Chapter 11: Mechanism Design

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1 Introduction

In previous chapters, we analyzed different games, allowing for players to be perfectly or imperfectly informed, to interact simultaneously or sequentially, once or in repeated interactions. However, the "rules of the game" were assumed to be given (exogenous). In this chapter, we analyze how the agent organizing the game can alter the rules of the game to maximize his objective function. A common example is that of auctions, where the auctioneer does not need to take the auction form as given (e.g., organizing a first-price auction) but can instead design the auction (the "rules of the game") to his advantage, which in this case often implies maximizing his expected revenue.

This "game design" can be critical for the auctioneer, or generally for any social planner, if he cannot observe the preferences of the individuals playing the game, such as the bidders' valuation of a good being sold by the auctioneer, citizens' preferences for a public project (e.g., a bridge), or firms' production costs in procurement contracts. In all of these settings a similar question emerges:

Which game design, often referred to as "mechanism," can induce individuals to choose a strategy profile that maximizes the social planner's objective function?

A related question considers that, since the social planner designs the game (mechanism), he could envision a setting where every individual is asked to *directly* report his private information (e.g., valuation for an object in an auction), and then the social planner could choose a socially preferred outcome using that information. The question in this "direct revelation mechanism" is whether individuals have incentives to truthfully report their private information if they anticipate the social planner will use it for policy decisions, such as taxes or subsidies.

In this chapter we explore the above questions, analyzing direct revelation mechanisms as the one described above, and indirect revelation mechanisms, whereby players choose a strategy (e.g., a bid in an auction) and the strategy profile arising in equilibrium determines an outcome which can be used to infer players' private information. We also study the relationship between both types of mechanisms, that is, under which conditions a "clever" mechanism designer could envision an indirect revelation mechanism (e.g., a particular type of auction) that produces the same equilibrium outcomes as that chosen by a social planner who could directly observe players' private information.

We then examine properties that are usually regarded as desirable for a mechanism, such as efficiency (the social planner cannot find any reallocation that could increase aggregate surplus);

[1] We appreciate the suggestions and comments of several students, specially Pak-Sing Choi and Kiriti Kanjilal.

incentive compatibility (so players have incentives to truthfully reveal their private information to the mechanism designer rather than misreporting it); participation constraint (i.e., every player's payoff from participating in the mechanism is weakly higher than his outside option); and budget balance (the mechanism does not run a deficit or surplus). Finally, we present different mechanisms often used in the literature, such as the Groves mechanism, the Vickrey-Clarke-Groves (VCG) mechanism, and the d'Aspremont, Gerard-Varet and Arrow (dAGVA) mechanism. We accompany our discussion with several examples from auction theory and public economics to facilitate the presentation.

2 Model: Mechanisms as Bayesian Games

Players: Each player $i = \{1, 2, ..., n\}$ privately observes his type $\theta_i \in \Theta_i$ which determines his preferences over the public project, or his willingness to pay for the object being sold in an auction. The profile of types for all n players, $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, is often called the "state." State θ is drawn randomly from the state space $\Theta \equiv \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. The draw of θ is according to some prior distribution $\phi(\cdot)$ over Θ. While the specific draw θ_i is player i's private information, the distribution $\phi(\cdot)$ is common knowledge among all players. Many applications assume that every player i has quasilinear preferences, which eliminates wealth effects. In particular, a common utility function considers that player i's utility is

$$v_i(x, t, \theta_i) = u_i(x, \theta_i) + t_i$$

where $u_i(x, \theta_i)$ indicates player i's utility from consuming x units of the good (e.g., public project or good being sold at an auction) given his individual preference for such good, as captured by parameter θ_i. Function $u_i(\cdot)$ could be increasing (decreasing) in $x \in X$ when x represents a good (bad, respectively), and concave or convex in x depending on the application we seek to study.[2]

Transfer t_i is the amount of money given to (or taken away from) individual i. Such a transfer can thus be positive, but can also be negative if money is taken away from individual i (e.g., he pays t_i to the central authority in order to fund the public project). An outcome is, then, represented as $y = (x, t_1, \cdots, t_N)$, which describes, for instance, the amount of public project to be provided, x, and the profile of transfers to each individual (which allows for some of them to be positive while other can be negative).

Mechanism Designer: The mechanism designer has the objective of achieving an outcome that depends on the types of players. For instance, the seller in an auction seeks to maximize his revenue without being able to observe the valuations that each bidder has for the good; or a government official considering the construction of a bridge would like to maximize a social welfare

[2]In auction settings, $x \in X$ represents the assignment of the object for sale, thus becoming a vector $x = (0, ..., 0, 1, 0, ..., 0)$ where 0 indicates that individual $1, .., i-1$ did not receive the object for sale, as so did individuals $i+1, ..., N$; while a 1 indicates that individual i received the object. For this reason, in auctions x is referred to as an assignment or allocation of the object.

function without observing the preferences of his constituents for that bridge. Hence, most of our subsequent discussion deals with the incentives that mechanism designers can provide to privately informed agents (e.g., bidders or citizens in the above two examples) so they voluntarily reveal their private information.

We assume that the mechanism designer does not have a source of funds to pay the players. That is, the monetary payments have to be self-financed, implying that $\sum_{i=1}^{n} t_i \leq 0$. When this condition holds strictly, $\sum_{i=1}^{n} t_i < 0$, the mechanism designer keeps some of the money that he raises from players; while if, instead, the condition holds with equality, $\sum\limits_{i=1}^{n} t_i = 0$, all negative transfers collected from some players end up distributed to other players, that is, the budget is balanced.

Since, as defined above, an outcome is represented as a vector $y = (x, t_1, \cdots, t_N)$, the set of outcomes is

$$Y = \left\{ (x, t_1, \cdots, t_N) : x \in X, t_i \in \mathbb{R} \text{ for all } i \in N, \sum_{i=1}^{n} t_i \leq 0 \right\}.$$

In words, an outcome is an alternative $x \in X$ and a transfer profile $(t_1, t_2..., t_n)$ such that transfers satisfy $\sum_{i=1}^{n} t_i \leq 0$. Finally, the mechanism designer's objective is given by a choice rule

$$f(\theta) = (x(\theta), t_1(\theta), \cdots, t_N(\theta)),$$

That is, for every profile of players' preferences $\theta \in \Theta$, the choice rule $f(\theta)$ selects an alternative $x(\theta) \in X$ and a transfer profile $(t_1(\theta), t_2(\theta), ..., t_n(\theta))$ satisfying $\sum_{i=1}^{n} t_i \leq 0$.

2.1 The Mechanism Game

Indirect revelation mechanism. *Definition.* An indirect revelation mechanism (IRM)

$$\Gamma = \{S_1, S_2, \ldots, S_n, g(\cdot)\}$$

is a collection of action sets $S_1, S_2, \ldots, S_n$, one for each player, and an outcome function $g : S_1 \times S_2 \times \cdots \times S_n \to Y$ mapping the actions chosen by the players into an outcome of the game.

In this context, a pure strategy for player i in the mechanism Γ is a function that maps his type $\theta_i \in \Theta_i$ into an action $s_i \in S_i$, that is, $s_i : \Theta_i \to S_i$. The payoffs of the players are then given by $v_i(g(s), \theta_i)$, which depend on the outcome that emerges from the game $g(s)$ when the action profile is s, and on player $i's$ type θ_i (e.g., his preferences for a public project).

Since the mechanism first maps players' types into their actions, and then their actions into a specific outcome, this type of mechanism is often referred to as "indirect revelation mechanism"; as depicted in figure 11.1a.

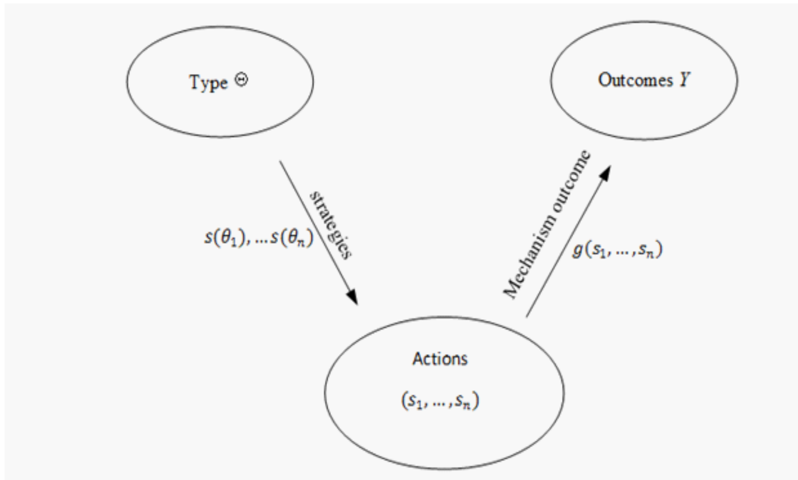

Figure 11.1(a). Indirect revelation mechanism.

In a special class of mechanisms we define next, each player i's strategy space S_i is restricted to coincide with his set of types, i.e., $S_i = \Theta_i$.

Direct revelation mechanism. *Definition.* A direct revelation mechanism (DRM) consists of $\Theta = (\Theta_1, \Theta_2, \cdots, \Theta_N)$ and a social choice function $f(\cdot)$ mapping every profile of types $\theta \in \Theta$, where $\theta = (\theta_1, \theta_2, \cdots, \theta_N)$, into an outcome $x \in X$,

$$f : \Theta \to X$$

As mentioned above, DRMs can be understood as a special class of mechanisms, in which each player i's strategy space S_i is restricted to coincide with his set of types, i.e., $S_i = \Theta_i$. In contrast, IRMs require that, first, every player i chooses a strategy $s_i \in S_i$, such as a bid or a production level, and then all players' strategies are mapped into an outcome. Figure 11.1b below depicts a DRM, which could be understood as directly connecting the two unconnected balloons in the upper part of figure 11.1(a) rather than doing the "de-tour" of first mapping strategies into actions, and then actions into outcomes.

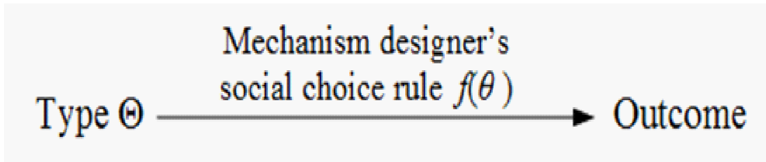

Figure 11.1(b). Direct revelation mechanism

2.2 Examples of DRMs

The following examples explore a setting where a seller (agent 0) seeks to sell an indivisible object to one of two buyers (agents 1 and 2) so that the set of players is $N = \{0, 1, 2\}$. The set of feasible outcomes is

$$X = \{(y_0, y_1, y_2, t_0, t_1, t_2) : y_i \in \{0, 1\} \text{ where } \sum_{i=0}^{2} y_i = 1 \text{ and } t_i \in \mathbb{R}\ \forall i \in N\};$$

In words, the object is assigned to either the seller, $y_0 = 1$, buyer 1, $y_1 = 1$, or buyer 2, $y_2 = 1$; and a transfer t_i is provided to player i, if $t_i > 0$, or a tax is imposed on him, if $t_i < 0$.[3] The utility that buyer $i = \{1, 2\}$ obtains from outcome x in the above set of feasible outcomes, i.e., $x \in X$, is

$$u_i(x_i, \theta_i) = \theta_i y_i + t_i$$

where θ_i represents the buyer's valuation for the object (which buyer i only enjoys if the object is assigned to him, i.e., $y_i = 1$), and t_i is the positive (or negative) transfer he receives (or pays).

Example 11.1 - Direct revelation mechanism. Consider a setting in which the seller asks buyers 1 and 2 to simultaneously and independently reveal their types (their valuation for the object), $\hat{\theta}_1$ and $\hat{\theta}_2$, and the seller assigns the object to the agent with the highest revealed valuation $\hat{\theta}_i$. Without loss of generality, we assume that if there is a tie, the object is assigned to buyer 1. More formally, for every profile of announced types, $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$, the assignment rule of this direct revelation mechanism is

$$y_0(\hat{\theta}) = 0$$

[3] At this point, we do not require the mechanism to be budget balanced, which would imply that positive and negative transfers offset each other at the aggregate level, $\sum_{i=1}^{3} t_i = 0$. We return to the budget balance property in further sections.

implying that the seller never keeps the object, and

$$y_i(\hat{\theta}) = \begin{cases} 1 & \text{if } \hat{\theta}_i \geq \hat{\theta}_j \\ 0 & \text{otherwise} \end{cases}$$

for every player $i \neq j$ where $i = \{1, 2\}$; while the transfer (or payment) rule is

$$t_i(\hat{\theta}) = -\hat{\theta}_i \cdot y_i(\hat{\theta}) \text{ where } i = \{1, 2\}$$

and

$$t_0(\hat{\theta}) = -[t_1(\hat{\theta}) + t_2(\hat{\theta})] = \hat{\theta}_1 \cdot y_1(\hat{\theta}) + \hat{\theta}_2 \cdot y_2(\hat{\theta})$$

In words, if player i reports a larger valuation than his rival, $\hat{\theta}_i \geq \hat{\theta}_j$, he is assigned the object, $y_i(\hat{\theta}) = 1$, paying a transfer equal to his reported valuation $\hat{\theta}_i$, i.e., $t_i(\hat{\theta}) = -\hat{\theta}_i \cdot 1 = -\hat{\theta}_i$. In contrast, his rival j does not receive the object, $y_j(\hat{\theta}) = 0$, thus entailing a zero transfer $t_j(\hat{\theta}) = 0$. Finally, the seller receives the sum of the transfers, which in this setting is equivalent to the transfer paid by the individual i who receives the object, that is, $t_0(\hat{\theta}) = -t_i(\hat{\theta}) = \hat{\theta}_i$. ■

Example 11.2 - Direct revelation mechanism (variation of Example 11.1). Buyer 1 and 2 report $\hat{\theta}_1$ and $\hat{\theta}_2$ to the seller, the seller assigns the object to the buyer with the highest announced report $\hat{\theta}_i$ (that is, we use the same allocation rule $y_i(\hat{\theta})$ for $i = \{0, 1, 2\}$ as in the previous example), but the payment rule now becomes

$$t_i(\hat{\theta}) = -\hat{\theta}_j \cdot y_i(\hat{\theta})$$

and

$$t_0(\hat{\theta}) = -[t_1(\hat{\theta}) + t_2(\hat{\theta})]$$

Intuitively, if player i reports a larger valuation than his rival, $\hat{\theta}_i \geq \hat{\theta}_j$, he is assigned the object, $y_i(\theta) = 1$, but pays the second highest reported valuation, $\hat{\theta}_j$. A similar argument extends to settings with N players, where $t_i(\hat{\theta}) = -\max_{j \neq i}\{\hat{\theta}_j\} \cdot y_i(\theta)$, i.e., player i, if he is assigned the object, pays a price equal to the highest competing reported valuation. ■

Example 11.3 - Procurement contract. Consider a seller (agent 0) and buyers 1 and 2, with the set of outcomes X being the same as that in all previous examples, and the same utility function. However, the assignment rule is now reversed, as the seller seeks to assign the service (e.g., public water management) to the firm reporting the lowest cost. That is, the assignment rule specifies

$$y_0(\hat{\theta}) = 0$$

implying that the seller never keeps the object, and

$$y_i(\hat{\theta}) = \begin{cases} 1 & \text{if } \hat{\theta}_i \leq \hat{\theta}_j \\ 0 & \text{otherwise} \end{cases} \quad \text{for every } i = \{1, 2\}$$

That is, the procurement contract is assigned to the firm announcing the lowest cost, $\hat{\theta}_i \leq \hat{\theta}_j$. Finally, the transfer rule coincides with that in Example 11.1 (if the winning agent is paid his costs) or with that in Example 11.2 (if the winning agent is paid the cost of the losing firm). ■

Example 11.4 - Funding a public project. A set of individuals $N = \{1, 2, \cdots, n\}$ seek to build a bridge. Let $k = 1$ indicate that the bridge is built, and $k = 0$ that it is not. The cost of the project is $C > 0$. Let t_i be a transfer to agent i, so $-t_i$ is a tax paid by agent i. The project is then built, $k = 1$, if total tax collection exceeds the bridge's total cost $C \leq -\sum_{i=1}^{n} t_i$, but it is not build otherwise. Alternatively, we can express this condition by writing $kC \leq -\sum_{i=1}^{n} t_i$ captures both the case in which the bridge is built and the case it is not, where k operates here as an "indicator function," being activated when $k = 1$ or inactive when $k = 0$.

The set of outcomes, X, in this setting is then

$$X = \left\{ (k, t_1, t_2, \cdots, t_n) : k \in \{0, 1\}, t_i \in \mathbb{R}, \text{and } kC \leq -\sum_{i=1}^{n} t_i \text{ where } i \in N \right\}$$

As in the outcomes sets considered in previous examples, X specifies the assignment rule k followed by transfer rule to each agent $i \in N$ (which can be taxes since $t_i \in \mathbb{R}$ is not restricted to be positive). Utility function for every agent i is

$$u_i(k, t_i, \theta_i) = k\theta_i + t_i$$

where θ_i can be interpreted as agent i's valuation of the project. Note that agent i only enjoys such a valuation if the bridge is built, $k = 1$, and that we allow for agent i to pay taxes if $t_i < 0$. ■

Example 11.5 - Direct revelation mechanism in the public project. In this case, the mechanism asks agents to directly report their types (i.e., their private valuation for the bridge). In other words, the game restricts every player i's strategy set to coincide with his set of types, $S_i = \Theta_i$. In this setting, the social choice function maps the reported (announced) profile of types $\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_n)$ into an assignment rule and a transfer rule. In particular, the assignment rule specifies

$$k(\hat{\theta}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \hat{\theta}_i \geq C \\ 0 & \text{otherwise} \end{cases}$$

i.e., the project is built if and only if the aggregate reported valuation of all agents exceeds the project's cost. In addition, the transfer rule of this mechanism is

$$t_i(\hat{\theta}) = -\frac{C}{n} k(\hat{\theta}).$$

In words, if the project is built, $k(\hat{\theta}) = 1$, every agent i bears an equal share of its cost, $\frac{C}{n}$; but if the project is not built $k(\hat{\theta}) = 0$, no agent has to pay anything, i.e., $t_i(\hat{\theta}) = 0$ for every agent i. ■

3 Implementation

3.1 Testing the implementability of SCF in direct revelation mechanisms

Let us test the implementability of the social choice function (SCF) described in Example 11.1 above. Suppose θ_1, $\theta_2 \sim U[0,1]$ and i.i.d. In order to test if truthfully reporting his type $\theta_1 = \hat{\theta}_1$, is a weakly dominant strategy for player 1, let's assume that player 2 truthfully reports his type, so his equilibrium strategy is $\hat{\theta}_2 \equiv s_2^*(\theta_2) = \theta_2$ and check for profitable deviations for player 1. (Recall that this is the standard approach to test whether a strategy profile is an equilibrium, where we fix the strategies of all $N-1$ players and check if the remaining player has incentives to deviate from the proposed equilibrium strategy.)

In particular, player 1 solves

$$\max_{\hat{\theta}_1} \; (\theta_1 - p) \cdot \text{prob}\{win\} = (\theta_1 - \hat{\theta}_1) \cdot \text{prob}\{\theta_2 \leq \hat{\theta}_1\}$$

where $\theta_1 - \hat{\theta}_1$ represents the margin that player 1 keeps by under-reporting his valuation of the object (which helps him obtain the good at a lower price), while $\text{prob}\{\theta_2 \leq \hat{\theta}_1\}$ denotes the probability that player 1 wins the object because he reveals a larger valuation than player 2 to the seller.

Since $\theta_2 \sim U[0,1]$, $\text{prob}\{\theta_2 \leq \hat{\theta}_1\}$ is $F(\hat{\theta}_1) = \hat{\theta}_1$, which reduces player 1's problem to

$$\max_{\hat{\theta}_1} \; (\theta_1 - \hat{\theta}_1) \cdot \hat{\theta}_1 = \theta_1 \hat{\theta}_1 - \hat{\theta}_1^2$$

Differentiating with respect to $\hat{\theta}_1$ yields $\theta_1 - 2\hat{\theta}_1 = 0$. Solving for $\hat{\theta}$, we obtain an optimal announcement of

$$\hat{\theta}_1 = \frac{\theta_1}{2}$$

(An analogous argument applies to player 2: if player 1 truthfully reports his type, $\hat{\theta}_1 = \theta_1$, then player 2's optimal report is $\hat{\theta}_2 = \frac{\theta_2}{2}$.) Hence, the SCF in Example 11.1 is not implementable as a DRM since it doesn't induce every player to truthfully report his type to the seller.

3.2 Incentive Compatibility

In our above discussion, player 1 shades his valuation in half, not truthfully reporting his type to the seller, so $\hat{\theta}_1 \equiv s_1^*(\theta_1) \neq \theta_1$. As suggested by Example 11.1, players may not have incentives to truthfully report their types in DRMs. This is, however, a desirable property that the mechanism designer will try to guarantee to extract information from agents. When a SCF induces privately informed players to truthfully report their types in equilibrium, we refer to such SCF as "Incentive Compatible." We can, nonetheless, consider two types of incentive compatibilities depending on whether truthtelling is a Bayesian Nash Equilibrium (BNE) of the incomplete information game, or a dominant strategy equilibrium. We separately study each case below. In both of them, consider a

DRM $D = ((\Theta_i)_{i \in N}, f(\cdot))$ where every player i submits a report $\hat{\theta}_i$ to the mechanism designer (e.g., social planner) and a SCF $f(\cdot)$ maps the profile of reported types $\hat{\theta} \equiv \left(\hat{\theta}_1, ..., \hat{\theta}_n\right)$ to an outcome $x \in X$, that is, $f(\hat{\theta}) = x$.

Bayesian Incentive Compatibility. *Definition***:** A SCF $f(\cdot)$ is Bayesian Incentive Compatible (BIC) if the DRM has a BNE where every player i's strategy is to thruthfully report his type, $s_i^*(\theta_i) = \theta_i$, for every type $\theta_i \in \Theta_i$.

That is, every player i finds truthtelling optimal, given his beliefs about his opponents' types, and given that all his opponents' strategies are truthtelling, $s_{-i}^*(\theta_{-i}) = \theta_{-i}$. More formally, BIC entails that for every player $i \in N$ and every type $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}\left[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i\right] \geq E_{\theta_{-i}}\left[u_i(f(\theta_i', \theta_{-i}), \theta_i)|\theta_i\right]$$

for every misreport $\theta_i' \neq \theta_i$. This inequality says that player i prefers to truthfully report his type θ_i, yielding an outcome $f(\theta_i, \theta_{-i})$ than misreporting his type to be $\theta_i' \neq \theta_i$, which would yield an outcome $f(\theta_i', \theta_{-i})$. Importantly, player i prefers to truthfully reveal his type θ_i in expectation, as he doesn't observe his rivals' types $\theta_{-i} \in \Theta_{-i}$. As a consequence, the above definition could allow player i to find truthtelling optimal for some values of his rivals' types θ_{-i}, but not for others as long as, in expectation, he prefers to truthfully report his type θ_i.

The following version of incentive compatibility is more demanding since it requires player i to find truthtelling optimal *regardless* of the specific realization of his rivals' types θ_{-i}, and *regardless* of his rivals' announcements. That is, the SCFs makes truthtelling a dominant strategy for every player $i \in N$.

Dominant Strategy Incentive Compatibility, *Definition***:** A SCF $f(\cdot)$ is Dominant Strategy Incentive Compatible (DSIC) if the DRM has a dominant strategy equilibrium where every player i's strategy is to thruthfully report his type, $s_i^*(\theta_i) = \theta_i$, for every type $\theta_i \in \Theta_i$.

Therefore, every player i finds truthtelling optimal regardless of his beliefs about his opponents' types, and independently on his opponents' strategies in equilibrium, i.e., both when they truthfully report their types, $s_{-i}^*(\theta_{-i}) = \theta_{-i}$, and when they do not, $s_{-i}^*(\theta_{-i}) \neq \theta_{-i}$. More formally, DSIC entails that for every player $i \in N$ and every type he may have $\theta_i \in \Theta_i$,

$$u_i(f(\theta_i, s_{-i}), \theta_i) \geq u_i(f(\theta_i', s_{-i}), \theta_i)$$

for every misreport $\theta_i' \neq \theta_i$, where $s_{-i} \in S_{-i}$. Then, DSIC is a more demanding property than BIC since DSIC requires that players find truthtelling optimal regardless of the specific types of their opponents and independently on their specific actions in equilibrium. In contrast, BIC asks that truthtelling is utility maximizing only in expectation and given that all other players are truthfully reporting their own types.

4 Implementation and the revelation principle

An indirect revelation mechanism (IRM) allows strategy spaces to differ from a direct announcement of types, i.e., $S_i \neq \Theta_i$, or to coincide, $S_i = \Theta_i$, for every player i. A DRM can then be interpreted as a special case of IRM where players' strategies are restricted to coincide with their type space, i.e., when $S_i = \Theta_i$ we only allow players to report a type (either truthfully or misreporting) but they cannot do anything else. In contrast, in an IRM players can potentially choose from a richer strategy space, such as submitting a bid. Once every player i chooses his strategy s_i, and a profile of strategies emerges $s = (s_1, s_2, ..., s_n)$, the IRM maps such strategy profile s into an outcome $g(s) = x$. The equilibrium that arises in the IRM has every player i choosing a strategy as a function of his privately observed type, $s_i^*(\theta_i)$, which yields equilibrium strategy profile $s^*(\theta) = (s_1^*(\theta_1), \cdots, s_n^*(\theta_n))$. This strategy profile entails an equilibrium outcome $g(s^*(\theta))$.

A natural question is whether the equilibrium outcome $g(s^*(\theta))$ emerging from the IRM, where every player chooses an action, ultimately giving rise to an outcome in the game, coincides with the outcome that the SCF selects in the DRM. This coincidence in outcomes, where the rules of the game in the IRM produce the same equilibrium outcome as the SCF in the DRM, is referred to as "implementability" since the IRM implements the same equilibrium outcome as the DRM. For completeness, we next explore two versions of implementability, first in dominant strategies and then in BNE. In both cases, consider a mechanism $M = ((S_i)_{i \in N}, g(\cdot))$, which we can interpret as an IRM, or more informally as a game, allowing player i to choose his strategy from strategy space S_i, and mapping every strategy profile s into an outcome of the game, $g(s)$.

4.1 Implementation in Dominant Strategies

A mechanism M implements the SCF $f(\cdot)$ in dominant strategy equilibrium if there is a weakly dominant strategy profile $s^*(\theta) = (s_i^*(\theta_1), \cdots, s_n^*(\theta_n))$ of the Bayesian game induced by mechanism M such that

$$g\left(s^*(\theta)\right) = f(\theta) \text{ for all } \theta \in \Theta$$

Example 11.6. Implementation in dominant strategies. Second-price auctions implement the SCF in Example 11.2 in weakly dominant strategy equilibrium. In particular, the strategy set for every bidder i is his set of feasible bids, which in the case of positive bids without a reservation price simplifies to $S_i = \mathbb{R}_+$.[4] In this context, we showed that every bidder i finds that a bid of $s_i(\theta_i) = \theta_i$ (bids coinciding with his valuation) constitutes a weakly dominant strategy in the second-price auction, i.e., he would choose it regardless of his opponents' valuations for the object and independently of their bidding profile s_{-i}. Hence, the object is assigned to the bidder submitting the highest bid, who pays a price equal to the second highest bid. Therefore, the equilibrium

[4] When a reservation price $r > 0$ exists, every player i's strategy profile is restricted to $S_i = [r, +\infty)$ since player i can only submit bids weakly above the reservation price r.

outcome emerging in the SPA coincides with that arising from the SCF in Example 11.2 where the social planner aksed each bidder i to report his valuation for the object, $\widehat{\theta}_i$. ■

4.2 Implementation in BNE

We say that a mechanism M implements the SCF $f(\cdot)$ in BNEs if there is a BNE strategy profile $s^*(\theta) = (s_i^*(\theta_1), \cdots, s_n^*(\theta_n))$ of the Bayesian game induced by mechanism M such that

$$g\left(s^*(\theta)\right) = f(\theta) \text{ for all } \theta \in \Theta$$

Example 11.7. Implementation in BNE. Recall that the SCF of Example 11.3 is BIC, meaning that, for every profile of types θ, every player i thruthfully reports his type in the BNE of the game, $s_i^*(\theta_i) = \theta_i$. In addition, we can use a first-price auction as an IRM which produces the same outcomes as the SCF of Example 11.3, namely, the bidder with the highest valuation for the object wins the auction, paying for it his submitted bid. Therefore, we can say that a first-price auction "implements" the SCF of Example 11.3 in BNE. ■

The above discussion suggests a connection between the outcomes of a DRM that induces truthtelling and an IRM. In particular, we might wonder if, for a given SCF, which maps profiles of types into socially desirable outcomes, we can design a clever game (a IRM) in which equilibrium play would yield the same outcome as that identified by the SCF. The answer is positive (although we discuss some disadvantages later), and it is known in the literature as the "Revelation Principle." The next sections separately present it for the cases of BNE and dominant strategies.

Figure 11.2 depicts the revelation principle by combining figures 11.1a and 11.1b. The upper part of the figure illustrates a DRM, thus mapping types into outcomes through a social choice function. The lower part, in contrast, takes an "indirect route" by first allowing every player to map his own type into a strategy, i.e., $s_i(\theta_i)$ for every player i, and then mapping the action profile into an outcome of the game. Informally, the Revelation Principle asks whether we can find game rules that provide players with the incentives to choose strategies that ultimately lead to outcomes coinciding with those selected by a social choice function.

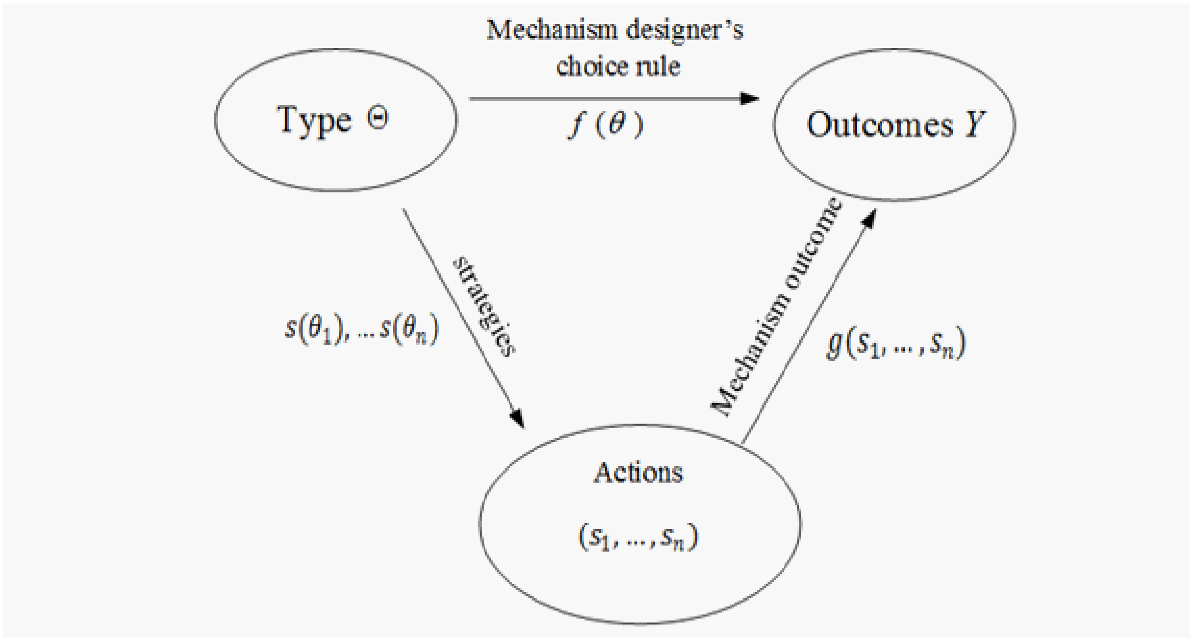

Figure 11.2. The Revelation Principle

4.3 Revelation Principle - I: BNE Approach

A mechanism M implements SCF $f(\cdot)$ in BNE if and only if $f(\cdot)$ is BIC.

Proof: Since the "if and only if" clause means that: (1) Mechanism M implements $f(\cdot)$ in BNE $\Rightarrow f(\cdot)$ is BIC; and (2) $f(\cdot)$ is BIC $\Rightarrow$ mechanism M implements $f(\cdot)$ in BNE, we next show both lines of implication.

($\Leftarrow$) If $f(\cdot)$ is BIC, then it can also be implementable in BNE by the DRM in which we restrict every player i's strategy set to coincide with his set of types, $S_i = \Theta_i$.

($\Rightarrow$) If mechanism M implements $f(\cdot)$ in BNE, then there is a BNE of the IRM $(s_1^*(\theta_1), \cdots, s_n^*(\theta_n))$ such that

$$g\left(s_1^*(\theta_1), \cdots, s_n^*(\theta_n)\right) = f(\theta_1, \cdots, \theta_n) \text{ for all } \theta.$$

Since strategy profile $(s_1^*(\theta_1), \cdots, s_n^*(\theta_n))$ is a BNE, then

$$E_{\theta_{-i}}\left[u_i\left(g\left(s_i^*(\theta_i), s_{-i}^*(\theta_i)\right), \theta_i\right)|\theta_i\right] \geq E_{\theta_{-i}}\left[u_i\left(g\left(s_i, s_{-i}^*(\theta_{-i})\right), \theta_i\right)|\theta_i\right]$$

for all $s_i \in S_i$, all $\theta_i \in \Theta_i$, and every player $i \in N$. Note that a deviating strategy s_i on the right-hand side of the inequality could be $s_i^*(\theta_i')$ so player i uses the same strategy function as in the left-hand side but evaluating it at a misreported type $\theta_i' \neq \theta_i$.

Combining the above two results yields

$$E_{\theta_{-i}}\left[u_i\left(f(\theta_i,\theta_{-i}),\theta_i\right)|\theta_i\right] \geq E_{\theta_{-i}}\left[u_i\left(f(\theta_i',\theta_{-i}),\theta_i\right)|\theta_i\right]$$

for all $\theta_i' \neq \theta_i$, all types $\theta_i \in \Theta_i$, and every player $i \in N$, which is exactly the condition that we need for SCF $f(\cdot)$ to be BIC. (Q.E.D.)

In words, player i prefers to thruthfully report his type when his opponents thruthfully report their own types.

4.4 Revelation Principle - II: DSIC Approach

A mechanism M implements SCF $f(\cdot)$ in dominant strategy equilibrium if and only if $f(\cdot)$ is DSIC.

Proof: In this case we also need to show both directions of the "if and only if" clause.

The first line of implication, $(\Leftarrow)$, is analogous to the first step of the above proof.

The second line of implication, $(\Rightarrow)$, is similar to the previous proof, but now every player i does not take expectations of his opponents' types, nor we need to fix his opponents' strategies in equilibrium, $s^*_{-i}(\theta_{-i})$, where they truthfully reveal their types. Instead, player i considers *any* strategy of his opponents, $s_{-i}(\theta_{-i})$.

As a practice, let us develop the premise of the above statement. If mechanism M implements $f(\cdot)$ in dominant strategy equilibrium (DSE), there exists a weakly dominant BNE, $(s_1^*(\theta_1),\cdots,s_n^*(\theta_n))$ such that

$$g\left(s_1^*(\theta_1),\cdots,s_n^*(\theta_n)\right) = f(\theta_1,\cdots,\theta_n) \text{ for all } \theta.$$

Since strategy profile $(s_1^*(\theta_1),\cdots,s_n^*(\theta_n))$ is a dominant strategy equilibrium of mechanism M, it must provide every player i with a weakly higher payoff than any other available strategy $s_i \neq s_i^*(\theta_i)$. That is,

$$u_i\left(g\left(s_1^*(\theta_1),s_{-i}(\theta_{-i})\right)\right) \geq u_i\left(g\left(s_i,s_{-i}(\theta_{-i})\right),\theta_i\right)$$

for all $\theta_i \in \Theta_i$, all $\theta_{-i} \in \Theta_{-i}$, all $s_{-i} \in S_{-i}$, and all $i \in N$. In words, player i does not have incentives to deviate, i.e., of choosing a strategy $s_i \neq s_i^*(\theta_i)$, for any type θ_i he may have, any profile of types his opponents may have, θ_{-i}, and for any strategy profile they may choose, $s_{-i}(\theta_{-i})$.[5]

Combining the above conditions yields

$$u_i\left(f(\theta_i,\theta_{-i}),\theta_i\right) \geq u_i\left(f(\theta_i',\theta_{-i}),\theta_i\right)$$

[5] As an example, the deviating strategy s_i on the right-hand side of the inequality could be $s_i^*(\theta_i')$ whereby player i uses the same function as in the left-hand side, $s_i^*(\theta_i)$, but evaluated at a misreported type $\theta_i' \neq \theta_i$. However, it can generally be any deviating strategy $s_i \neq s_i^*(\theta_i)$. Alternatively, player i could deviate from $s_i^*(\theta_i)$ by mapping his true type θ_i into a different strategy function, such that $s_i'(\theta_i) \neq s_i^*(\theta_i)$ for any type θ_i.

which exactly coincides with the condition that we need for the SCF to be DSIC. (Q.E.D.)

In summary, the revelation principle in its two versions tells use that

$$\text{A mechanism } M \text{ implements } f(\cdot) \text{ in BNE} \Longleftrightarrow f(\cdot) \text{ is BIC}$$

$$\text{A mechanism } M \text{ implements } f(\cdot) \text{ in DSE} \Longleftrightarrow f(\cdot) \text{ is DSIC}$$

where DSE denotes "dominant strategy equilibrium." Hence, if a mechanism is not BIC or DSIC (i.e., telling the truth is not an equilibrium in the DRM), then we cannot find a clever game or institutional setting (an IRM) that implements such a SCF $f(\cdot)$ in the BNE of the game (or DSE, respectively). Alternatively, if a mechanism M is BIC, we can find an IRM that implements $f(\cdot)$ in BNE. Similarly, if a mechanism is DSIC, we can find an IRM that implements $f(\cdot)$ in DSE.

5 Efficiency

In most of the sections hereafter we consider the following quasilinear preferences

$$v_i(k, \theta_i) = u_i(k, \theta_i) + w_i + t_i$$

where $k \in K$ describes, as usual, the assignment rule. For instance, $k = \{0, 1\}$ can represent that a public project is implemented, $k = 1$, or not, $k = 0$. In auctions, $k_i = \{0, 1\}$ indicates whether player $i = \{0, 1, 2, ..., N\}$ receives the object being sold if $k_i = 1$, or not, if $k_i = 0$. Types are denoted by $\theta_i \in \Theta_i$; wealth is strictly positive, $w_i > 0$; and $t_i > 0$ denotes that player i receives a net transfer while $t_i < 0$ indicates that he pays to the system.

5.1 Allocative efficiency

We say that a SCF $f(\theta) = (k(\theta), t_1(\theta), \cdots, t_n(\theta))$ satisfies *allocative efficiency* if, for every profile of types $\theta \in \Theta$, the allocation function $k(\theta)$ satisfies

$$k(\theta) \in \underset{k \in K}{\arg\max} \sum_{i \in N} u_i(k, \theta_i)$$

That is, $k(\theta)$ allocates objects in an auction (or public projects in procurement) to maximize aggregate payoffs for every profile of types, θ.[6] The following examples test whether the allocation function $k(\theta)$ in two different SCF satisfies allocative efficiency.

[6] Allocative efficiency is then analogous to Pareto efficiency. However, since most mechanism design problems deal with the allocation of property rights (e.g., auctions and procurement contracts) and the implementation of public projects, we normally use the concept of allocative efficiency.

Example 11.8 - Public project with an allocative efficient SCF. Consider a setting with two agents, each with two types $\Theta_i = \{20, 60\}$ for every player $i = \{1, 2\}$. Every agent's utility function is

$$u_i(k, \theta_i) = k \times (\theta_i + t_i)$$

which indicates that if the project is not implemented, $k = 0$, agents' utilities are zero; but if it is implemented, $k = 1$, every agent i bears a cost $t_i < 0$ (that is, a tax) which we discuss below. Consider the following allocation function

$$k(\theta_1, \theta_2) = \begin{cases} 0 & \text{if } \theta_1 = \theta_2 = 20 \\ 1 & \text{otherwise} \end{cases}$$

indicating that, if both individuals' valuations are low (20), the project is not implemented; but if the valuation of at least one individual is high (60), the project is implemented. In this context, consider the transfer function

$$t_i(\theta_1, \theta_2) = \begin{cases} 0 & \text{if } \theta_1 = \theta_2 = 20 \\ -25 & \text{otherwise} \end{cases}$$

which essentially says that, if the project is implemented, every player i bears the same cost (a tax of \$25); but otherwise every player pays zero.

Table 11.1 considers all type profiles (θ_1, θ_2) (one per row), and the utilities that agents enjoy from the above allocation function $k(\theta)$ and transfer function $t_i(\theta_1, \theta_2)$.

(θ_1, θ_2)	$k(\theta)$	$u_1(0, \theta_1)$	$u_2(0, \theta_2)$	$u_1(1, \theta_1)$	$u_2(1, \theta_2)$	$u_1(1, \theta_1) + u_2(1, \theta_2)$
$(20, 20)$	0	0	0	$20 - 25 = -5$	$20 - 25 = -5$	-10
$(20, 60)$	1	0	0	$20 - 25 = -5$	$60 - 25 = 35$	30
$(60, 20)$	1	0	0	$60 - 25 = 35$	$20 - 25 = -5$	30
$(60, 60)$	1	0	0	$60 - 25 = 35$	$60 - 25 = 35$	70

Table 11.1. Allocations and utility levels for each profile of types.

Hence, the SCF is allocative efficient. To see this, note that for profile of types $(\theta_1, \theta_2) = (20, 20)$ (in the first row), the total utility of implementing the public project is negative and thus lower than that of not implementing it (which is zero). The allocation function $k(\theta)$ correctly selects $k(\theta) = 0$ since in this case not implementing the public project is welfare maximizing. In contrast, for all remaining type profiles (rows 2 to 4), the total welfare from implementing the project is positive, and thus larger than from not implementing it. In all of these type profiles, the allocation function selects $k(\theta) = 1$, thus implementing the project. ■

Example 11.9 - Public project with an allocative inefficient SCF. Consider the same quasilinear preference as in Example 11.8, but assume now that the project is costless, i.e., $t_i(\theta) = 0$

for all profiles of types $\theta \in \Theta$ and every player $i \in N$. Given such a change in the transfer function, the SCF is no longer allocative efficient. For the SCF to be allocative efficient, it should implement the project, $k(\theta) = 1$, regardless of the type profile θ. For instance, when $(\theta_1, \theta_2) = (20, 20)$, the allocation function determines that $k(\theta) = 0$, which yields $\sum_{i \in N} u_i(0, \theta) = 0 + 0$. However, implementing it, $k(\theta) = 1$, would yield in this case a total welfare of $\sum_{i \in N} u_i(1, \theta) = 20 + 20 = 40$. Intuitively, even if both agent assign a low value to the project, the aggregate value they obtain is still positive (40), which exceeds its development cost (zero in this case). Since the allocation function $k(\theta)$ described above does not implement the project when both individuals' valuations are low, i.e., when $(\theta_1, \theta_2) = (20, 20)$, we can conclude that allocation function $k(\theta)$, and thus the SCF, are not allocative efficient. ■

6 Examples of common mechanisms

We next present some mechanisms extensively used in the theoretical and applied literature. In particular, we are interested in showing that the SCF they implement satisfies allocative efficiency, i.e., we cannot find alternative outcomes that could increase social surplus; and DSIC, i.e., agents find it optimal to truthfully reveal their private information θ_i to the mechanism designer independently on what their rivals do.

6.1 Groves mechanism

Let $f(\theta) = (k(\theta), t_1(\theta), \cdots, t_n(\theta))$ be a SCF that satisfies allocative efficiency. Then $f(\cdot)$ satisfies DSIC if transfer functions can be represented by

$$t_i(\theta_i, \theta_{-i}) = \sum_{j \neq i} u_j \left(k(\theta), \theta_j\right) + h_i(\theta_{-i})$$

where $h_i : \Theta \to \mathbb{R}$ is an arbitrary function.

Intuitively, the transfer that player i receives depends on the utility that all other agents $j \neq i$ experience from the complete profile of announced types, θ, indicating the externality that player i's announcement causes on their well-being, plus a function $h_i(\theta_{-i})$ which is independent on player i's announcement. To see this point, note that, if player i changes his report from θ_i to θ_i', the allocation changes from $k(\theta_i, \theta_{-i})$ to $k(\theta_i', \theta_{-i})$, changing his transfer by exactly the utility change that he imposes on other agents, that is,

$$\begin{aligned} t_i(\theta_i, \theta_{-i}) - t_i(\theta_i', \theta_{-i}) &= \left[\sum_{j\neq i} u_j\left(k(\theta_i, \theta_{-i}), \theta_j\right) + h_i(\theta_{-i})\right] - \left[\sum_{j\neq i} u_j\left(k(\theta_i', \theta_{-i}), \theta_j\right) + h_i(\theta_{-i})\right] \\ &= \sum_{j\neq i} u_j\left(k(\theta_i, \theta_{-i}), \theta_j\right) - \sum_{j\neq i} u_j\left(k(\theta_i', \theta_{-i}), \theta_j\right) \\ &= \sum_{j\neq i} \left[u_j\left(k(\theta_i, \theta_{-i}), \theta_j\right) - u_j\left(k(\theta_i', \theta_{-i}), \theta_j\right)\right] \end{aligned}$$

Let us now show that such a transfer function entails DSIC.

Proof: By contradiction. Suppose that a SCF $f(\cdot)$ satisfies allocative efficiency and its transfer function can be represented à la Groves as stated above, but it is *not* DSIC. That is, there is at least one agent i who finds misreporting to be profitable, that is,

$$u_i\left(f(\theta_i', \theta_{-i}), \theta_i\right) > u_i\left(f(\theta_i, \theta_{-i}), \theta_i\right)$$

in at least one of his types $\theta_i \in \Theta_i$, and for at least one profile of his rivals' types $\theta_{-i} \in \Theta_{-i}$, where $\theta_i' \neq \theta_i$. Given quasilinearity, we can expand this inequality yielding

$$u_i\left(k(\theta_i', \theta_{-i}), \theta_i\right) + t_i(\theta_i', \theta_{-i}) + w_i > u_i\left(k(\theta_i, \theta_{-i}), \theta_i\right) + t_i(\theta_i, \theta_{-i}) + w_i$$

We can now plug the transfer from the Groves theorem,

$$t_i(\theta_i', \theta_{-i}) = \sum_{j\neq i} u_j\left(k(\theta_i', \theta_{-i}), \theta_j\right) + h_i(\theta_{-i})$$

and similarly for $t_i(\theta_i, \theta_{-i})$. Hence, the above inequality becomes

$$u_i\left(k(\theta_i', \theta_{-i}), \theta_i\right) + \underbrace{\sum_{j\neq i} u_j\left(k(\theta_i', \theta_{-i}), \theta_j\right)}_{t_i(\theta_i', \theta_{-i})} > u_i\left(k(\theta_i, \theta_{-i}), \theta_i\right) + \underbrace{\sum_{j\neq i} u_j\left(k(\theta_i, \theta_{-i}), \theta_j\right)}_{t_i(\theta_i, \theta_{-i})}$$

which simplifies to

$$\sum_{i\in N} u_i\left(k(\theta_i', \theta_{-i}), \theta_i\right) > \sum_{i\in N} u_i\left(k(\theta_i, \theta_{-i}), \theta_i\right)$$

entailing that the SCF $f(\cdot)$ is not allocative efficient since it doesn't maximize total surplus, i.e., allocation $k(\theta_i', \theta_{-i})$ yields a larger social welfare. Hence, if SCF $f(\cdot)$ is AE and transfers can be expressed à la Groves, the SCF is DSIC. (Q.E.D.)

6.2 VCG mechanism

Vickrey-Clarke-Groves mechanism (VCG) constitutes a special class of Groves mechanisms described in the previous section, in which the function $h_i(\theta_{-i})$ takes the form

$$h_i(\theta_{-i}) = -\sum_{j\neq i} u_j\left(k_{-i}(\theta_{-i}), \theta_j\right) \quad \text{for all } \theta_{-i} \in \Theta_{-i}, \text{and for all } i \in N$$

where $k_{-i}(\theta_{-i})$ denotes the allocation that the SCF selects when considering all agents $j \neq i$, i.e., as if player i's preferences as captured by parameter θ_i were ignored when determining the allocation k.

Hence, the transfer becomes

$$\begin{aligned} t_i(\theta) &= \sum_{j\neq i} u_j\left(k(\theta), \theta_j\right) + h_i(\theta_{-i}) \\ &= \sum_{j\neq i} u_j\left(k(\theta), \theta_j\right) - \underbrace{\sum_{j\neq i} u_j\left(k_{-i}(\theta_{-i}), \theta_j\right)}_{\text{Clarke } h_i(\theta_{-i}) \text{ function}} \text{ for all } i \in N \end{aligned}$$

Intuitively, the first term represents the total value that all $j \neq i$ agents obtain when the seller (mechanism designer) considers player i's preferences, thus determining allocation $k(\theta)$. The second term, in contrast, describes the total value that they obtain when the seller ignores player i's preferences, so the allocation becomes $k_{-i}(\theta_{-i})$. Therefore, the difference between both terms captures the effect that considering player i's preferences has on the mechanism's allocation, and thus on all other agents' utility. In this sense, the VCG mechanism is pivotal, as the preferences of individual i's preferences play a pivotal role at finding the transfer he receives (pays) as a function of the positive (negative) externality that his preferences impose on all other players.

Example 11.10 - One example of a VCG mechanism. Consider 5 individuals participating in a DRM in which they are asked to reveal their valuations for a good, which are

$$v_1 = 20,\ v_2 = 15,\ v_3 = 12,\ v_4 = 10,\ v_5 = 6$$

And consider an allocation function $k(\theta)$ that assigns the object to the individual reporting the highest valuation. If VCG mechanism is used, player 1's transfer would be

$$\begin{aligned} t_1(\theta) &= \sum_{j\neq 1} u_j\left(k(\theta), \theta_j\right) - \sum_{j\neq 1} u_j\left(k_{-1}(\theta_{-1}), \theta_j\right) \\ &= 0 - 15 = -15 \end{aligned}$$

In the first term, the allocation rule considers the valuation of all the bidders. Then, the object would be assigned to bidder 1 (the individual with the highest valuation), entailing a value of 0+0+0+0=0 to the other $j \neq 1$ bidders. The second term, in contrast, ignores bidder 1's preferences (valuation), thus assigning the object to bidder 2 (as he is now the player with the highest valuation). Bidder 2's utility from receiving the good is 15, implying that the sum of valuations is now 15+0+0+0=15. The difference between the two terms yields a transfer of $t_1(\theta) = 0 - 15 = -15$, thus indicating that player 1 pays 15, i.e., the second largest valuation. Intuitively, the price that player 1 pays coincides with the negative externality that his presence generates on player 2, since the latter woudl receive the object should the former not participate in the mechanism.

A similar argument applies to all other players. However, since their valuations are lower than that of player 1, their transfers become

$$t_i(\theta) = (20 + 0 + 0 + 0) - (20 + 0 + 0 + 0) = 0 \text{ for every player } i \neq 1.$$

In words, the object is assigned to player 1 in these settings, thus yielding the same profile of utilities when individual $i \neq 1$ participates, $20 + 0 + 0 + 0$, and when individual $i \neq 1$ does not participate, $20 + 0 + 0 + 0$. In particular, player 1 obtains 20 as he receives the object, and all other individuals $i \neq 1$ do not receive the object. Intuitively, their decision to participate in the mechanism does not produce an externality on other players, thus yielding a nil transfer $t_i(\theta) = 0$ for all θ and all $i \neq 1$.

Importantly, the VCG mechanism (which is an example of DRM) leads to the same outcome (the object is allocated to the bidder with highest valuation) and transfer profile (the individual receiving the object pays a transfer equal to the valuation of the individual with the second highest valuation, while everyone else pays zero) as the SPA (which is an IRM). ■

Example 11.11 - Another example of VCG mechanisms. Consider the same players as in Example 11.10, with the same valuations for the good. However, allow for 3 identical items to be available in the auction. Each bidder wants only one item. In this context, the transfer to player 1 becomes

$$\begin{aligned} t_1(\theta) &= \sum_{j \neq 1} u_j \left(k(\theta), \theta_j\right) - \sum_{j \neq 1} u_j \left(k_{-1}(\theta_{-1}), \theta_j\right) \\ &= (15 + 12) - (15 + 12 + 10) = -10 \end{aligned}$$

When the valuation profiles of all players are taken into account in the allocation rule, $k(\theta)$, the three available items are assigned to the players with the highest valuation: player 1, 2 and 3. The first term, however, measures the utility that players $j \neq 1$ obtain from such an allocation, i.e., the valuations of players 2 and 3, (15 + 12). In the second term, we still measure the utility of players $j \neq 1$ but ignoring player 1's preferences. In this case, the three items go to the three remaining players with the highest valuations (players 2, 3 and 4) yielding a total utility of (15 + 12 + 10). As a result, the transfer that player 1 has to pay is $-\$10$, indicating that, if his preferences were considered he would impose a negative externality of $-\$10$ on the remaining players. Specifically, this externality captures the utility loss that player 4 suffers as he would get one object when player

1's preferences are ignored (when player 4 enjoys a utility of 10) but he does not receive any object when the preferences of player 1 are considered.[7] ■

7 Participation constraints

Thus far we assumed that all agents participated in the mechanism, as if participation was compulsory by some government regulation. But what if participation is voluntary? We then need to add participation constraints (PC) to each agent with type θ_i.

We next present different approaches to write the PC, depending on the information that the agent knows when the PC constraint is defined:

- Before he knows his type (ex-ante stage);
- After knowing his type, but without observing his opponents' type θ_{-i} (interim stage); and
- After knowing his type and that of all other individuals (ex-post stage).

Intuitively, you may think of a construction company planning to submit a bid for a procurement project. Before knowing the details of the project, the company operates at the ex-ante stage since it does not know its own costs from participating in the project; after receiving more information about the project and investigating how costly it will be, the firm operates at the interim stage; and once the project is assigned (and potentially completed) the firm is at the ex-post stage since at that point the company is likely observing other firms' technologies too.

Let $\overline{u}_i(\theta_i)$ denote agent i's reservation utility.[8] The PC in the above three stages becomes

$$\begin{aligned}
&\text{Ex-ante PC:} \quad E_\theta[u_i(g(\theta_i, \theta_{-i}), \theta_i)] \geq E_{\theta_i}[\overline{u}_i(\theta_i)] \\
&\text{Interim PC:} \quad E_{\theta_{-i}}[u_i(g(\theta_i, \theta_{-i})|\theta_i)] \geq \overline{u}_i(\theta_i) \text{ for all } \theta_i \\
&\text{Ex-post PC:} \quad u_i(g(\theta_i, \theta_{-i}), \theta_i) \geq \overline{u}_i(\theta_i) \text{ for all } (\theta_i, \theta_{-i})
\end{aligned}$$

At the ex-ante stage, individual i takes expectations of both his own type, θ_i, and his rivals', θ_{-i}, since he could not observe his own type yet. At the interim stage, he only takes the expectations of his rivals' types, θ_{-i}; while at the ex-post stage he does not need to take expectations since all type profiles $\theta = (\theta_i, \theta_{-i})$ have been revealed. As you can anticipate, for any SCF

$$\text{Ex-post PC} \Rightarrow \text{Interim PC} \Rightarrow \text{Ex-ante PC}$$

[7] For other examples of VCG mechanisms, see Tadelis (2013), pp. 298-299.

[8] Player i's reservation utility from not participating in the mechanism is here assumed to be a function of his type θ_i (e.g., cost in an alternative market) but is not a function of his opponent's type, θ_{-i}. In some applications, however, player i's reservation utility could be a function of both θ_i and θ_{-i}, written as $\overline{u}(\theta_i, \theta_{-i})$, if this player is affected by externalities originating from other players whose amount (or severity) depends on other players' types θ_{-i}.

which occurs because the ex-post definition is more demanding (for all (θ_i, θ_{-i}) pairs) than the interim definition (for all θ_i types), and both are more demanding than the ex-ante definition.[9] In the following subsections we apply the above PC definitions to different settings, such as a VCG mechanism, and a Groves mechanism, among others.

7.1 Participation constraints in the VCG mechanism

Example 11.14 - PC in a public good project. Consider a society with two individuals $N = \{1, 2\}$. A public project is either implemented or not, $k = \{0, 1\}$, and both individuals' private valuations for the project are drawn from $\Theta_1 = \Theta_2 = \{20, 60\}$. Finally, the total cost of building the project is 50.

In this setting, the set of feasible outcomes is

$$X = \{(k, t_1, t_2) : k = \{0, 1\},\ t_1, t_2 \in \mathbb{R},\ -(t_1 + t_2) \leq 50\},$$

that is, allocation rules $k = \{0, 1\}$ and transfer rules that guarantee total payments of \$50. Consider the allocation function we considered in Example 11.8 (where the project is implemented if at least the valuation of one individual is 60), which we reproduce below:

$$k^* (\theta_1, \theta_2) = \begin{cases} 0 & \text{if } \theta_1 = \theta_2 = 20 \\ 1 & \text{otherwise} \end{cases}$$

and define the same valuation function as in Example 11.8,

$$v_i (k^* (\theta_1, \theta_2), \theta_i) = k^* (\theta_1, \theta_2) \cdot (\theta_i - 25) \quad \text{for all } \theta_1, \theta_2$$

where. from previous sections, such allocation rule is allocative efficient. From the Groves' theorem, we know that if the transfer function is "à la Groves" then the resulting SCF satisfies DSIC. Let us now check if, despite being DSIC, such SCF violates ex-post PC. In particular, assume that reservation utility is $\overline{u}_i (\theta_i) = 0$ for all types θ_i and for every player i. Hence, for ex-post PC to hold, we need

$$u_i (g (\theta_i, \theta_{-i}), \theta_i) \geq 0 \quad \text{for every } \theta_1 \text{ and } \theta_2$$

In the case that $(\theta_1, \theta_2) = (20, 60)$, such condition requires $v_1 (k^* (20, 60), 20) \geq 0$, which in this case is

$$k^* (20, 60) \cdot (20 - 25) = 1 \cdot (20 - 25) = -5 \ngeq 0$$

[9] Continuing with the above example about the construction company, we can say that the firm may have incentives to participate in the procurement project before knowing its costs from the project (its type θ_i) at the ex-ante stage, but choose to not participate in the project once it observes its costs (which could be relatively high) at the interim stage. Using the same argument, a firm choosing to submit a bid in the procurement auction (after observing a relatively low cost at the interim stage) may ultimately obtain a negative utility level at the ex-post stage once it observes its rivals' costs.

entailing that ex-post PC does not hold. ■

7.2 Participation constraints in Clarke mechanism

Clarke mechanisms satisfy ex-post PC if they satisfy the following properties:

1. Reservation utility is zero, $\overline{u}_i(\theta_i) = 0$ for every type θ_i;
2. The mechanism satisfies "choice set monotonicity": The set of feasible outcomes X weakly grows in N. The intuition behind this assumption is that the choice set X becomes wider ("richer" set of possible outcomes) as more agents enter the population.
3. The mechanism satisfies "no negative externality": Formally, player i obtains a positive utility when his preferences θ_i are ignored, $v_i\left(k^*_{-i}(\theta_{-i}), \theta_i\right) \geq 0$ for every type θ_i, where allocation $k^*_{-i}(\theta_{-i})$ is allocative efficient for all types θ_i, all $\theta_{-i} \in \Theta_{-i}$, and every player i. In words, player i obtains a positive value from the allocation that emerges when his preferences are ignored. Otherwise, the preferences of all other agents would lead to an allocation $k^*_{-i}(\theta_{-i})$ that imposes a negative externality on player i.

Let us next show why the above three properties help guarantee that the Clarke mechanism satisfies ex-post PC.

Proof: Recall that, given the transfer function in the Clarke mechanism, the utility function $u_i(g(\theta), \theta)$ becomes

$$\begin{aligned}
u_i(g(\theta), \theta) &= v_i(k^*(\theta), \theta) + t_i(\theta_i, \theta_{-i}) \\
&= v_i(k^*(\theta), \theta) + \underbrace{\left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)\right]}_{t_i(\theta_i, \theta_{-i})} \\
&= \underbrace{\sum_j v_j(k^*(\theta), \theta_j)}_{\substack{\text{From the first two terms} \\ \text{in the above expression}}} - \underbrace{\sum_{j \neq i} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)}_{\text{Last term in above expression}}
\end{aligned}$$

From choice set monotonicity (Assumption 2), the set of possible outcomes considering agent i's preferences must be weakly larger than that when ignoring agent i's preferences, implying that the choice with agent i, $k^*(\theta)$, must generate the same or more total value than the choice without him, $k^*_{-i}(\theta_{-i})$, that is, $\sum_j v_j(k^*(\theta), \theta_j) \geq \sum_j v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)$. Subtracting $\sum_{j \neq i} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)$ on both sides, we obtain

$$\sum_j v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right) \geq \sum_j v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right) - \sum_{j \neq i} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)$$

In addition, the right-hand side simplifies to $v_i(k^*_{-i}(\theta_{-i}), \theta_i)$ since the first term, $\sum_j v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)$, includes utility of agent i in the summation operator while the second term does not. Therefore, the above expression reduces to

$$u_i(g(\theta), \theta) \geq v_i(k^*_{-i}(\theta_{-i}), \theta_i) \geq 0$$

where the last inequality, "≥ 0", originates from the "no negative externality" property (Assumption 3). Finally, since the reservation utility is $\overline{u}_i(\theta_i) = 0$ for every type θ_i by Assumption 1, we can write $u_i(g(\theta), \theta) \geq \overline{u}_i(\theta_i)$ for all θ, as required for the SCF to satisfy ex-post PC. (Q.E.D.)

8 Linear utility

We now consider a special case of the quasi-linear utility environment, where every player i's utility function is

$$u_i(x, \theta_i) = \underbrace{\theta_i v_i(k)}_{v_i(k,\theta_i) \text{ in quasilinear environment}} + m_i + t_i$$

Indeed, the only difference with respect to the quasilinear environment is that the generic $v_i(k, \theta_i)$ function is now $v_i(k, \theta_i) = \theta_i v_i(k)$. For simplicity, we assume that: (1) Types are drawn from intervals of real numbers $[\underline{\theta}_i, \bar{\theta}_i] \subset \mathbb{R}$, where $\underline{\theta}_i < \bar{\theta}_i$, such as $[0, 1]$ in many applications; and (2) types are i.i.d. with positive densities for all types θ_i.

In this context, consider a SCF $f(\theta) \equiv (k(\theta), t_1(\theta), \cdots, t_N(\theta))$, and define expected transfers and valuations as follows:

1. $\bar{t}_i(\hat{\theta}_i) \equiv E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$, that is, agent i's expected transfer when he reports $\hat{\theta}_i$ and all other agents truthfully report their types. As a practice, note that agent i's expected transfer from thruthfully reporting his type θ_i is then written as $\bar{t}_i(\theta_i)$, since we evaluated $\bar{t}_i(\hat{\theta}_i)$ at $\hat{\theta}_i = \theta_i$.

2. $\bar{v}_i(\hat{\theta}_i) \equiv E_{\theta_{-i}}[v_i(\hat{\theta}_i, \theta_{-i})]$, that is, agent i's expected valuation (in a quasilinear environment where valuations are defined as $v_i(k, \theta_i)$) when he reports $\hat{\theta}_i$ and all other agents truthfully report their types. Again, we can then express his expected value from truthtelling as $\bar{v}_i(\theta_i)$.

3. $u_i(\hat{\theta}_i|\theta_{-i}) \equiv E_{\theta_{-i}}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i] = E_{\theta_{-i}}[\bar{v}_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] = \theta_i \bar{v}_i(\hat{\theta}_i) + \bar{t}_i(\hat{\theta}_i)$, that is, agent i's expected utility (in a linear environment) when he reports $\hat{\theta}_i$ while all other agents truthfully report their types. Finally, if agent i truthfully reports his type θ_i, i.e., $\hat{\theta}_i = \theta_i$, his expected utility becomes

$$u_i(\theta) \equiv u_i(\theta_i|\theta_i) = \theta_i \bar{v}(\theta_i) + \bar{t}_i(\theta_i)$$

We next present under which conditions a SCF in this linear environment satisfies BIC; a result originally presented by Myerson (1981).

8.1 Myerson's Characterization Theorem

In a linear environment, a SCF is BIC if and only if for every individual i,

1. $\overline{v}_i(\theta_i)$ is weakly increasing in his own type θ_i, and
2. Function $u_i(\theta_i)$ can be expressed as

$$u_i(\theta_i) = u_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)\,ds \quad \text{for all} \quad \theta_i \in \Theta_i$$

Intuitively, this theorem says that we can identify all SCFs satisfying BIC in two steps: First, identify allocation functions $k(\theta)$ that lead every agent i's expected valuation function $\overline{v}_i(\theta_i)$ to be weakly increasing in his type θ_i; second, among these allocation functions, choose the expected transfer function $\bar{t}_i(\theta_i)$ that entails an expected utility which can be expressed in terms of the second condition of the theorem.

We can rewrite the utility function of agent i as

$$\begin{aligned} u_i(\theta_i) &= u_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)\,ds \\ &= \underbrace{\bar{t}_i(\underline{\theta}_i) + \underline{\theta}_i \overline{v}_i(\underline{\theta}_i)}_{u_i(\underline{\theta}_i)} + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)\,ds \end{aligned}$$

where $\bar{t}_i(\underline{\theta}_i)$ is a constant. Since the utility function is linear, $u_i(\theta_i) = \theta_i \overline{v}_i(\theta_i) + \bar{t}_i(\theta_i)$, the transfer function becomes $\bar{t}_i(\theta_i) = u_i(\theta_i) - \theta_i \overline{v}_i(\theta_i)$. Inserting the above expression $u_i(\theta_i)$ into this transfer function, we find

$$\begin{aligned} \bar{t}_i(\theta_i) &= u_i(\theta_i) - \theta_i \overline{v}_i(\theta_i) \\ &= \underbrace{\bar{t}_i(\underline{\theta}_i) + \underline{\theta}_i \overline{v}_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)\,ds}_{u_i(\theta_i)} - \theta_i \overline{v}_i(\theta_i) \end{aligned}$$

Since many studies in auction theory and industrial organization consider linear environments, Myerson's characterization result has been applied to problems in many fields. We next present one of the most famous applications, in auction theory, to show that, under relatively general conditions, the seller's expected revenue coincides across different auction formats. For compactness, this result is often referred to as the "Revenue Equivalence Theorem."

8.2 Revenue equivalence theorem

Consider $N \geq 2$ risk neutral bidders so we operate in an environment of linear utility functions. Every bidder i's type θ_i is drawn from the interval $[\underline{\theta}_i, \bar{\theta}_i]$, where $\underline{\theta}_i < \bar{\theta}_i$ and $\phi_i(\cdot) > 0$ for all $\theta_i \in$

$[\underline{\theta}_i, \bar{\theta}_i]$ with independent distribution of valuations among buyers. Consider two auction formats, such as the first- and second-price auction. If the BNE of these two auctions yield, for all profiles of types $\theta = (\theta_1, \cdots, \theta_I)$,

a) The same assignment rule $(y_1(\theta), y_2(\theta), \cdots, y_N(\theta))$, and

b) The same value of $u_1(\underline{\theta}_1)$, $u_2(\underline{\theta}_2)$, $\cdots$, $u_N(\underline{\theta}_N)$, where $u_i(\underline{\theta}_i)$ is the expected utility for buyer i evaluated at his lowest valuation when every player truthfully reveal his type.

Then the seller's expected revenue is the same in both auction formats.

Before proving this result, let us briefly discuss what it means. The revenue equivalence theorem says that, if two auction formats with risk neutral bidders and independent valuations assign the object to the same bidder (or bidders) and generate the same expected utility for individual i has the lowest valuation for the object, then they must generate the same expected revenue for the seller.

Proof: From the Revelation Principle we have that the SCF that implements the BNE of any auction format is BIC.

The seller's expected revenue is given by the sum of expected transfers, i.e., $\sum_{i=1}^{N} E[-\bar{t}_i(\theta_i)]$. We initially find $E[-\bar{t}_i(\theta_i)]$

$$E[-\bar{t}_i(\theta_i)] = \int_{\underline{\theta}_i}^{\bar{\theta}_i} -\bar{t}_i(\theta_i)\phi_i(\theta_i)\, d\theta_i$$

Since utility is given by $u_i(\theta_i) = \bar{y}_i(\theta_i)\theta_i + \bar{t}_i(\theta_i)$ in this linear environment, we can solve for the expected transfer $\bar{t}_i(\theta_i)$, which yields $\bar{t}_i(\theta_i) = u_i(\theta_i) - \bar{y}_i(\theta_i)\theta_i$. Multiplying by -1 on both sides, we obtain $-\bar{t}_i(\theta_i) = \bar{y}_i(\theta_i)\theta_i - u_i(\theta_i)$, which implies that the above expression becomes

$$E[-\bar{t}_i(\theta_i)] = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \underbrace{[\bar{y}_i(\theta_i)\theta_i - u_i(\theta_i)]}_{-\bar{t}_i(\theta_i)} \phi_i(\theta_i)\, d\theta_i$$

and since $u_i(\theta_i) = u_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}(s)\, ds$, then $E[-\bar{t}_i(\theta_i)]$ becomes

$$E[-\bar{t}_i(\theta_i)] = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\bar{y}_i(\theta_i)\theta_i - \underbrace{\left(u_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} y(s)\, ds \right)}_{u_i(\theta_i)} \right] \phi_i(\theta_i)\, d\theta_i$$

Taking $u_i(\underline{\theta}_i)$ out of the integral operator, yields

$$E[-\bar{t}_i(\theta_i)] = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\underbrace{\bar{y}_i(\theta_i)\theta_i}_{\text{Term } A} - \int_{\underline{\theta}_i}^{\theta_i} y(s)\, ds \right] \phi_i(\theta_i)\, d\theta_i - u_i(\underline{\theta}_i) \qquad (B)$$

Applying integration by parts in Term A, we obtain[10]:

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}_i}^{\theta_i} [\overline{y_i}(s)\, ds]\phi_i(\theta_i)\, d\theta_i = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)\, d\theta_i - \int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)\Phi_i(\theta_i)\, d\theta_i$$
$$= \int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)(1-\Phi_i(\theta_i))\, d\theta_i$$

Substituting this result inside expression (B) yields:

$$E[-\bar{t}_i(\theta_i)] = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\bar{y}_i(\theta_i)\theta_i - \bar{y}_i(\theta_i)\frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right]\phi_i(\theta_i)\, d\theta_i$$
$$= \int_{\underline{\theta}_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i)\left[\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right]\phi_i(\theta_i)\, d\theta_i - u_i(\underline{\theta}_i)$$

which represents the expected transfer from bidder i. Finally, summing over all N bidders, we find[11]

$$\sum_{i=1}^{N} E[-\bar{t}_i(\theta_i)] = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \cdots \int_{\underline{\theta}_N}^{\bar{\theta}_N} \underbrace{\sum_{i=1}^{N} \bar{y}_i(\theta_i)}_{\text{Term } C} \underbrace{\left[\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right] \prod_{i=1}^{N} \phi_i(\theta_i)\, d\theta_N \cdots d\theta_i}_{\text{Term } D} - \underbrace{\sum_{i=1}^{N} u_i(\underline{\theta}_i)}_{\text{Term } E}$$

Therefore, if the BNE of two different auction formats have:

1. the same probabilities of assigning the object to each bidder, $(\overline{y}_1(\theta_i), \cdots, \overline{y}_N(\theta_N))$, as captured by term C; and

2. the same expected utilities when bidder i has the lowest valuation for the good, $u_i(\underline{\theta}_i)$, for every player i, as measured by term E,

we can see that the above expression generates the same expected revenue for the seller. Indeed, by property (1) term C is constant across two auctions formats; and by property (2) so is term E. Finally, term D is unaffected by the rules of the auction since its is just given by the distribution of bidders' valuations for the good. For instance, if valuations are uniformly distributed, $\theta_i \sim U[0,1]$ for every player i, term $\left[\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right] \prod_{i=1}^{N} \phi_i(\theta_i)$ simplifies to $(2\theta_i - 1)$. (Q.E.D.)

Example 11.15. Applying the revenue equivalence theorem in the first- and second-price auction. Consider the DRM version of the first-price auction presented in Example 1.1, where every bidder reports his type θ_i, the seller assigns the object to the bidder reporting the highest value, and this bidder pays a transfer equal to the value he submitted. Similarly, consider

[10] In order to apply integration by parts, a common trick is to first recall the derivative of the product of two functions $h(x)$ and $g(x)$, that is, $(h \cdot g)' = h'g + hg'$, or alternatively $hg' = (h \cdot g)' - h'g$. Integrating both sides yields $\int hg'dx = hg - \int h'gdx$. For our current example, let $h(x) = \int_{\underline{\theta}_i}^{\theta_i} \overline{y}_i(s)ds$, $g'(x) = \varphi_i(\theta_i)d\theta_i$, $h'(x) = \overline{y}_i(\theta_i)$ and $g(x) = \Phi_i(\theta_i)$. Plugging these functions and rearranging yields the above result.

[11] In this expression, we moved the summation signs inside the integral because types $(\theta_1, ..., \theta_N)$ are independently distributed.

the DRM version of the second-price auction presented in Example 1.2, where every bidder reports his type θ_i, the seller assigns the object to the individual with the highest reported value, but the latter pays a transfer equal to the second-highest reported value.

Comparing these two DRMs, we can see that they satisfy the conditions in this theorem since: (1) the allocation rule in both DRMs coincides, i.e., the bidder reporting the highest value receives the object; and (2) the expected utility of the bidder with the lowest valuation, $u_i(\underline{\theta}_i)$, when truthfully reporting his type $\underline{\theta}_i$, coincides in both auctions (his utility is zero in both auction formats since this bidder does not win the object). Hence, we can conclude that the DRM version of the first- and second-price auction generate the same expected revenue for the seller. ■

This is a useful theorem. Consider, for instance, the first- and second-price auction. In a context where bidders' valuations are uniformly distributed, $\theta_i \sim U[0,1]$ for every player i, it is easy to show that the expected revenue in both auction formats coincide and is equal to $\frac{N-1}{N+1}$, where $N \geq 2$ denotes the number of bidders. In a setting with a more general distribution function, such as $\theta_i \sim [\underline{\theta}_i, \bar{\theta}_i]$ where $\underline{\theta}_i < \bar{\theta}_i$ and $\phi_i(\cdot) > 0$ for all θ_i, where densities $\phi_i(\cdot)$ are i.i.d., showing that the first- and second-price auction generate the same expected revenue for the seller becomes more technically challenging. A similar issue arises when comparing expected revenues of other auction formats, such as all-pay auctions or auctions with reservation prices or entry fees. In contrast, the application of the revenue equivalence theorem is more straightforward than computing the expected revenue from each auction format and then confirm if they coincide. Instead, we only need to check if properties (1) and (2) hold in the two auction formats that we seek to compare.

9 Optimal Bayesian Mechanism

Let us now put ourselves in the shoes of mechanism designer, e.g., the seller of an object in an auction, or a regulatory agency that does not observe the production cost of firms in a regulated industry. As mechanism designers, we now seek to select a feasible SCF that maximizes a certain objective function, such as welfare or total revenue. But, what do we mean when we say "feasible SCF" in the context? We focus on those SCFs satisfying both BIC and PC and denote them as F^*.

We next search for optimal mechanisms. That is, SCFs that maximize the mechanism designer's objective function (e.g., expected profits or social welfare) subject to feasibility, i.e., $f(\cdot) \in F^*$, and thus satisfying BIC and PC. For illustration purposes, we conduct this search in the principal-agent problem. While we analyzed this problem in Chapter 10 (Contract Theory), we considered that the principal (firm manager) offers a menu of contracts to the agent (worker), and the agent responds choosing the contract that maximizes his own utility given his privately observed type, i.e., an IRM. We next show that the principal can design a DRM in which the agent has incentives to truthfully report his type to the principal.

9.1 Solving the Principal-Agent problem using Mechanism Design

Model. Consider the principal-agent problem we discussed in Chapter 10, where the agent's utility function is

$$u(e, t_1, \theta) = t_1 + \theta g(e)$$

where t_1 denotes the transfer that the principal pays to the agent (e.g., salary); $g(e)$ represents the agent's disutility of effort, which originates at zero, and is increasing and convex in effort e, that is, $g(0) = 0$, $g(e) > 0$ for all $e > 0$, $g'(e) > 0$ and $g''(e) > 0$ for all effort levels $e > 0$; and $\theta \in [\underline{\theta}, \bar{\theta}]$ indicates the agent's type which is a negative number, $\underline{\theta} < \bar{\theta} \leq 0$. Intuitively, lower values of θ reflect a larger disutility of effort while higher realizations of θ denote that the agent suffers a small disutility from effort (e.g., close to zero). Unlike in Chapter 10, where we mostly considered discrete types, we allow the agent's type to be drawn from a continuous cumulative distribution function $\Phi(\cdot)$ with positive density $\phi(\cdot) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. In addition, assume that $\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}$ is weakly increasing in θ (we return to the intuition behind this assumption below).

The principal's (agent 0) utility is

$$u_0(e, t_0) = x(e) + t_0$$

where $x(e)$ represents the profits that the principal obtains from the agent's effort level e, which is increasing in effort and concave, that is, $x'(e) > 0$ and $x''(e) < 0$ for all $e \geq 0$. Intuitively, a larger effort by the agent increases the firm's profits, but a decreasing rate. Transfer t_0 denotes the wage payment that the principal provides to the agent for his effort, and is negative as we show below.

DRM. Since the principal is considering SCFs that satisfy BIC, the agent must be provided incentives to truthfully reveal his type. We can then invoke the Revelation Principle so that the principal, rather than designing an IRM, can more easily design a DRM in which the agent is induced to truthfully announce his type θ, and then the principal maps it into the SCF

$$f(\theta) = (e(\theta), t_0(\theta), t_1(\theta))$$

where $e(\theta)$ plays the role of the outcome function, thus being analogous to $k(\theta)$ in our previous discussions; while $t_0(\theta)$ and $t_1(\theta)$ are transfer functions to the principal and the agent, respectively. For simplicity, we assume that the mechanism must be budget balanced, entailing that all transfers to the agent originate from the principal, i.e., $-t_0(\theta) = t_1(\theta)$ for all types θ, which helps us reduce the three elements of the above SCF to only two, i.e., $f(\theta) = (e(\cdot), t_1(\cdot))$. Informally, once we find the transfer that the agent receives, $t_1(\cdot)$, we can infer the payment that the principal must provide, $t_0(\theta)$, since the two coincide in absolute value. Therefore, the principal's objective function is $x(e) + t_0$ and $-t_0(\theta) = t_1(\theta)$, we can rewrite it as $x(e) - t_1$.

Symmetric information. As a benchmark, consider that the principal is perfectly informed about the agent's type, θ, his problem would be

$$\max_{e(\cdot),t_1(\cdot)} \quad x(e(\theta)) - t_1(\theta)$$

$$\text{subject to } \; t_1(\theta) + \theta g(e(\theta)) \geq \bar{u} \; \text{ for all } \theta \tag{PC}$$

which, as in other models where players are symmetrically informed, is only subject to the voluntary participation constraint of the agent (PC), but does not require BIC to induce the agent to truthfully report his type, θ, as the principal now observes this information. The PC constraint must bind (otherwise the principal could lower the transfer $t_1(\theta)$ to the agent), implying that we can use $t_1(\theta) = \bar{u} - \theta g(e(\theta))$, and insert it into the principal's objective function as follows

$$\max_{e(\cdot)} \quad x(e(\theta)) \underbrace{- [\bar{u} - \theta g(e(\theta))]}_{-t_1(\theta)}$$

which only includes one choice variable, e. Differentiating with respect to effort e yields $x'(e^*(\theta)) + \theta g'(e^*(\theta)) = 0$, or

$$\underbrace{x'(e^*(\theta))}_{\text{Marginal profit}} = \underbrace{-\theta g'(e^*(\theta))}_{\text{Marginal cost of inducing effort}}$$

where $e^*(\theta)$ represents the profit-maximizing effort function under symmetric information. Intuitively, the principal increases effort until the marginal profit he obtains, $x'(e^*(\theta))$, coincides with the marginal increase in wages the agent needs to exert this additional effort, $-\theta g'(e^*(\theta))$. Figure 11.3 separately depicts these two effects as a function of effort, e, on the horizontal axis. First, the marginal profit from additional effort, $x'(e^*(\theta))$, is decreasing in e since $x''(e) \leq 0$ by assumption. In contrast, the marginal cost of inducing more effort, $-\theta g'(e^*(\theta))$, is increasing in e because $g''(e) \geq 0$ by definition. Therefore, $x'(e^*(\theta))$ and $-\theta g'(e^*(\theta))$ cross only once, at effort level $e^*(\theta)$, as depicted on the figure.

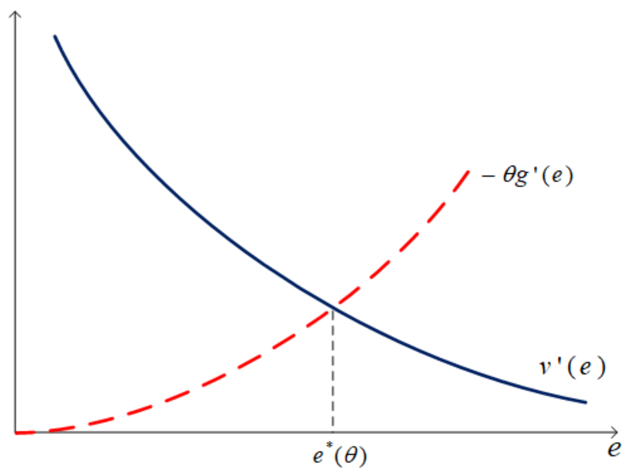

Figure 11.3. Equilibrium effort level under symmetric information.

Asymmetric information. When the principal does not observe the agent's type, θ, he maximize his expected utility, $E_\theta\left[x(e(\theta)) - t_1(\theta)\right]$, by choosing a SCF $f(\cdot) = (e(\cdot), t_1(\cdot))$, i.e., he chooses a pair of effort and wage, that solves

$$\max_{e(\cdot),t_1(\cdot)} \quad E_\theta\left[x(e(\theta)) - t_1(\theta)\right]$$

subject to $f(\cdot)$ being feasible, i.e., $f(\cdot) \in F^*$

Since agent's utility is linear, we can use the notation presented in the section on linear utility to simplify our problem. In particular, let $e(\theta)$ play the role of $k(\theta)$ in previous sections, so that we can use $g(e(\theta))$ rather than $\overline{v}_1(k(\theta))$. We can then represent the agent's expected utility from truthfully reporting his type, θ, as

$$U_1(\theta) = t_1(\theta) + \theta g(e(\theta))$$

Solving for transfer $t_1(\theta)$, yields

$$t_1(\theta) = U_1(\theta) - \theta g(e(\theta))$$

Plugging $t_1(\theta)$ in the principal's objective function, we obtain

$$\max_{e(\cdot),U_1(\cdot)} \quad E_\theta\left[x(e(\theta)) - \left(\underbrace{U_1(\theta) - \theta g(e(\theta))}_{t_1(\theta)}\right)\right]$$

$$\text{subject to} \;\; f \in F^*$$

(Note the change in choice variables below the max operator, from $(e(\cdot), t_1(\cdot))$ to $(e(\cdot), U_1(\cdot))$, since $t_1(\cdot)$ is now absent from the program.)

How can we express the feasibility constraint, $f(\cdot) \in F^*$, in a more tractable way? Feasibility entails both BIC and PC. For the first property, recall that, from Myerson's characterization theorem, in a linear environment, a SCF $f(\cdot)$ is BIC if and only if:

1. $\bar{U}_1(\theta)$ is weakly increasing in θ. In our principal-agent context, that entails $g(e(\theta))$ being weakly increasing in θ. But since $g'(e) > 0$ by definition, this amounts to the agent's effort $e(\theta)$ being weakly increasing in his type θ.[12]

2. $U_i(\theta_i) = U_i(\underline{\theta}_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s)\, ds$ for all θ_i, which in our principal-agent context, where effort is costly, implies that $U_1(\theta) = U_1(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} g(e(s))\, ds$ for all θ.

From the above conditions, we know how to express BIC, but how can we express PC? That property is actually easier to represent than BIC. In particular, for PC to hold we need that

$$U_1(\theta) \geq \bar{u} \;\; \text{for all types } \theta$$

That is, the expected utility that the agent (player 1) obtains from participating in the mechanism, when he truthfully reveals his type θ, is larger than his reservation utility level $\bar{u}$.

Summarizing, the principal's problem can be expressed as follows

$$\max_{e(\cdot), U_1(\cdot)} \;\; E_\theta \left[x(e(\theta)) - U_1(\theta) + \theta g(e(\theta)) \right]$$

$$\text{subject to 1) } e(\theta) \;\; \text{is weakly increasing in } \theta$$

$$\text{2) } U_1(\theta) = U_1(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} g(e(s))\, ds \;\; \text{for all } \theta$$

$$\text{3) } U_1(\theta) \geq \bar{u} \;\; \text{for all } \theta$$

where the first two constraints guarantee BIC (thanks to Myerson's characterization theorem), and the third constraint guarantees PC.

Simplifying the principal's problem. Before taking first-order conditions, a common trick in this type of problem is simplifying it as much as possible. First, note that if constraint (2) holds, then $U_1(\theta) \geq U_1(\underline{\theta})$ since $\int_{\underline{\theta}}^{\theta} g(e(s))\, ds$ is positive for all θ. Hence, constraint (3) holds for every type θ if it holds for the worker with the lowest cost of effort $\underline{\theta}$, i.e., $U_1(\underline{\theta}) \geq \bar{u}$. We can then replace

[12] To see this point, you can express $g(e(\theta))$ being weakly increasing in θ as $\frac{\partial g}{\partial e} \frac{\partial e}{\partial \theta} \geq 0$. Since $g'(e) > 0$ by definition, $\frac{\partial g}{\partial e} > 0$, entailing that the second term in the above derivative must satisfy $\frac{\partial e}{\partial \theta} \geq 0$, i.e., effort must be weakly increasing in the agent's type θ. Intuitively, workers with a value of θ closer to zero (from below) experience a smaller disutility from effort and thus exert a larger effort level in equilibrium.

constraint (3) for its version evaluated at the lowest type $\theta = \underline{\theta}$,

$$U_1(\underline{\theta}) \geq \bar{u}$$

which we denote as constraint (3)'. Second, we can substitute $U_1(\theta)$ in the objective function from constraint (2), as such constraint holds with equality, yielding a slightly reduced program:

$$\max_{e(\cdot),U_1(\underline{\theta})} \quad E_\theta \left[x(e(\theta)) \underbrace{-U_1(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} g(e(s))\, ds}_{-U_1(\theta)} + \theta g(e(\theta)) \right]$$

subject to

1) $e(\theta)$ is weakly increasing in θ

3)$'$ $U_1(\underline{\theta}) \geq \bar{u}$

(Note the change in choice variables below the max operator, from $U_1(\theta)$ to $U_1(\underline{\theta})$ since now $U_1(\theta)$ is absent from objective function and constraints.)

Expanding the integral in the objective function yields

$$\max_{e(\cdot),U_1(\underline{\theta})} \int_{\underline{\theta}}^{\bar{\theta}} [x(e(\theta)) + \theta g(e(\theta))]\phi(\theta)\, d\theta - \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} [g(e(s))\, ds]\phi(\theta)\, d\theta}_{\text{Term } A} - \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} U_1(\underline{\theta})\phi(\theta)\, d\theta}_{\text{Term } B}$$

subject to

1) $e(\theta)$ is weakly increasing in θ

3)$'$ $U_1(\underline{\theta}) \geq \bar{u}$

Note that $U_1(\underline{\theta})$ is a constant, and thus the last term of the objective function (Term B) becomes

$$\int_{\underline{\theta}}^{\bar{\theta}} U_1(\underline{\theta})\phi(\theta)\, d\theta = U_1(\underline{\theta})$$

Likewise, we can use integration by parts to simplify the second term of the principal's objective function (Term A), which yields[13]

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} [g(e(s))\, ds]\phi(\theta)\, d\theta = \int_{\underline{\theta}}^{\bar{\theta}} [g(e(\theta)) - \Phi(\theta) g(e(\theta))]\, d\theta$$
$$= \int_{\underline{\theta}}^{\bar{\theta}} [g(e(\theta))(1 - \Phi(\theta))]\, d\theta$$

We can now substitute our simplification back into the second term of the principal's objective function, to obtain

$$\max_{e(\cdot),U_1(\underline{\theta}_1)} \int_{\underline{\theta}}^{\bar{\theta}} [x(e(\theta)) + \theta g(e(\theta))]\, \phi(\theta)\, d\theta - \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} [g(e(\theta))(1 - \Phi(\theta))]\, d\theta}_{\text{Term } A} - \underbrace{U_1(\underline{\theta})}_{\text{Term } B}$$

$$= \int_{\underline{\theta}}^{\bar{\theta}} \{[x(e(\theta)) - \theta g(e(\theta))]\phi(\theta)\, d\theta - g(e(\theta))(1 - \Phi(\theta))\}\, d\theta - U_1(\underline{\theta})$$

subject to

1) $e(\theta)$ is weakly increasing in θ

3)′ $U_1(\underline{\theta}) \geq \bar{u}$

and factoring out $g(e(\theta))$ from the objective function yields

$$\max_{e(\cdot),U_1(\underline{\theta})} \int_{\underline{\theta}}^{\bar{\theta}} \left[x(e(\theta)) + \left(\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)}\right) g(e(\theta))\right] \phi(\theta) d\theta - U_1(\underline{\theta})$$

subject to

1) $e(\theta)$ is weakly increasing in θ

3)′ $U_1(\underline{\theta}) \geq \bar{u}$

Finally, note that the PC constraint (3)' must hold with equality; otherwise the principal could still reduce $U_1(\underline{\theta})$ further, still extracting more surplus from the agent with the lowest θ. We can

[13] Recall that the formula for integrations by parts: Starting from the derivative of the product of two functions $f(x)$ and $h(x)$, i.e., $(fh)' = f'h + fh'$, we rearrange it as $fh' = (fh)' - f'h$. Hence, integrating on both sides, we obtain $\int f(x)h'(x)\, dx = f(x)h(x) - \int f(x)h'(x)\, dx$. Let $f(x) = \int_{\underline{\theta}}^{\bar{\theta}} [g(e(\theta))\, d\theta]$, $f'(x) = g(e(\theta))d\theta$, $h(x) = \Phi(\theta)$, and $h'(x) = \phi(\theta)d\theta$.where $\Phi(\theta)$ represents the cdf of the distribution. Applying integration by parts on the second term of the objective function, yields

$$\int_{\underline{\theta}}^{\bar{\theta}} \underbrace{\int_{\underline{\theta}}^{\theta} [g(e(s))\, ds]}_{f} \underbrace{\phi(\theta)\, d\theta}_{h'} = \underbrace{\left(\int_{\underline{\theta}}^{\theta} [g(e(s))\, ds]\right)}_{f} \underbrace{\Phi(\theta)|_{\underline{\theta}}^{\bar{\theta}}}_{h} - \int_{\underline{\theta}}^{\theta} \underbrace{\Phi(\theta)}_{h} \underbrace{g(e(\theta))}_{f'}\, d\theta$$

then use $U_1(\underline{\theta}) = \bar{u}$ into the objective function to obtain the following reduced program:

$$\max_{e(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left[x(e(\theta)) + \left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)} \right) g(e(\theta)) \right] \phi(\theta) d\theta - \underbrace{\bar{u}}_{=U_1(\underline{\theta})}$$

subject to 1) $e(\theta)$ is weakly increasing in θ

which has only one choice variable, $e(\cdot)$, since neither the objective function nor the (single) constraint depends on $U_1(\underline{\theta})$ any more.

As in similar applications, we can now solve the unconstrained program, i.e., ignoring constraint (1), and later on show that our results indeed satisfy constraint (1).

Results. Differentiating with respect to e yields

$$x'(e(\theta)) + \left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)} \right) g'(e(\theta)) = 0$$

or, rearranging,

$$\underbrace{x'(e(\theta))}_{\text{Marginal profits}} = \underbrace{-\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)} \right) g'(e(\theta))}_{\text{Marginal cost of inducing effort}}$$

as depicted in figure 11.4. First, $x'(e(\theta))$ is decreasing in e since $x''(e) < 0$ by definition; as in figure 11.3. Second, term $-\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)} \right)$ is positive given that $\theta < 0$ by assumption, and $g'(e(\theta))$ is increasing in e since $g'' > 0$ by definition (cost of effort is increasing and convex). Hence, the right-hand side of the above first-order condition is increasing in effort. Intuitively, curve $x'(e(\theta))$ depicts the principal's marginal increase in profits from inducing a larger effort from the agent, while curve $-\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)} \right) g'(e(\theta))$ denotes the marginal cost that principal must bear from inducing more effort. This cost includes the compensation to the agent, as the latter experiences a larger disutility from effort that needs to be compensated with a higher wage, as captured by $g'(e(\theta))$. However, this cost also includes the "information rent" that the principal needs to pay to all agents with type $\theta \geq \underline{\theta}$ in order for them to truthfully report their types. (We discuss this point in more detail below.)

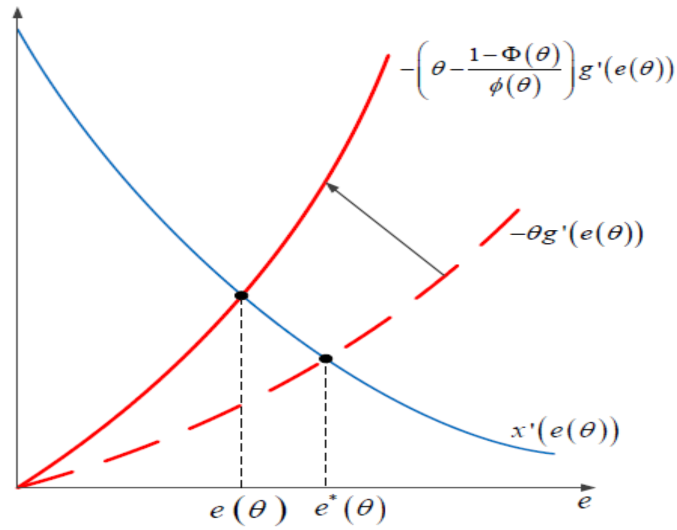

Figure 11.4. Equilibrium effort level under symmetric and asymmetric information.

Comparison: Symmetric vs. Asymmetric information. For comparison purposes, the figure plots the right-side of the first-order condition under symmetric information, $-\theta g'(e)$, and that under asymmetric information, $-\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right) g'(e)$.[14] In particular,

$$-\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right) > -\theta$$

which simplifies to $\frac{1-\Phi(\theta)}{\phi(\theta)} > 0$ given that $\phi(\theta) > 0$ and $\Phi(\theta) \in [0,1]$ for all θ. Therefore, the effort level that the principal induces under asymmetric information is *smaller* than under symmetric information. This is true for all agents with types $\theta < \bar{\theta}$, but not for the most efficient type of worker, $\theta = \bar{\theta}$, since evaluating the first-order condition under incomplete information at $\theta = \bar{\theta}$ yields

$$x'(e(\bar{\theta})) - \left(\bar{\theta} - \frac{1-\Phi(\bar{\theta})}{\phi(\bar{\theta})}\right) g'(e(\bar{\theta})) = 0$$

where $\bar{\theta} - \frac{1-\Phi(\bar{\theta})}{\phi(\bar{\theta})} = \bar{\theta} - \frac{1-1}{\phi(\bar{\theta})} = \bar{\theta}$ since $\Phi(\bar{\theta}) = 1$ (i.e., full cumulated probability at the highest type). Therefore, the first-order condition simplifies to

$$x'(e(\bar{\theta})) = -\bar{\theta} g'(e(\bar{\theta}))$$

[14] Recall that the left-hand side of the first-order condition under symmetric and asymmetric information coincide, as depicted in the decreasing curve in figure 11.3. The right-hand side of these first-order conditions differ, as illustrated in the two upward sloping curves in this figure.

which coincides with that under symmetric information for $\theta = \bar{\theta}$. This is a usual result in principal-agent and screening problems:

- "*No distortion at the top*": The "top" agent (in this context, the worker with the smallest disutility of effort, $\theta = \bar{\theta}$), suffers no distortion relative to his symmetric information. His utility in equilibrium also coincides with that in the symmetric information context.
- *"Downward distortion*" for all other types of agents, $\theta < \bar{\theta}$, since their efforts are lower under asymmetric information than under symmetric information, $e^*(\theta) < e(\theta)$. We already discussed this downward distortion in previous chapters, where the principal sought to reduce the incentives of the top worker to choose the contract meant for other workers. That is, in the IRMs considered in Chapter 10, the principal designed the menu of contracts to guarantee that incentive compatibility conditions hold for every type of worker. In the DRM we examined now, the principal also seeks to guarantee that the BIC condition holds, which entails inducing all types of workers (including the top worker) to truthfully report their types. In this context, the principal decreases the effort for workers with types $\theta < \bar{\theta}$, and reduces the transfer they receive, making misreporting unprofitable for the top worker (with $\theta = \bar{\theta}$), ultimately reducing the information rents that agents with $\theta < \bar{\theta}$ retain.

Effort increases in θ (Monotonicity of effort). Before claiming that, for a given θ, the effort level $e(\theta)$ that solves the above first-order condition is a solution of the principal's problem, we still need to check if it satisfies the (so far ignored) constraint (1). Recall that, to check if this constraint holds, we only need to confirm that the agent's effort $e(\theta)$ weakly increases in his type θ. We check this constraint by using the above first-order condition, which we start by rewriting it as follows:

$$x'(e(\theta)) + J(\theta)g'(e(\theta)) = 0$$

where, for compactness, we denote $J(\theta) \equiv \theta - \frac{1-\Phi(\theta)}{\phi(\theta)}$. We can now differentiate with respect to θ, and apply the chain rule to obtain

$$x''(e(\theta)) \cdot e'(\theta) + J'(\theta)g'(e(\theta)) + J(\theta)g''(e(\theta))e'(\theta) = 0$$

where $J'(\theta) \equiv \frac{\partial J(\theta)}{\partial \theta}$. Factoring out $e'(\theta)$ we find

$$e'(\theta)\left[x''(e(\theta)) + J(\theta)g''(e(\theta))\right] = -J'(\theta)g'(e(\theta))$$

or,

$$e'(\theta) = -\frac{J'(\theta)g'(e(\theta))}{x''(e(\theta)) + J(\theta)g''(e(\theta))} = -\frac{(+)(+)}{(-)+(-)(+)} = -\frac{(+)}{(-)} = (+)$$

In the numerator, $J'(\theta) > 0$ holds by assumption, as well as $g' > 0$. In the denominator, $x'' < 0$, $J(\theta) < 0$, $\theta < 0$, and $g'' > 0$ all hold by definition. Therefore, the agent's effort $e(\theta)$ weakly increases in his type θ.

Example 11.16 - Finding equilibrium effort in the principal-agent problem. Consider the above principal-agent model, and assume that types are uniformly distributed, $\theta \sim U[-1, 0]$. The cumulative distribution function is then $\Phi(\theta) = \theta$ and its density is $\phi(\theta) = 1$, yielding a virtual valuation of

$$J(\theta) = \theta - \frac{1 - \Phi(\theta)}{\phi(\theta)} = \theta - \frac{1 - \theta}{1} = 2\theta - 1$$

which is increasing in θ. In addition, assume that the principal's profits from effort are given by concave function $x(e) = \ln e$, where $x'(e) = \frac{1}{e}$; and that the agent's disutility of effort is represented by a (weakly) convex function $g(e) = 2e$, where $c'(e) = 2$.

Symmetric information. We can then evaluate the above first-order condition under symmetric information, obtaining

$$\frac{1}{e} = -2\theta$$

which, solving for e, yields an optimal effort of $e^{SI}(\theta) = -\frac{1}{2\theta}$, which is positive since $\theta < 0$ by definition; where superscript SI denotes "symmetric information." In addition, $e^{SI}(\theta)$ is decreasing in θ, i.e., when the disutility from effort becomes closer to zero the agent exerts more effort, but the worker's effort decreases as his disutility increases (i.e., as θ approaches -1).

Asymmetric information. Similarly operating, we can evaluate the first-order condition under asymmetric information to find

$$\frac{1}{e} = -(2\theta - 1)2$$

which, solving for e, yields an optimal effort of $e^{AI}(\theta) = -\frac{1}{4\theta-2}$, which is also positive since $\theta < 0$ by definition; and where superscript AI denotes "asymmetric information." In this case we also find that effort $e^{AI}(\theta)$ is decreasing in θ. In addition, effort under asymmetric information is lower than under symmetric information, $e^{AI}(\theta) < e^{SI}(\theta)$ since $-\frac{1}{4\theta-2} < -\frac{1}{2\theta}$ simplifies to $\theta < 1$, which is true given that $\theta \sim U[-1, 0]$. ■

9.1.1 Virtual valuations and information rents

The term $J(\theta) = \theta - \frac{1-\Phi(\theta)}{\phi(\theta)}$, or more often represented as $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ is commonly known as the principal's "virtual valuation" of assigning more effort to an agent with type θ. Consider the effect of increasing more effort to agent θ. On one hand, such higher effort allows the principal to increase his profits by $x'(e)$. However, in order to induce such additional effort the agent must now receive a larger transfer to compensate for his larger disutility of effort, $\theta \cdot g'(e)$. Until this point, we just described the trade-off that the principal experiences under a symmetric information setting. Under asymmetric information, however, a new effect emerges. In particular, from constraint (2), an increase in the effort from agent θ entails a larger transfer to all types above θ for them to truthfully report their types (rather than reporting a type equal to θ, as this agent now receives a more generous transfer). Specifically, since their probability mass is $1 - F(\theta)$, the total expected cost of increasing the effort from agent θ is

$$[\theta f(\theta) - (1 - F(\theta))] \, g'(e(\theta))$$

which, dividing by $f(\theta)$, can be expressed as

$$\left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) g'(e(\theta))$$

as in our above results.

9.1.2 Increasing or decreasing virtual valuations

Our previous discussion assumed that the virtual valuation $J(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$ is weakly increasing in θ. This assumption holds as long as the hazard rate

$$\frac{f(\theta)}{1 - F(\theta)}$$

is weakly increasing in θ. Intuitively, the probability of drawing an agent with type θ, given that we previously drew agents with types larger than θ, is increasing in θ. As we saw in the previous example where θ was uniformly distributed, the virtual valuation $J(\theta)$ became $J(\theta) = 2\theta - 1$, and thus was increasing in θ. A similar argument applies to other typical distributions such as the normal and exponential. For instance, in the case of the exponential distribution, $F(\theta) = 1 - \exp(-\lambda\theta)$ where $\lambda > 0$, we obtain a density of $f(\theta) = \lambda \exp(-\lambda\theta)$, which yields a virtual valuation of

$$J(\theta) = \theta - \frac{1 - [1 - \exp(-\lambda\theta)]}{\lambda \exp(-\lambda\theta)} = \theta - \frac{1}{\lambda}$$

which is increasing in θ for all values of parameter λ. Similarly, if $F(\theta) = \theta^\alpha$ where $\alpha \geq 1$, then $f(\theta) = \alpha\theta^{\alpha-1}$, entailing that the virtual valuation becomes

$$J(\theta) = \theta - \frac{1 - \theta^\alpha}{\alpha\theta^{\alpha-1}} = \frac{\theta(\alpha + 1) - \theta^{1-\alpha}}{\alpha}$$

with first-order derivative $J'(\theta) = \frac{\alpha+1-(1-\alpha)\theta^{-\alpha}}{\alpha}$, which is positive for all θ since $\alpha \geq 1$ by definition. In other settings, however, $J(\theta)$ could have strictly decreasing segments. This occurs, for example, if types are drawn from $F(\theta) = \theta^\alpha$ but parameter α satisfies $\alpha < 1$, since the derivative $J'(\theta)$ found above is not necessarily positive for all types θ.[15] When $J(\theta)$ is strictly decreasing for some values of θ, we can apply "ironing" techniques, which intuitively uses a monotonic transformation of virtual valuation $J(\theta)$ that is either flat or increasing in θ rather than using the original virtual

[15] For instance, if $\alpha = 1/3$, the virtual valuation becomes $J(\theta) = 4\theta - 3\,(\theta)^{2/3}$, with derivative $J'(\theta) = 4 - \frac{2}{\theta^{1/3}}$. Setting $J'(\theta) = 0$ and solving for θ, we find $\theta = 1/8$. Hence, $J(\theta)$ is first decreasing in θ, reaches a minimum at $\theta = 1/8$, and then becomes increasing in θ, i.e., $J'(\theta) > 0$ for all $\theta > 1/8$.

valuation (which can have strictly decreasing segments).[16]

10 Exercises

1. **Grobes-Loeb mechanism.** Consider a divisible public good y. For simplicity, assume that the cost function is linear in y, $c(y) = y$. Every agent $i \in N$ enjoys a benefit $b_i(y, \theta) = \theta_i \sqrt{y}$ from y units of the public good, and $\theta_i > 0$ denotes agent i's valuation for the public good.

   (a) Find the socially optimal amount of the public good, y^{SO}.

   (b) Consider a DRM where every agent reports his valuation of the public good θ_i, and then outcome $(y^{SO}, c_1, c_2, ..., c_N)$ is implemented, where $c_i = \frac{1}{4}\theta_i \sum_{i=1}^{N} \theta_i$ represents agent i's cost share. Show that this DRM is not strategyproof.

   (c) Let us now consider an alternative DRM, suggested by Groves and Loeb, with the following transfer function

   $$t_i = \frac{1}{4}\theta_i^2 + \frac{1}{2(N-2)} \sum_{j,k \neq i,\, j<k} \theta_j \theta_k$$

   In this context, every individual i's utility function is

   $$u_i(y, \theta) = b_i(y, \theta) - t_i = \theta_i \sqrt{y} - t_i$$

   where the firm term, $\theta_i \sqrt{y}$, represents the benefit that agent i enjoys from the public good (as in previous sections of the exercise); while the second term, t_i, denotes the tax he pays when $t_i > 0$ or the transfer he receives when $t_i < 0$. Assume that the public project must be budget balanced, so that total tax collection is weakly larger than the cost of the project, $\sum_{i=1}^{N} t_i \geq c(y)$. Answer the following questions about this mechanism.

      i. Show that, with the transfer function defined above, the cost of the project is exactly covered by tax collection, that is, $\sum_{i=1}^{N} t_i = c(y)$.

      ii. Show that the above mechanism satisfies DSIC. In words, every agent i has incentives to truthfully report his type θ_i regardless of the reported types of all other players, θ_{-i}.

      iii. Analyze if the mechanism satisfies ex-post participation constraint.

2. **Asymmetric virtual valuations**. Consider two agents with cdfs $F_1(\theta_1) = a_1 (\theta_1)^2$ for agent 1, and $F_2(\theta_2) = a_2 (\theta_2)^2$ for agent 2, and where coefficients satisfy $a_1 \neq a_2$.

   (a) Find the virtual valuation of every agent i and check if it is increasing or decreasing in θ_i.

[16] For more details on this ironing technique, see Bolton and Dewatripont (2005, pp. 88-93).

(b) Compare virtual valuations $J_1(\theta_1)$ and $J_2(\theta_2)$ in the case that $a_1 = a_2 = a$, and in the case that $a_1 = 4$ and $a_2 = 2$.

3. **Regulating two utility companies** A regulator is responsible for two public utility companies located in separate geographic areas, e.g., two plants of water distribution. Each utility produces a fixed amount of output (normalized at $q = 1$) and has a cost function

$$C_i = \alpha + \beta_i - e_i$$

where α can be interpreted as a common shock affecting both firms' costs, and β_i is an idiosyncratic shock only affecting firm i's costs (e.g., rainfall in the area where firm i is located). Assume that the realization of β_i is independent of β_j. Furthermore, the effort that firm i exerts, $e_i \geq 0$, helps reduce its total cost.

In addition, social welfare is

$$\sum_i [S - (1+\lambda)(C_i + t_i) + \pi_i]$$

where S is a constant surplus that the regulator obtains from the fixed amount of output produced by each plant; t_i is the net transfer paid by the regulator to firm i; $\pi_i = t_i - g(e_i)$ is firm i's rent; and $\lambda > 0$ is the shadow cost of raising public funds (since the transfer paid to firm i originates from setting distortionary taxes in other markets). The reservation utility of each firm is 0. The effort function $g(e_i)$ originates at zero, $g(0) = 0$, and it is strictly increasing and convex in effort, i.e., $g' > 0$ and $g'' > 0$. The firm observes both α and β_i when signing the contract with the regulator.

(a) *Complete information.* Assume the regulator observes all components of each cost function, i.e., the realization of parameters α and β_i. Determine the optimal efforts, rents, and transfers.

(b) *Idiosyncratic shocks.* Suppose that there are no common shocks, i.e., $\alpha = 0$, and that both firms and regulator know this (i.e., it is common knowledge among all players). However, the regulator does not observe the realization of the idiosyncratic shock β_i. Instead, the regulator has prior beliefs that the realization of β_i is low, i.e., $\beta_i = \beta^L$, with probability p and high, $\beta_i = \beta^H$, with probability $1-p$, where $\beta^H > \beta^L$. In this setting, the regulator offers a menu of contracts (t^H, C^H) and (t^L, C^L) that maximize expected social welfare (as in a standard two-type screening problem). Write the social planner's problem, subject to the firm's incentive compatibility conditions and the participation constraints. Identify which of these four constraints are binding at the optimum. [*Hint*: It might be useful to start writing the ICs and PCs before writing the social planner's objective function. In addition, you may want to recognize that the cost function in this case is $C_i = \beta_i - e_i$, which helps us write effort as $e_i = \beta_i - C_i$.]

(c) Differentiate with respect to C^L and C^H in the regulator's program that you described in part (b), and obtain the optimal contract that the regulator offers to the firm.

(d) *Common shock.* Consider now the opposite case as in part (b). That is, suppose that $\beta_i = 0$ is common knowledge, i.e., idiosyncratic shocks are absent, but that α is known only to the firms and unobserved the regulator. In addition, assume that the regulator offers the following transfer

$$t_i = -\left(C_i - C_j\right) + g\left(e^*\right),$$

where every firm exerts the effort level that solves $g'\left(e^*\right) = 1$. Show that under this transfer every firm is induced to choose the effort level that solves $g'\left(e^*\right) = 1$ in the Nash equilibrium of the game. Then, find the equilibrium rents. [*Hint*: You don't need to solve the regulator's problem again. Instead, the transfer from the regulator is already given in the above expression, so you only need to solve the firm's problem of choosing an optimal effort level.]

(e) Argue that if the regulator uses the transfer in part (c), social welfare coincides with that under full information. Explain. [*Hint*: No math is necessary, and your explanation can be really short.]

4. **DRM between a government official and an expert.** Consider the president of a country (P) facing a binary decision $k \in \{-1, 1\}$, e.g., whether to sign a bill ($k = 1$) or not ($k = -1$). Similarly as in cheap-talk games, his utility depends on whether his decision coincides with the state of nature (so he deviates as little as possible from the true state of nature, θ). In particular, considering that the state of nature is also binary, $\theta \in \{-1, 1\}$, the president's utility is

$$u_P(k, \theta) = \begin{cases} 0 & \text{if } k = \theta, \text{ and} \\ -1 & \text{otherwise} \end{cases}$$

For simplicity, assume that both states of nature are equally likely. Before choosing k, the president talks to an expert (E). Specifically, the expert privately observes a noisy signal s of the true state of nature θ, where $s \sim N(\theta, 1)$, and given that signal s, the expert sends a message $m \in \mathbb{R}$ to the president. Hence, the president's choice can be described as a function of the message he receives from the expert, $k(m)$. The expert's utility is

$$u_E(k, \theta) = \begin{cases} 0 & \text{if } k = \theta, \\ -q & \text{if } \theta = 1 \text{ but } k = -1, \text{ and} \\ -(2 - q) & \text{if } \theta = -1 \text{ but } k = 1 \end{cases}$$

where parameter $q \in (0, 2)$ can be interpreted as the expert's bias towards one type of error. For instance, when $q = 0.1$, the expert's utility is relatively high (low) when the president chooses $k = -1$ ($k = 1$) when the true state of nature was $\theta = 1$ ($\theta = -1$, respectively). The opposite argument applies when $q \to 2$.

Finally, note that the above description represents an indirect revelation mechanism, in which the expert sends a message $m \in \mathbb{R}$ to the president, and the president responds with allocation function $k(m) \in \{-1, 1\}$, entailing the above utilities for the president and the expert depending on the profile of (k, θ)-pairs.

(a) Consider a direct revelation mechanism in which the expert announces his privately observed signal s, and for each signal s, the efficient allocation function $k^*(s) \in \{-1, 1\}$ that maximizes the president's expected utility, $E_\theta[u_P(k, \theta)]$. That is, $k^*(s) = 1$ if and only if $E_\theta[u_P(1, \theta)|s] \geq E_\theta[u_P(-1, \theta)|s]$. Find under which conditions on the signal s, $k^*(s) = 1$. [*Hint:* This should be short.]

(b) Assume in this part of the exercise that the expert's bias is exactly $q = 1$. Show that the above allocation rule $k^*(s)$ is incentive compatible. [*Hint:* This is easy, you don't need to do any math.]

(c) For the remainder of the exercise, assume that the expert's bias is exactly $q = \frac{1}{2}$. Consider an indirect revelation mechanism in which the expert sends a message m to the president, and that the president responds using the following allocation rule

$$k(m) = \begin{cases} 1 & \text{if } m \geq K, \text{ and} \\ -1 & \text{otherwise} \end{cases}$$

This is actually a simple *cutoff rule*: If the message m that the expert sends is sufficiently high (i.e., it is higher or equal to a cutoff $K \in \mathbb{R}$), the president responds signing the bill; but if the message is lower than cutoff K the president responds not signing the bill.

i. Show that this mechanism is incentive compatible if and only if the cutoff is exactly $K = \frac{\log 3}{2}$. [*Hint:* Recall that signals are normally distributed, implying a density function of $f(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right)$. Then, $\Pr\{s|\theta\} = f(s - \theta)$, which entails $\frac{f(s-\theta)}{f(s+\theta)} = \exp(2s)$.]

ii. Provide a short verbal explanation of your result.

5. **Procurement auction with/without external effects** Consider a town mayor inviting N firms to bid in a procurement contract that will allocate to the selected firm the right of water distribution for all town residents. The efficiency in implementing the project, θ_i, is observable to bidder i, but not other bidders or the procurer. Its distribution, $\theta_i \sim U[0, 1]$, is common knowledge. Bidders are regarded as more efficient when their efficiency parameter, θ_i, increases.

The cost of bidder i to implement the contract is $C_i(q_i, \theta_i)$, which is increasing and convex in output q_i, and decreasing and convex in efficiency θ_i. Each bidder has a quasilinear utility function,

$$U(q_i, \theta_i) = t_i(q_i) - C_i(q_i, \theta_i),$$

where $t_i(q_i)$ represents the transfer that the bidder receives from the procurer when the bidder produces q_i units of output (i.e., water). For simplicity, assume that bidders have zero reservation utility. The procurer's welfare function is

$$V(q_i) - (1 + \lambda) t_i(q_i)$$

where $V(q_i)$ denotes the value that the procurer assigns to q_i units of output, while λ captures the shadow cost of raising public funds (as the procurer needs to raise distortionary taxes in order to pay for the transfer $t_i(q_i)$ to bidder i).

(a) **No Externalities**. Let us assume that a bidder's (i.e. firm) output decision has no effects on the costs of other firms. Answer the following questions.

   i. In your opinion, what is the sign of the cross partial derivative, $\frac{\partial^2 C_i(q_i, \theta_i)}{\partial q_i \partial \theta_i}$? Justify.

   ii. Setup the procurer's program that induces participation and revelation of the bidders.

   iii. Solve for the optimal output and transfer of bidder i.

(b) **With Externalities**. Let us now assume that bidder i's cost depends on the output of other bidders. Specifically, consider $C_i(q_i, q_{-i}, \theta_i)$, where vector $q_{-i} = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_N)$ can be understood as the externalities from other bidders on bidder i, which can be positive or negative. Also, assume that $\frac{\partial^2 C_i(q_i, q_{-i}, \theta_i)}{\partial q_{-i} \partial \theta_i} \geq 0$ such that the efficiency of bidder i attenuates the effect of positive externalities other bidders impose on him.

   i. Write down the welfare maximization program of the procurer. (You may assume that the IC_i and IR_i conditions still hold so that you need not repeat them again.)

   ii. Solve for the optimal output and transfer of bidder i.

(c) Compare the results between parts (a) and (b) of the exercise. Also draw a figure to illustrate the difference in optimal output. (For simplicity, you may assume that externalities are positive).

(d) *Parametric example.* Let us now assume a parametric form for the value and cost functions in a setting with 2 bidders. In particular, assume that the cost function of bidder i is

$$C_i(q_i, q_j, \theta_i) = \frac{(q_i)^2}{2(r + \theta_i)} - \alpha q_j \qquad (4)$$

where $r > 0$ is a constant to make sure that cost will not diverge when efficiency approaches zero, and α calibrates the degree of externality the other bidder j impose on bidder i. We normalize this parameter, $\alpha \in [-1, 1]$, such that if (i) $\alpha \in (0, 1]$, externality is positive thus reducing the cost of bidder i; (ii) $\alpha = 0$, no externality on bidder i;

and (iii) $\alpha \in [-1, 0)$, the externality is negative thus increasing the cost of bidder i. Furthermore, the value that the procurer assigns to the output of bidder i is $V(q_i) = q_i$. To facilitate your calculations, let us assume that $r = \frac{1}{2}$ and $\lambda = \frac{1}{10}$, and solve for the optimal output and transfer of bidder i, considering two cases: (1) $\alpha = 0$, and (2) $\alpha = \frac{1}{3}$, and afterwards we will compare our results.

6. **Optimal Auction of Monopoly Rights** Consider the following model based on the article by Dana and Spier (1994)[17]. A regulator sells the monopoly right in an industry among two firms, $i = \{1, 2\}$. Assume that the social planner's objective is to maximize social welfare:

$$\begin{aligned} W &= CS + \sum_i (\pi_i - t_i) + \lambda \sum_i t_i \\ &= CS + \sum_i \pi_i - (1 - \lambda) \sum_i t_i \end{aligned}$$

where CS denotes consumer surplus; $PS = \sum_i (\pi_i - t_i)$ is the producer surplus net of transfers, where π_i represents the profit of firm i before paying a transfer t_i to the regulator; and λ represents the shadow cost of raising public funds through distortionary taxation, where $\lambda > 1$. Intuitively, a larger transfer allows the regulator to reduce taxes in other markets, thus reducing their distortionary effects.

The regulator determines which firm (or firms) obtain the production license, and the transfer from every firm i to the regulator, t_i for all $i = \{1, 2\}$. Once one (or both) firms obtain a production license it freely determines its profit-maximizing output. Each firm privately observes its fixed cost of production, θ_i, which is drawn from $[\underline{\theta}, \overline{\theta}]$ with cdf $\Phi(.)$ and positive density $\phi(\theta_i)$ for all θ_i and all $i = \{1, 2\}$. For simplicity, assume that $\frac{\Phi(\theta_i)}{\phi(\theta_i)}$ is increasing in θ_i. Both firms face a common marginal cost $c < 1$, and a demand function $p(x)$, where $p'(x) < 0$ and $p''(x) \leq 0$,, which is common knowledge among all players.

(a) *Complete Information.* Assuming that the regulator observes the profile $\theta = (\theta_1, \theta_2)$, describe the optimal contract.

(b) *Incomplete Information.* Consider now a setting in which the regulator cannot observe the profile $\theta = (\theta_1, \theta_2)$. Using the Revelation Principle we can restrict attention to truth-telling equilibrium in direct-revelation mechanisms. Hence, the government's mechanism $\{t, p\}$ including:

  (i) A transfer from each firm $t_i(\hat{\theta}_1, \hat{\theta}_2)$ for all $i = \{1, 2\}$; and

  (ii) A probability of implementing each market structure $p^i(\hat{\theta}_1, \hat{\theta}_2)$, where $i = \{1, 2, d\}$ denotes each of the three possible market structures, $\hat{\theta}_1$ and $\hat{\theta}_2$ are the announcements made by each firm to the regulator in the direct-revelation mechanism, and $p = (p^1, p^2, p^d)$ indicates the probability that firm 1, 2, or both, obtain the license.

[17] Dana, J.D. and Spier, K.E. 1994. "Designing a private industry: Government auctions with endogenous market structure." *Journal of Public Economics.* 53 (1). 127-147.

Analyze the optimal contract in this incomplete information context.

(c) *Parametric example.* Your above analysis left results in a relatively general format. Assume now that the inverse demand function is $p(x) = 1 - x$, total costs are $TC = \theta_i + cx$, where $c < 1$ represents marginal costs. Evaluate your above results using these parametric functions.

(d) *Numerical Example.* Consider that the marginal cost is $c = \frac{1}{4}$, and that types are uniformly distributed in $[0, 1]$, so that $\Phi(\theta_1) = \theta_1$ and $\Phi(\theta_2) = \theta_2$. (Densities are, therefore, $\phi(\theta_1) = 1$ and $\phi(\theta_2) = 1$.) In addition, assume that the realization of θ_1 is $\theta_1 = \frac{1}{2}$ and that of θ_2 is $\theta_2 = \frac{1}{3}$. Evaluate your previous results using these numerical values.

7. [**Pollution abatement as a mechanism**] Consider a polluting firm and a regulator, such as the Environmental Protection Agency (EPA) in the U.S., designing policies to reduce such pollution. In particular, assume that pollution, x, causes a damage measured by the damage function $D(x)$, which is strictly increasing and convex in pollution, i.e., $D' > 0$ and $D'' \geq 0$. The firm's cost function is represented by $C(x, \theta)$, being strictly decreasing and convex in pollution, $C_x < 0$ and $C_{xx} \geq 0$, (i.e., the firm invests less in clean technologies as its pollution increases) and increasing in the firm's inefficiency parameter θ, $C_\theta > 0$; alternatively, a lower value of θ represents a lower total cost. The firm privately observes the realization of parameter θ, but the regulator does not. Finally, assume that costs satisfy the single-crossing property $C_{x\theta} < 0$, and $C_{\theta xx} \geq 0$.

(a) Show that if the government has coercive power, it can obtain the socially optimal amount of pollution $x^*(\theta)$ by giving the firm a transfer equal to a constant minus the damage cost $D(x)$. How does this scheme link with the Groves mechanism?

(b) Suppose that the firm can refuse to participate (it has property rights and is free to pollute if it wants to). Can the first-best outcome described in part (a) still be implemented if the government cares about the sum of consumer and producer surplus? Next, suppose that the government faces a shadow cost of public funds $\lambda > 0$, so that its objective function is

$$W = -D(x) - (1 + \lambda)t + (t - C(x, \theta))$$

(up to a constant).Derive the optimal incentive scheme (Note: The IR level may be type-dependent.) Perform the analysis as if it were type independent and check ex post that everything is fine.

(c) *Parametric example.* Assume a damage function $D(x) = x^2$, where $D' = 2x > 0$ and $D'' = 2 > 0$ as required; and a cost function $C(x, \theta) = \frac{\theta}{x^2}$ where $C_\theta = \frac{1}{x^2} > 0$, $C_{x\theta} = -\frac{2}{x^3} < 0$, $C_x = -\frac{2\theta}{x^3} < 0$, $C_{xx} = \frac{6\theta}{x^4} > 0$, and $C_{\theta xx} = \frac{6}{x^4} > 0$, as required.

Evaluate the FOC you found in part (b) using these $D(x)$ and $C(x,\theta)$ functions, and find the optimal pollution level $x^*(\theta)$, and the optimal transfer $t_i^*(\theta)$.

8. **Applying mechanism design in a monopoly problem** Consider a monopolist facing a single consumer with utility function $u = \theta q - t$, where q denotes units of the good sold by the monopolist and t is the transfer to the monopolist (i.e., lump-sum payment to the monopolist for the q units that the consumer receives, rather than a price per unit). Intuitively, parameter $\theta > 0$ represents the marginal utility that the consumer obtains from every additional unit of the good he consumes. The monopolist has cost function $\frac{cq^2}{2}$, where $c > 0$, and offers a sales contract to the consumer. For simplicity, assume that the consumer has reservation utility 0.

   (a) *Complete information.* Let us start considering that the monopolist observes the realization of parameter θ. Find the optimal transfer and consumption under complete information, $(t(\theta), q(\theta))$.

   (b) *Incomplete information.* Suppose from now on that the monopolist has incomplete information about θ, which takes the value θ^L with probability p^L and θ^H with probability p^H. Assume that $\theta^L > p^H\theta^H$. The monopolist's expected profit is

$$p^L\left(t^L - c\frac{\left(q^L\right)^2}{2}\right) + p^H\left(t^H - c\frac{\left(q^H\right)^2}{2}\right)$$

   Compute the optimal contract. Show that the equilibrium utility of type-θ^H consumer is $\overline{S} = \frac{(\theta^H - \theta^L)(\theta^L - p^H\theta^H)}{cp^L}$.

   (c) Suppose now that the consumer can purchase at the fixed cost f an alternative (bypass) technology that allows him to produce any amount q of the same good at cost $\frac{\tilde{c}q^2}{2}$. Suppose for simplicity that the consumer can consume only the monopolist's good or the alternative good (but not a mix of both), and that

$$\frac{\left(\theta^H\right)^2}{2\tilde{c}} - f > \overline{S} > 0 > \frac{\left(\theta^L\right)^2}{2\tilde{c}} - f$$

   Is the transfer that you found in part (b) still optimal for the monopolist? Discuss what may be optimal for monopolist. In particular, why it may be optimal to have $cq^H > \theta^H$. For example, consider what happens when f decreases from $\frac{\theta^{H2}}{2\tilde{c}} - \overline{S}$.

9. **Designing optimal taxation.** Consider a government needing to raise a fixed sum \$$S$ through income tax. There are two types of workers, high productivity (H) and low productivity (L), and the output (gross income) produced by each is given by

$$q^k = \theta^k e^k, \text{ where } k = H, L$$

where e^k is the amount of effort exerted by a worker of type k and the productivity parameter satisfies $\theta^H > \theta^L$. Hence, for a given effort level, the high-productivity worker generates a larger amount of output than the low-productivity worker. The utility function of a worker with type k is

$$v^k = q^k - t^k - g(e^k)$$

where t^k is the tax on a worker of type k, and $g(\cdot)$ is a strictly increasing and convex function in effort, i.e., $g' > 0$ and $g'' > 0$. The government has no interest in the inequality of utility outcomes and so just seeks to maximize the expected social welfare

$$W = pv^H + [1-p]v^L$$

where p is the proportion of H-type workers.

(a) *Complete information.* If the government was perfectly informed about the worker's type, find the socially optimal taxes, and the associated output levels.

(b) *Parametric example (Complete information).* Assume that the cost of effort function is $g(e^k) = (e^k)^2$, so its derivatives are $g' = 2e^k \geq 0$ and $g'' = 2 > 0$; as required. Evaluate the FOCs found in part (b) for the complete information context assuming that productivity parameters are $\theta^H = 1$ and $\theta^L = \frac{1}{2}$. Find the optimal values of q^H and q^L.

(c) *Incomplete information.* Assuming that the government cannot observe the worker's type, write the government's objective function in terms of q^H, q^L, p, and S.

(d) Using the government's objective function you identified in part (d), write down the government's optimization problem.

(e) Find the solution to the government's problem in part (d). Compare your answer to the complete information solution found in part (b).

(f) *Parametric example (Incomplete information).* Assume the same cost of effort function as in the parametric example developed in part (c), $g(e^k) = (e^k)^2$, and the same set of productivity parameters $\theta^H = 1$ and $\theta^L = \frac{1}{2}$. In addition, consider that both types of workers are equally likely, i.e., $p = \frac{1}{2}$. Find the optimal values of q^H and q^L in the incomplete information setting. Then, find the optimal y^H and y^L, where $y^k \equiv q^k - t^k$.

10. **Virtual valuations.** Consider an auction between two bidders, 1 and 2; and let θ_i denote bidder i's privately observed valuation for the good

(a) *Symmetric distributions.* Assume that θ_i is drawn from a uniform distribution, $\theta_i \sim U[0,1]$. Find the virtual valuation $J_i(\theta_i)$ of every bidder i, and determine which bidder (if any) receives the object in an optimal auction (i.e., an auction maximizing the seller's expected revenue).

(b) *Asymmetric distributions.* Let us now assume that the valuation of bidder 1, θ_1, behaves according to cdf $F_1(\theta_1) = (\theta_1)^2$, whereas that of bidder 2 is $F_2(\theta_2) = 2\theta_2 - (\theta_2)^2$; and $\theta_i \in [0, 1]$ for every bidder i. Find the virtual valuation $J_i(\theta_i)$ of every bidder i, and determine which bidder (if any) receives the object in an optimal auction.

11. **Virtual valuations-II.** Consider a mechanism with virtual valuation $J(\theta_i) = \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$. Show that if the density function $f(\theta_i)$ is increasing in θ_i, then the virtual valuation $J(\theta_i)$ must also be increasing in θ_i.

12. **Virtual valuations-III.** Show that if the distribution of types for agent i dominates that agent j in hazard rate terms, then agent i also dominates agent j in terms of their virtual valuations.

13. **Public Good Provision.** Imagine that you and your colleagues want to buy a coffee machine for your office. Suppose that some of you may be heavily addicted to coffee and are willing to pay more for the machine than the others. However, you do not know your colleagues' willingness to pay for the machine. The cost of the machine is C. We would like to find a decision rule in which (i) each individual reports a valuation (i.e., direct mechanism), and (ii) the coffee maker is purchased if and only if it is efficient to do so. Let us next analyze if it is possible to find a cost-sharing rule which gives incentive for everyone to report his valuation truthfully.

    In particular, assume n individuals, each of them with private valuation $\theta_i \sim U(0, 1)$. The allocation function is binary $y \in \{0, 1\}$, i.e., the coffee machine is purchased or not. Let t_i be the transfer from individual i, implying a utility of

    $$u_i(y, \theta_i, t_i) = y\theta_i - t_i$$

    Let $i \in \{1, ..., n\}$ denote the individuals, and let $i = 0$ denote the original owner of the good.

    (a) What is the efficient assignment rule, $y^*(\theta_1, ..., \theta_n)$?

    (b) *Equal-share rule.* Consider the following equal-share rule: When the public good is provided, the cost is equally divided by all n individuals.

    i. Before starting any computation, what would you expect - whether each individual would overstate or understate their valuation?

    ii. Confirm that the transfer rule is written by:

    $$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

    iii. Let $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})$ be individual i's payoff when i reports $\tilde{\theta}_i$ instead of his true valuation θ_i, while the others truthfully report their valuations θ_{-i}. Show that

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

iv. Let $U_i\left(\tilde{\theta}_i|\theta_i\right)$ be individual i's expected payoff when he reports $\tilde{\theta}_i$ instead of the true valuation θ_i. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

v. Suppose that i's private valuation θ_i satisfies $\theta_i > \frac{C}{n}$. Assuming that the others are telling the truth, what is the best response for i? What if $\theta_i < \frac{C}{n}$? Is this mechanism strategy-proof? Is this mechanism Bayesian incentive compatible?

(c) *Proportional payment rule.* Consider now the proportional payment rule:

$$t_i(\theta) = \frac{\theta_i C}{\sum_j \theta_j} y^*(\theta)$$

where every individual i pays a share of the total cost equal to the proportion that his reported valuation signifies out of the total reported valuations.

i. Under this rule, what would you expect - whether each individual would overstate or understate the valuation?

ii. Show that the utility of reporting $\tilde{\theta}_i$ is now

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

iii. For simplicity, suppose two individuals, $n = 2$ and a total cost of $C = 1$. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \tilde{\theta}_i\left(\theta_i - \log\left(\tilde{\theta}_i + 1\right)\right)$$

iv. Is this mechanism strategy-proof? Is it Bayesian incentive compatible?

v. Which way is everyone biased, overstate or understate? What is the intuition?

(d) *VCG mechanism.* Let us consider now the VCG mechanism. Recall that the efficient rule $y^*(\theta)$ determines that the coffee machine is bought if and only if total valuations satisfy $\sum_i \theta_i \geq C$. Remember that we need to include the original owner of the public good; $i = 0$. Then, the total surplus when the valuation of individual i is considered in $\theta = (\theta_1, \theta_2, ..., \theta_n)$ is

$$\sum_{j\neq i} v_j\left(y^*(\theta), \theta_j\right) = \begin{cases} \sum_{j\neq i}\theta_j & \text{if } \sum_j \theta_j \geq C \\ C & \text{if } \sum_j \theta_j < C \end{cases}$$

while total surplus when the valuation of individual i is ignored, θ_{-i}, is

$$\sum_{j\neq i} v_j\left(y^*(\theta_{-i}),\theta_j\right) = \begin{cases} \sum_{j\neq i}\theta_j & \text{if } \sum_{j\neq i}\theta_j \geq C \\ C & \text{if } \sum_{j\neq i}\theta_j < C \end{cases}$$

The only difference in total surplus arises from the allocation rule which specifies that, when θ_i is considered, the good is purchased if and only if $\sum_j \theta_j \geq C$, whereas when θ_i is ignored, the good is bought if and only if $\sum_{j\neq i}\theta_j \geq C$. Hence, the VCG transfer is

$$\begin{aligned} t_i^*(\theta) &= -\left(\sum_{j\neq i} v_j(y^*(\theta),\theta_j) - \sum_{j\neq i} v_j(y^*(\theta_{-i}),\theta_j)\right) \\ &= \begin{cases} C-\sum_{j\neq i}\theta_j & \text{if } \sum_{j\neq i}\theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Intuitively, player i pays the difference between everyone else's valuations, $\sum_{j\neq i}\theta_j$, and the total cost of the good, C. Such a payment, however, only occurs when aggregate valuations exceed the total cost, $\sum_j \theta_j \geq C$, and thus the good is purchased, and when the valuations of all other players do not yet exceed the total cost of the good, $\sum_{j\neq i}\theta_j < C$, so the difference $C-\sum_{j\neq i}\theta_j$ is paid by player i in his transfer.

i. Show that in this mechanism player i's utility from reporting a valuation $\tilde{\theta}_i \neq \theta_i$ is

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i,\theta_{-i}) &= v_i\left(y^*\left(\tilde{\theta}_i,\theta_{-i}\right),\theta_i\right) - t_i^*\left(\tilde{\theta}_i,\theta_{-i}\right) \\ &= \begin{cases} 0 & \text{if } \tilde{\theta}_i + \sum_{j\neq i}\theta_j < C \\ \sum_j \theta_j - C & \text{if } \sum_{j\neq i}\theta_j < C \leq \tilde{\theta}_i + \sum_{j\neq i}\theta_i \\ \theta_i & \text{if } C \leq \sum_{j\neq i}\theta_j \end{cases} \end{aligned}$$

ii. Is this mechanism strategy-proof? Is this Bayesian incentive compatible?

iii. For simplicity, suppose two individuals, $n=2$, and a total cost of $C=0.5$. Compute y^*, t_1^* and t_2^* for the following (θ_1,θ_2) pairs.

θ_1	θ_2
0.1	0.3
0.3	0.3
0.3	0.8
0.8	0.8

iv. Show that the expected revenue from this mechanism is $E\left[t_1^*(\theta_1,\theta_2)+t_2^*(\theta_1,\theta_2)\right] = \frac{1}{6} \simeq 0.167$. Based on what you calculated in part (iii), is this problematic?

14. **[Implementation of Efficient Public Good Provision by Charging Pivotal Agents]** Suppose that agent i's value for a good being auctioned, θ_i, is a random variable with support

$[0, \beta_i]$. Each agent submits a bid $\tilde{\theta}_i$. The public good (which costs C to produce) is produced if total bids are larger than the production cost, $\sum_j \tilde{\theta}_j \geq C$. If this condition is not satisfied the agents pay nothing. If $C - \sum_{j \neq i} \tilde{\theta}_j - \beta_i \leq 0 < C - \sum_{j \neq i} \tilde{\theta}_j$ the public good is produced if and only if agent i's value is sufficiently high. Such an agent is said to be "pivotal." Define a transfer

$$t(\tilde{\theta}_{-i}) = \max \left\{ 0, C - \sum_{j \neq i} \tilde{\theta}_j \right\}$$

If agent i is pivotal and has submitted a bid above this transfer he pays t_i. Otherwise agent i pays nothing.

(a) Show that if agent i bids his value, his payoff is a function of $\sum_{j \neq i} \theta_j$.

(b) Draw the payoff graph considering that: (i) $\tilde{\theta}_i < \theta_i$, and (ii) $\tilde{\theta}_i > \theta_i$.

(c) Explain why it is a dominant strategy for agent i to bid his value.

15. **Using mechanism design in monopoly pricing** Consider a monopolist with costs $c > 0$ and multiple consumers with types $\theta > 0$. The consumers have utility functions $\theta v(x) - t$ where x is the amount of the good consumed and $v'(\cdot) > 0$ $v''(\cdot) < 0$. θ is distributed across the support $[\underline{\theta}, \bar{\theta}]$ with $\bar{\theta} > \underline{\theta} > 0$ distributed with a cdf $\Phi(\cdot)$ with positive density $\phi(\cdot) > 0$. In addition, consider the buyer, who we will denote as agent $i = 1$. His utility function is

$$u^1(\theta, x, t) = \theta v(x) - t$$

Hence, its first-order derivative with respect to θ is $u^1_\theta(\theta, x, t) = v(x)$ and its second order derivative with respect to $x\theta$ is $u^1_{x\theta}(\theta, x, t) = v'(x) > 0$. We thus have that the single-crossing property is satisfied, i.e., the marginal utility of additional units of x is increasing in the buyer's type θ.

Next, consider the seller, who we will denote as agent $i = 0$. His utility function is

$$u^0(\theta, x, t) = t - c \cdot x$$

Using the Revelation Principle we can focus only on Direct Revelation Mechanisms $f(\theta) = (x(\theta), t(\theta))$ that solves the seller's maximization problem:

$$\max_{x(\theta), t(\theta)} \quad E[t(\theta) - c \cdot x(\theta)]$$

subject to the SCF $f(\theta) = (x(\theta), t(\theta))$ being Bayesian Incentive Compatible (BIC) and Individually Rational (IR). Let's denote by $U(\theta) = \theta v(x(\theta)) - t(\theta)$ the expected utility of the buyer when truthfully revealing his type θ.

(a) Set up the monopolist's constrained maximization problem.

(b) Solve the monopolist's problem, i.e., find an optimal output function $x(\theta)$.

(c) If you did not check for sufficiency of the optimal $x(\theta)$ in part (b), check for it now.

(d) Interpret the first-order condition for the optimal $x(\theta)$ found in part (b), evaluating them for the individual with the highest valuation and all other individuals.

(e) *Parametric example.* Assume that valuations are distributed according to a uniform distribution, $\theta \sim U[0,1]$, that $v(x) = \ln x$, and that $c = 1/4$. Evaluate the first-order condition for the optimal $x(\theta)$ found in part (b).

16. **Mutual insurance system, from Cabrales et al., 2003.**[18] Consider that there are N farms in the country of Andorra (a small country in the Pyrenees, between Spain and France). Farms have historically participated in a mutual insurance system called *La Crema*. Each farm has an initial wealth, w_i, which is a one-dimensional aggregator of insurable assets such as farm, barn, cowshed, stable, etc. For simplicity, assume that when an insurable event, such as fire, occurs, all the insurable assets are consumed so that farm i ends up with zero wealth in the absence of insurance. Therefore, there are two states of nature associated with each farm i, $s_i = 1$ corresponding to a fire outbreak and $s_i = 0$ corresponding to no outbreak, respectively. As a consequence, there are $S = 2^N$ possible states for different combination of fire outbreaks, where $s \in S$ denotes the list of farms in which a fire breaks out. For example, if $s = \{6, 8, 9\}$, then farms 6, 8 and 9 are burnt, while the other farms remain intact.

Farm owners can insure themselves against fire outbreak by participating in *La Crema*. The mechanism works as follows. Each farm announces an amount to be insured, q_i, and the sum of reported values across all farms is $Q = \sum_{i=1}^{N} q_i$. When a fire occurs with a list of farms s, let Q_s be the residual value of remaining farms whose assets are not consumed by fire. For example, $Q_s = Q_{\{6,8,9\}}$ represents the sum of reported values of all farms but 6, 8 and 9, such that the total value consumed by fire is $Q - Q_{\{6,8,9\}}$. In such case, farm $i \in s$ receives a compensation of the reported value of his assets as a fraction of the residual value, that is, $q_i \frac{Q_s}{Q}$. For the other farms that remain intact, farm $j \notin s$ pays a contribution of the reported value of his assets as a fraction of the value lost, that is $q_j \frac{Q - Q_s}{Q}$. In the case that no farms are burnt, $s = \varnothing$, no transfers are made across farms. Also, when all farms are burnt, $s = N$, all farms are burnt, no residual value $Q - Q_s$ is left for re-distribution among farm owners.

(a) Show that the above mechanism satisfies budget balance.

(b) Let p_i be the probability of a fire outbreak for farm i, which is assumed to be independently and identically distributed across all farms. Also assume that farms have the same initial wealth, that is, $w_i = w_j = w$ for all $i \neq j$, $i, j \in N$, which is common knowledge among all farms. Assuming that at most one farm will be burnt, does a risk-adverse farm i participate?

[18] Cabrales A., Calvó-Armengol A. and Jackson M. (2003). La Crema: A Case Study of Mutual Fire Insurance. *Journal of Political Economy,* 111(2), pp. 425-58.

(c) Let $u_i(w) = \sqrt{w}$. Will $N = 3$ farms with an initial wealth of $w = 1$ and probabilities of fire outbreak, $p_1 = \frac{1}{12}$, $p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{3}$ respectively, participate?

(d) Assume that $w_i \neq w_j$, and that the initial wealth is private information for farm i only. However, every farm i knows the common distribution of initial wealth, that is, $w_j \sim F(w)$ where $w \in [0, W]$, such that the expected wealth of farm $j \neq i$ is given by $\overline{w} = \int_0^W w dF(w)$. Assuming all farm participate, is it incentive compatible for every risk-neutral farm i, where $u_i(w_i) = w_i$, to report its wealth, that is, $q_i = w_i$? (For simplicity, you may assume that N is sufficiently large so that any reporting by farm i has no impact on the aggregate reported wealth, Q.)

(e) Let $w_1 = 4$, $w_2 = 3$ and $w_3 = 2$, will farm i over-report or under-report its wealth?

(f) Let us allow any number of farms to be burnt. Assume symmetry in wealth $w_i = w_j = w$ and also in the probabilities of fire outbreak $p_i = p_j = p$, and that both parameters are observable. Does a risk-neutral farm i participate? Show your results.

i. What is the expected wealth of farm i without insurance?

ii. Assume that k farms are burnt that do not include farm i. How much is farm i expected to contribute?

iii. Assume that k farms are burnt that include farm i. How much is farm i expected to be compensated?

iv. What is the expected wealth of farm i with insurance?

v. Compare the expected wealth with and without insurance. Does this system work?

vi. Assume a probability $p = \frac{1}{6}$. Will farm i participate?

17. **Emission fees and mechanisms, Duggan and Roberts, 2003.**[19]**.** Consider an industry with N polluting firms producing a homogenous good. Let the profit function of firm i be $\pi_i(q_i) = \ln q_i$, which is increasing and concave in its pollutants q_i. The social cost from pollution is

$$C(q_1, \cdots, q_n) = \sum_{i=1}^{n} \frac{\gamma_i}{2} q_i^2,$$

which is also increasing but convex in the pollutants q_i emitted by firm i. Finally, a regulator (e.g., government agency) considers the following welfare function

$$W(q_1, \cdots, q_n) = \sum_{i=1}^{n} \pi_i(q_i) - C(q_1, \cdots, q_n)$$

(a) *Complete information.* Assume that the regulator can observe pollution levels and sets an emission fee t_i per unit of emissions. Find the following:

i. Firm i's profit-maximizing pollution level as a function of fee t_i, $q_i(t_i)$.

[19] Duggan J. and Roberts J. (2003). Implementing the Efficient Allocation of Pollution. *American Economic Review,* 92(4), pp. 1070-78.

ii. The socially optimal pollution from firm i, q_i^{SO}.

iii. The emission fee t_i that induces firm i to produce q_i^{SO}, i.e., the fee t_i that solves $q_i(t_i) = q_i^{SO}$.

(b) *Incomplete information.* Assume that the level of pollution is unobservable to the regulator but observable among all firms. Then, the regulator can devise a circular monitoring mechanism, in which firm i reports the observed pollution level of firm $i-1$, $\overline{q}_{i-1}$, firm $i-1$ reports the observed pollution of firm $i-2$, $\overline{q}_{i-2}$, and firm 1 reports that of firm n, $\overline{q}_n$. This allows the regulator to set an emission fee per unit of pollution

$$t_i = \frac{\partial C\left(\overline{q}_i, q_{-i}\right)}{\partial q_i},$$

where $\overline{q}_i$ denotes firm i's pollution (reported by firm $i+1$), and q_{-i} represents the true pollution level of all other firms. In addition, firm i faces a penalty of $\left(\overline{q}_{i-1} - q_{i-1}\right)^2$ for misporting his neighbor's pollution level not at q_{i-1}.

i. Will firm i misreport the output of firm $i-1$? Why or why not?

ii. Write down firm i's profit-maximization problem and solve for its optimal output.

iii. Find the tax revenue generated by the mechanism, and the social cost of pollution.

(c) *Numerical example.* Consider the case of three firms, where $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{1}{9}$ and $\gamma_3 = \frac{1}{16}$, respectively. What are the socially optimal levels of pollution and the corresponding optimal emission fees? What are the social costs of pollution from each firm? And the total taxes paid by each firm?

18. **Procurement auctions with perfect monitoring.** Consider N bidders bidding for a procurement contract, for example, a conservation project that restores the wetland to provide a habitat for migratory birds. Each bidder i, where $i \in \{1, \cdots, n\}$, is endowed with a certain acreage of farmland available for conservation. We summarize the expertise of bidder i in implementing the conservation project and the biodiversity value of his farmland into a uni-dimensional measure of efficiency, θ_i, which is observable to bidder i but not to the other bidders or the procurer. This efficiency parameter is uniformly distributed, $\theta_i \sim U[0,1]$, is common knowledge among bidders. Bidder i's cost of implementing the project is determined by the acreage of farmland dedicated to wetland conservation q_i (herein denoted as the "input") and his efficiency θ_i, that is,

$$C_i\left(q_i, \theta_i\right) = \frac{\left(q_i\right)^2}{2\left(1+\theta_i\right)}$$

where the cost of conserving zero units of land is $C_i\left(0, \theta_i\right) = 0$, for there is no fixed cost. The cost of conservation is decreasing in the efficiency parameter, θ_i.

Each bidder i has a quasilinear utility function,

$$U(q_i, \theta_i) = t_i(q_i) - C_i(q_i, \theta_i),$$

where $t_i(q_i)$ represents the transfer he receives from the procurer. For simplicity, we assume a zero reservation utility for bidder i. The procurer's valuation of bidder i's conservation is $V(q_i) = A \ln q_i$, where $A > 0$ indicates the intensity of the procurer's valuation.

(a) First, we examine the properties of the cost function $C_i(q_i, \theta_i)$. Show that conversion cost is: (1) increasing and convex in the input q_i; (2) decreasing and convex in efficiency θ_i; and (3) that it satisfies the Spence-Mirrlees sorting condition (also known as the single-crossing condition).

(b) Second, we investigate the properties of the procurer's valuation function $V_i(q_i)$. Show that the valuation function is increasing and concave in input q_i.

(c) *Case 1. Perfect monitoring and observable efficiency.* In case 1, assume that both efficiency and input are observable. Write down the individual rationality constraint for bidder i to participate. Let λ, where $0 \leq \lambda \leq 1$, be the shadow cost of raising public funds. What is the procurer's welfare function? Solve for the optimal input and transfer of bidder i.

(d) *Case 2. Perfect monitoring, unobservable efficiency.* In case 2, assume that input is observable but efficiency is not. That is, the procurer can perfectly observe the amount of land being conserved by every bidder, but cannot observe each bidder's efficiency parameter. Write down the procurer's problem in this setting, and solve for the optimal input and transfer of bidder i.

19. **Procurement auctions with imperfect monitoring.** Consider the procurement auction analyzed in Exercise #18. However, we will now allow for monitoring to be imperfect, that is, the procurer cannot perfectly observe the amount of land they conserve (i.e., whether bidder i fully implements the contract by conserving q_i or a smaller amount $\hat{q}_i < q_i$). In this context, the procurer needs to monitor every bidder i to deter him from investing less than the contracted level of input, or not investing at all. In particular, let α_i be the probability of monitoring, where $0 \leq \alpha_i \leq 1$. For simplicity, we assume that bidder i will be detected with 100% certainty if he is monitored, and will not be detected if not monitored. Hence, if bidder i cheats, he is detected with probability α_i, receiving a transfer $t_i(\hat{q}_i)$ which is a function of his level of under-investment $\hat{q}_i$, where $\hat{q}_i < q_i$. Last, assume that the monitoring cost is $m_i(q_i) = \frac{\gamma}{2}(q_i)^2$, where $\gamma \in [0,1]$, implying that the cost of monitoring is increasing and convex in q_i, that is, $\frac{\partial m_i}{\partial q_i} = \gamma q_i \geq 0$ and $\frac{\partial^2 m_i}{\partial (q_i)^2} = \gamma \geq 0$.

(a) *Case 3: Imperfect monitoring, observable of efficiency.* Consider the opposite scenario of Case 2 in Exercise #18, that is, the procurer can observe bidders' efficiency but cannot perfectly observe the amount of land they conserve.

i. What is the expected utility of bidder i if he cheats?

ii. What is the expected utility of bidder i if he does not cheat?

iii. What is the incentive compatibility condition for bidder i not to cheat?

iv. What is the individual rationality condition for bidder i not to refrain from participation?

v. What conditions do we need on probability α_i so that bidder i participates and does not cheat?

vi. Write down the procurer's welfare maximization problem and the constraints.

vii. What is the transfer function of the procurer? And, what is the optimal level of bidder i's cheating?

viii. What is the procurer's optimal monitoring probability α_i?

ix. Using your above results, solve for the optimal input and transfer of bidder i.

(b) *Case 4. Imperfect monitoring, unobservable efficiency.* We finally assume that both efficiency and input are not observable by the procurer.

i. Write down the procurer's welfare maximization problem and the constraints.

ii. What is the optimal level of bidder i's cheating?

iii. What is the procurer's optimal monitoring probability?

iv. Find the optimal input and transfer of bidder i.

(c) *Numerical example.* Consider parameter values $A = 3$, $\lambda = \frac{1}{10}$, and $\gamma = \frac{1}{20}$. Solve for the optimal input, transfer and stochastic monitoring probability in Cases 1-4 when efficiency is low at $\theta_i = \frac{1}{4}$ and high at $\theta_i = \frac{3}{4}$ respectively. Compare your results.