payoff need to satisfy:

$4 + 6q > 0$, or $q > -\frac{2}{3}$ when firm 2 chooses h

$6q > 4$, or $q > \frac{2}{3}$ when firm 2 chooses m, and

$6 - 2q > 4$, or q < 1 when firm 2 chooses l

- The first and last condition on probability q hold because q must satisfy $q \in (0,1)$ by assumption.
- The second condition, $q > \frac{2}{3}$ restricts the range of probability weights we can use in randomization $qL + (1-q)M$.
- Intuitively, player 1 must assign a sufficiently high weight to strategy L.

Randomization

- Therefore, any randomization with $q > \frac{2}{3}$, helps us show that strategy H is strictly dominated, so we can delete it from Matrix 2.8 to obtain Matrix 2.10.

		Firm 2		
		h	m	l
Firm 1	M	4,6	0,10	6,4
	L	10,0	6,4	4,6

Matrix 2.10. Applying IDSDS and allowing for randomizations –III

Randomization

We can now move to Firm 2 to note that h is strictly dominated by m since h yields a strictly lower payoff than m does:

- When firm 1 chooses M in top row (where $10 > 6$), and
- When firm 1 chooses L in the bottom row (where $4 > 0$).

Randomization

We can now move to Firm 2 to note that h is strictly dominated by m since h yields a strictly lower payoff than m does:

- When firm 1 chooses M in top row (where $10 > 6$), and
- When firm 1 chooses L in the bottom row (where $4 > 0$).

After deleting this strictly dominated strategy for firm 1, we are left with:

		Firm 2	
		m	l
Firm 1	M	0,10	6,4
	L	6,4	4,6

Matrix 2.11. Applying IDSDS and allowing for randomization – IV

Randomization

- We can now move again to firm 1, noticing that it no longer has strictly dominated strategies.
  - In particular, M yields a strictly lower payoff than L when firm 2 chooses m, *but* a strictly higher payoff than L when firm 2 chooses l.
- If we move to firm 2, we cannot find strictly dominated strategies for this player either.
  - When firm 1 chooses row M, firm 2 is better off selecting m; *but* when firm 1 chooses row L, firm 2 is better off selecting l.

		Firm 2	
		m	l
Firm 1	M	0,10	6,4
	L	6,4	4,6

Randomization

- In summary, the application of IDSDS did not have a bite when we assumed players could not randomize.
  - In that context, our equilibrium prediction would have been the game as a whole, yielding 3x3=9 different outcomes.
- Allowing for randomizations, however, helped us discard one strategy for each player as being strictly dominated, leaving us with a 2×2 matrix (4 different equilibrium outcomes), that is:

$$IDSDS = \{(M,m),(M,l),(L,m),(L,l)\}$$

What if IDSDS has no bite?

In some games,

- Players may have no strictly dominated strategies that we can delete using IDSDS.
- Ok, we can try by allowing for randomizations.
- What if, still, we cannot delete any strictly dominated strategies?
- Examples include the Battle of the Sexes game and the Game of Chicken, among others.

		Criminal	
		Street A	*Street B*
Police	*Street A*	10,0	-1,6
	Street B	0,8	7,-1

What to do?

- Apply Nash Equilibrium! (Chapters 3-5)

Evaluating IDSDS as a concept

1. Existence? Yes.
   - When we apply IDSDS to any game, we find that at least one equilibrium exists.
2. Uniqueness? No.
   - IDSDS does not necessarily provide a unique equilibrium outcome for all games since more than one cell may survive IDSDS.
3. Robust to small payoff perturbations? Yes.
   - If we change the payoff of one of the players by a small amount, IDSDS still yields the same equilibrium outcome(s).
4. Socially optimal? No.
   - The application of IDSDS does not necessarily yield socially optimal outcomes.
   - Prisoner's Dilemma game, where IDSDS provides us with a unique equilibrium prediction, (Confess, Confess), which does not coincide with the strategy profile that maximizes the sum of players' payoffs, (Not Confess, Not Confess).

Weakly Dominated Strategies

- **Definition:** Player i finds that the strategy s_i weakly dominates another strategy s_i' if:
  1. $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for every strategy profile s_{-i} player $i's$ rivals, and
  2. $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for *at least one* strategy profile s_{-i}.

  Therefore, choosing s_i provides player *i* with:
  - the same or higher payoff than s_i' against all her rivals' strategies, but…
  - a strictly higher payoff than s_i' for at least one of her rivals' strategies.

  Technically, requirement #2 in this definition avoids a "total tie" between player *i*'s payoffs in s_i and s_i'.

Weakly Dominated Strategies

- **Definition:** Player i finds that the strategy s_i weakly dominates another strategy s_i' if:
  1. $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for every strategy profile s_{-i} player $i's$ rivals, and
  2. $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for *at least one* strategy profile s_{-i}.

  Graphical interpretation in a two-player game (potentially with more than two rows and columns):

  - Player 1 finds that two of her rows yield the same payoff against every column of player 2, except for one column.
  - Similar argument applies for player 2.

Weakly Dominated Strategies – Example 2.6

		Firm 2		
		h	*m*	*l*
	H	2,2	3,2	5,0
Firm 1	M	2,3	3,2	2.5,2.5
	L	0,5	2.5,2.5	3,3

Matrix 2.12. Finding weakly dominated strategies

- Firm 1 finds that row H weakly dominates M because:
  - When firm 2 chooses h, 2=2.
  - When firm 2 chooses m, 3=3.
  - When firm 2 chooses l, 5>2.5.
- Recall if, firm 1's payoffs coincided across all columns (i.e., 5=5 in column l), then we couldn't claim that H weakly dominates M because Requirement #2 in the definition wouldn't be satisfied.

Does weakly dominated mean strictly dominated?

$$s_i \text{ is strictly dominated} \Rightarrow s_i \text{ is weakly dominated}$$
$$\nLeftarrow$$

- In Matrix 2.12, strategy L is strictly dominated by H, since L yields an unambiguously lower payoff for firm 1 than H does, regardless of the column that firm 1 chooses.
- This means that the requirement 2 of the weak dominance definition holds for all firm 2's strategies, meaning that L is also weakly dominated by H.
- However, strategy M is weakly dominated by H but M is not strictly dominated by H.

Deletion order matters in IDWDS

IDWDS may yield different equilibrium predictions depending on which player we start with

- Starting with <u>Firm 1</u>

		Firm 2		
		h	m	l
Firm 1	H	2,2	3,2	5,0
	L	0,5	2.5,2.5	3,3

Matrix 2.13a. Applying IDWDS starting with firm 1- First step

Deletion order matters in IDWDS

IDWDS may yield different equilibrium predictions depending on which player we start with

- Starting with Firm 1
- It is straightforward to see that firm 1 finds that H strictly dominates L, which helps us delete row L, and obtain Matrix 2.13b.

		Firm 2		
		h	m	l
Firm 1	H	2,2	3,2	5,0

Matrix 2.13b. Applying IDWDS starting with firm 1- Second step

Deletion order matters in IDWDS

IDWDS may yield different equilibrium predictions depending on which player we start with

- Starting with Firm 1
- Turning now to firm 2, it finds that h strictly dominates l, so we can delete the right-hand column and obtain Matrix 2.13c.

		Firm 2	
		h	m
Firm 1	H	2,2	3,2

Matrix 2.13c. Applying IDWDS starting with firm 1- Third step

- At this point, we cannot delete any further strategies as being weakly or strictly dominates for either firm, yielding an equilibrium outcome:

$$IDWDS = \{(H,h),(H,m)\}$$

Deletion order matters in IDWDS

IDWDS may yield different equilibrium predictions depending on which player we start with

- Starting with Firm 2
- We find that column h weakly dominates m since it yields the same payoff as m does when firm 1 chooses H, but generates a strictly higher payoff when firm 1 selects M or L. After deleting column m, we obtain Matrix 2.14a:

		Firm 2	
		h	l
	H	2,2	5,0
Firm 1	M	2,3	2.5,2.5
	L	0,5	3,3

Matrix 2.14a. Applying IDWDS starting with firm 2- First step

Deletion order matters in IDWDS

IDWDS may yield different equilibrium predictions depending on which player we start with

- Starting with Firm 2
- Still analyzing firm 2, we find that h strictly dominates l, which leaves us with the reduced Matrix 2.14b:

		Firm 2
		h
Firm 1	H	2,2
	M	2,3
	L	0,5

Matrix 2.14b. Applying IDWDS starting with firm 2- Second step

Deletion order matters in IDWDS

IDWDS may yield different equilibrium predictions depending on which player we start with

- Starting with Firm 2
- We can now turn to firm 1, which finds that H strictly dominates L, yielding Matrix 2.14c:

		Firm 2
		h
Firm 1	H	2,2
	M	2,3

Matrix 2.14c. Applying IDWDS starting with firm 2- Third step

- At this point, we cannot delete any other strategies for firm 1, as none are strictly or weakly dominated, which yields the equilibrium prediction:

$$IDWDS = \{(H,h),(M,h)\}$$

IDSDS vs. IDWDS

We say that the set of strategy profiles surviving IDWDS is a subset of those surviving IDSDS

$$s \text{ survives } IDWDS \Rightarrow s \text{ survives } IDSDS$$
$$\not\Leftarrow$$

Example

Firm 2

		h	m
Firm 1	H	2,2	3,2
	M	2,3	3,2

Matrix 2.15. Applying IDSDS in Matrix 2.11

IDSDS vs. IDWDS

- Firm 1 cannot delete any strictly dominated strategies from Matrix 2.15, and neither can Firm 2, yielding the four equilibrium outcomes:

$$IDSDS = \{(H,h),(H,m),(M,h)(M,m)\}$$

- Therefore, the application of IDWDS has more bite than IDSDS, producing more precise equilibrium predictions.
- However, IDWDS depends on the order of deletion (e.g. starting with player 1 or 2), thus limiting its applicability.

Strictly Dominant Strategy

Definition: Strictly Dominant Strategy

- Player i finds that strategy s_i is strictly dominant if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

<u>for every strategy $\boldsymbol{s_i' \neq s_i}$</u>, and
for every strategy profile s_{-i} of player $i's$ rivals.

- Intuitively, all her available strategies yield a strictly lower payoff than s_i regardless of her rivals' behavior.
- This means that we can delete all strategies $s_i' \neq s_i$ as being strictly dominated for player i, leaving her with a single (undominated) strategy.
- In other words, she must be choosing s_i.

Strictly Dominant Strategy

Definition: Strictly Dominant Strategy

- Player i finds that strategy s_i is strictly dominant if

$$u_i(s_i, s_{-i}) > (s_i', s_{-i})$$

for every strategy $s_i' \neq s_i$, and
for every strategy profile s_{-i} of player $i's$ rivals.

- *Examples*:
  - Confess in Prisoner's Dilemma game is strictly dominant for both players.
  - No player has no strictly dominant strategies in BoS or Chicken games.
  - In matrix 2.1:
    - Firm 1 finds H to strictly dominate m and l, so we can claim that H is strictly dominant for firm 2.
    - Same argument with firm 2, which finds h to be strictly dominant.

Strictly Dominant Strategy and Strictly Dominant Equilibrium (SDE)

Definition: Strictly Dominant Equilibrium (SDE)

- A strategy profile $s^{SD} = \left(s_i^{SD}, s_{-i}^{SD}\right)$ is a strictly dominant equilibrium (SDE) if every player i finds her strategy, s_i^{SD}, to be strictly dominant.

- An SDE is the only strategy surviving IDSDS after just one round of deleting strictly dominated strategies.
  - Example: Matrix 2.1.
    - H (h) is strictly dominant for firm 1 (2, respectively), implying that (H,h) is an SDE.
    - Firm 1 (2) finds that M and L (m and l) are strictly dominated by H (h), implying that firm 1 (2) can delete M and L (m and l) from its rows (columns), leaving us with (H,h) as the unique strategy profile surviving IDSDS.
      - [As a curiosity, note that we only needed to apply Rationality, not Common Knowledge of Rationality]

Example SDEs

- Examples:
  - Prisoner's Dilemma. Confess is strictly dominant strategy for every player, implying that (C,C) is SDE.
  - Battle of the Sexes. No player has strictly dominant strategy, implying that no SDE exists.
  - Game of Chicken. No player has strictly dominant strategy, implying that no SDE exists.

		Firm 2	
		h	l
Firm 1	H	4, 4	0, 2
	M	1, 4	2, 0
	L	0, 2	0, 0

Matrix 2.2a. IDSDS yields a unique equilibrium.

Example SDEs

- More examples:
  - Consider Matrix 2.2 again.
  - Earlier in this chapter, we showed that a unique strategy profile survives IDSDS.
  - What about SDE? There is none, because no player has a strictly dominant strategy.
  - While we found strictly dominated strategies for every player, we cannot find strictly dominant strategies for any player.
  - Same argument applies to Examples 2.3 and 2.4.

		Firm 2	
		h	l
	H	4, 4	0, 2
Firm 1	M	1, 4	2, 0
	L	0, 2	0, 0

Matrix 2.2a. IDSDS yields a unique equilibrium.

Example SDEs

- Therefore, we can conclude that:

$$(s_i, s_{-i}) \text{ is an SDE} \Rightarrow (s_i, s_{-i}) \text{ survives IDSDS}$$
$$\not\Leftarrow$$

- For examples of the first line of implication, consider the Prisoner's Dilemma game.
- For examples of the second line (of not implication), consider (F,F) or (O,O) in the Battle of Sexes game.

Evaluating SDE as a solution concept

1. Existence? No.
   - When we seek to find strictly dominant strategies for each player, in our search of a SDE, we may find games where one or more players do not have a strictly dominant strategy, implying that an SDE does not exist.
2. Uniqueness? Yes
   - While a SDE may not exist in some games, when we find one, it must be unique.
3. Robustness to small payoff perturbation? Yes.
   - If we change the payoff of one of the players by a small amount $(\varepsilon \to 0)$, SDE still yields the same equilibrium prediction.
4. Socially optional? No.
   - SDE of a game does not need to be socially optimal.
   - Example: Prisoner's Dilemma game, the SDE is (Confess, Confess), which does not coincide with the strategy profile that maximizes the sum of players' payoffs.

Evaluating SDE as a solution concept

	IDSDS	IDWDS	SDE
Existence	Yes	Yes	No
Uniqueness	No	No	Yes
Robustness of payoff changes	Yes	Yes	Yes
Pareto optimal	No	No	No

- In summary:
  - IDSDS and IDWDS exhibit similar properties, although IDWDS has “more bite.”
  - Relative to IDSDS and IDWDS, SDE is so much more demanding that it may not exist. But if it exists, it is unique.