Chapter 1: Introduction to Games and their Representation

Game Theory:

An Introduction with Step-by-Step Examples

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Introduction/Roadmap

- What’s game theory.
- Main elements in strategic settings (in games).
- Two main graphical approaches to represent games:
  - Matrices
  - Game trees
- Introducing Imperfect Information in Game Trees
- Identifying Equilibrium Behavior

What is game theory?

Game Theory studies the interaction between a group of rational agents who behave strategically.

Let's analyze each underlined element of this definition.

1. *Group of individuals*.
    - Game theory studies interaction between a group of agents.
    - Scenarios with one individual or firms are analyzed with individual decision making, not with games.

What is game theory?

2. *Rationality*. Every agent seeks to maximize her objective function.
   - Common knowledge of rationality. In a two-player game:
     - Player 1 seeks to maximize her payoff,
     - Player 2 seeks to maximize her payoff,
     - Player 1 knows that player 2 seeks to maximize her payoff,
     - Player 2 knows that player 1 seeks to maximize her payoff,
     - Player 1 knows that player 2 knows that player 1 seeks to maximize her payoff,…
     - And so on, ad infinitum.
     - Intuitively, everyone can put herself in the shoes of her rival, anticipating her moves.

   - Common knowledge of rationality in "The Princess Bride," movie scene: www.youtube.com/watch?v=9s0UURBihH8.

What is game theory?

3. *Strategic behavior*. Every agent seeks to maximize a well-defined objective function.
    - The objective function can be a utility function of an individual, a profit function of a firm, or a social welfare function of an entire country.
    - This allows for the agent to be selfish (if his objective function only includes his own payoffs) or altruistic (if it contains payoffs from other individuals).
    - Examples of other-regarding preferences:
        - Envy and guilt aversion in Fehr and Schmidt (1999) and Bolton and Ockenfels (2000),
        - meaning that players can be both altruistic and strategic at the same time.

Main elements in a game

1. *Players*. Set of agents, such as individuals, firms, or countries, interacting in a game.
   - For instance, if two firms compete in an industry, we say that the number of players is *N*=2.
   - And if a generic number firms interact, we write that the number of players is $N \geq 2$.
2. *Strategy.* Complete contingent plan, describing which actions a player chooses in each possible situation (each contingency) that she faces along the game.
   - In a two-player game with firms A and B, we denote firm A's strategy as s_A, where $s_A \in S_A$, meaning that firm A selects a specific strategy s_A (such as a price of $12) from a set of available strategies S_A, known as the "strategy set."

Main elements in a game

2. *Strategy (cont'd).*
   - Strategies can be understood as an instruction manual: a player opens the manual, looks for the page describing the actions other players chose, and the manual indicates how to respond, recommending that she chooses a specific action s_A.
   - **Discrete strategies**. $S_A = \{6,10\}$ if binary, $S_A = \{1,2, \ldots\}$ otherwise.
     - Examples are output levels, or other strategies that require being natural numbers.
   - **Continuous strategies**. $S_A = [0,10]$ if bounded meaning that the agent can choose non-integer amounts from 0 to 10 (e.g., $s_A = 5.7$) , or $S_A = [0, +\infty)$ or $S_A = \mathbb{R}_+$ if unbounded.
     - Examples can be prices or bids in an auction, where dollars can be split into cents, or cents of a cent.

Main elements in a game

2. *Strategy (cont'd).*
   - **Symmetric strategy sets**. If $S_i = S_j = S$ for every player $j \neq i$.
     - Example: firms competing in the same industry have access to the same technology, so every firm chooses an output level from the same strategy set $S = [0,10]$.
   - **Asymmetric strategy sets**. If $S_i \neq S_j$ for at least one player $j \neq i$.
     - Example: firms competing in the same industry have access to different technologies, so $S_i = [0,10]$ but $S_j = [0,75]$.

Main elements in a game

2. *Strategy (cont'd).*

- **Strategy profile**. A list describing the strategies that each player selects, $s = (s_1, s_2, \dots, s_N)$.
  - Examples: $S = (12{,}8)$, meaning that firm A chooses $s_A = 12$ and firm B selects $s_B = 8$ units of output.
  - In a setting with N players, a strategy profile is $s \equiv (s_1, s_2, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_N)$.
  - For compactness, we write $s = (s_i, s_{-i})$ where $s_{-i} \equiv (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$ denotes the strategy profile chosen by player *i*'s rivals.
- Example:
  - In a setting with four firms, $s = (12{,}8{,}10{,}13)$, so $s = (12, s_{-1})$ where $s_{-1} = (8{,}10{,}13)$ denotes the output of firm 1's rivals.

Main elements in a game

2. *Strategy (cont'd).*

- Therefore, strategy profile $s = (s_i, s_{-i})$ is an element in the Cartesian product $S_i \times S_{-i}$.
  - In the case of a two-player game where every player has binary strategy sets, this Cartesian product has 2×2 different strategy profiles. Check!
  - Still in a two-player game, where player 1 has k available strategies (i.e., the cardinality of S_1 is k) and player 2 has m available strategies, there are $k \times m$ strategy profiles.
- If players choose from ***continuous*** strategy spaces, where every $S_i = \mathbb{R}_+$, then:
  - $S_A \times S_B = \mathbb{R}_+^2$ in the case of two players (A and B); and
  - $S_1 \times S_2 \times \cdots \times S_N = \mathbb{R}_+^N$ in the case of N players.

Two graphical approaches

Matrices

		Player 2	
		h	l
Player 1	H	4,4	0,-7
	L	8,1	2,2

Matrix 1.1. Example of a two-player game.

- Player 1 typically chooses rows and is referred to as the “row player” (strategies denoted with uppercase letters, H and L)
- Player 2 chooses columns and is known as the “column player” (strategies denoted with lowercase letters, h and l)

Two graphical approaches

Matrices

		Player 2	
		h	l
Player 1	H	4,4	0,-7
	L	8,1	2,2

Matrix 1.1. Example of a two-player game.

- In Matrix 1.1, if player 1 chooses high prices, H, while player 2 picks low price, l, their payoff corresponds to $(0, -7)$,
  - Indicating that player 1's payoff is 0 and player 2's is -7.
  - Negative payoffs may represent, for instance, that the player's revenue if lower than her costs, thus losing money.

Two graphical approaches

Matrices

		Player 2	
		h	l
Player 1	H	4,4	0,-7
	L	8,1	2,2

Matrix 1.1. Example of a two-player game.

- This matrix shows that every player has only two available strategies, implying that:

$$S_1 = \{H, L\} \text{ and } S_2 = \{h, l\}$$

  so each player faces a binary decision.

Two graphical approaches

Matrices

Player 2

		h	l
Player 1	H	4,4	0,-7
	L	8,1	2,2

Matrix 1.1. Example of a two-player game.

- In other games, however:
  - A player could have more strategies than her rival, a rich strategy space with hundreds of possible strategies to choose from, or
  - Continuous strategy spaces (such as when firms compete in prices).

Two graphical approaches

Matrices

		Player 2	
		h	l
Player 1	H	4,4	0,-7
	L	8,1	2,2

Matrix 1.1. Example of a two-player game.

- When players interact choosing their strategies simultaneously but the number of available strategies in S_i or S_j, or both, makes the matrix too large to represent graphically…
  - we will analyze the game keeping in mind that players interact in a context which is, essentially, equivalent to that of a matrix form.

Two graphical approaches

Game tree

- When players act sequentially (one after the other), we frequently use game trees.
  - They can help illustrate who acts when,
  - representing which information a player observes in each part of the game.

Two graphical approaches

Game tree

- Figure 1.1a depicts a game tree, with player 1 acting first at the node, which we can interpret as the "root" of the tree.
- The first mover is often referred to as the leader, and player 2, who observes player 1's choice and responds with her own action, is known as the follower.

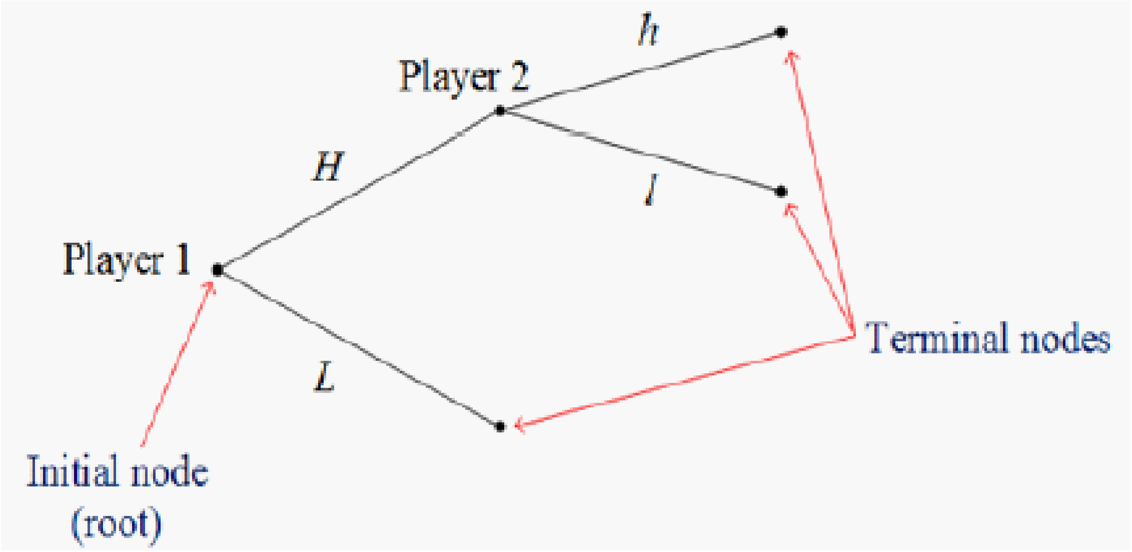

Figure 1.1a. Example of a game tree

Two graphical approaches

Game tree

- In Figure 1.1a, player 1 can choose between H and L.
- If player 1 chooses L, the game is over and payoffs are distributed.
- But if she chooses H, player 2 must respond with either h or l.

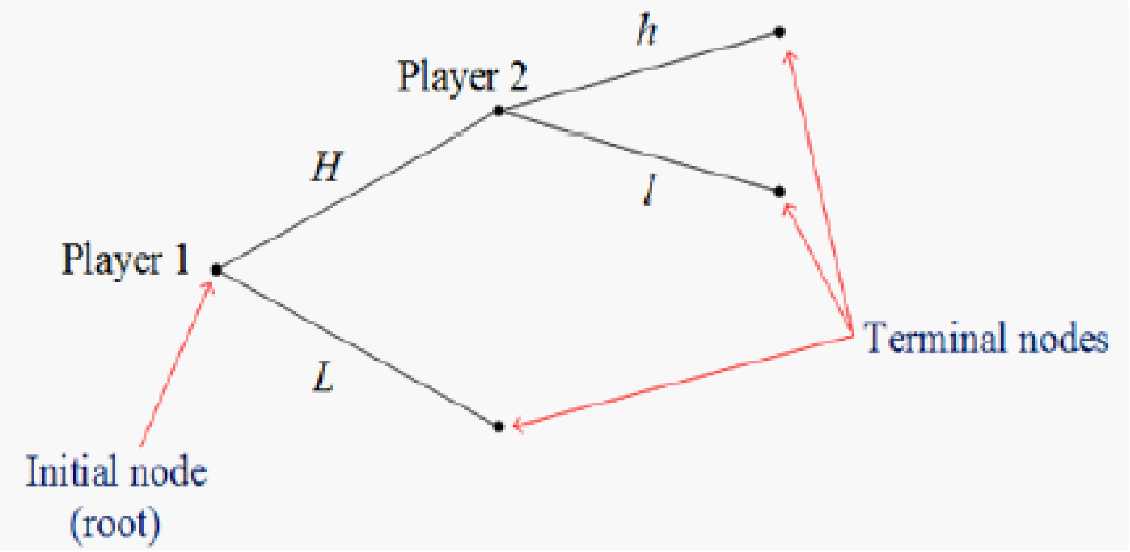

Figure 1.1a. Example of a game tree

Two graphical approaches

Game tree

- The nodes at the end of the game tree are denoted as "terminal nodes".
  - They list the payoffs that players earn at each strategy profile.
- For instance, if the game proceeds through H and then l, we say that the strategy profile is (H, l).
  - In the context of game trees, it can also be understood as a "path of play."

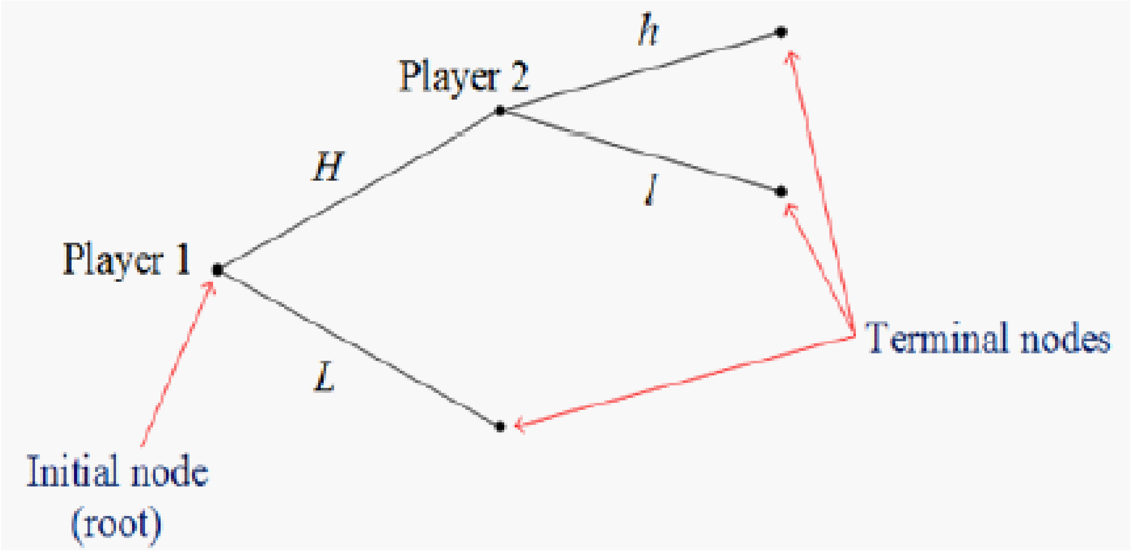

Figure 1.1a. Example of a game tree

Two graphical approaches

Game tree, example of strategy profile.

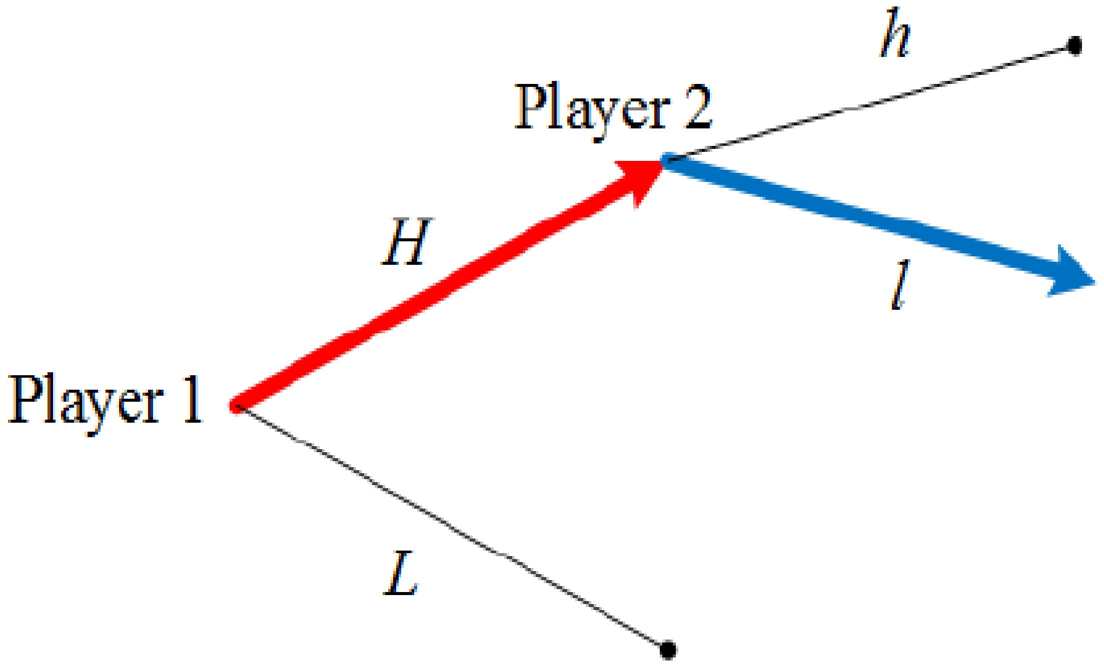

Figure 1.1b. Strategy profile (H, l).

Two graphical approaches

Game tree, larger tree.

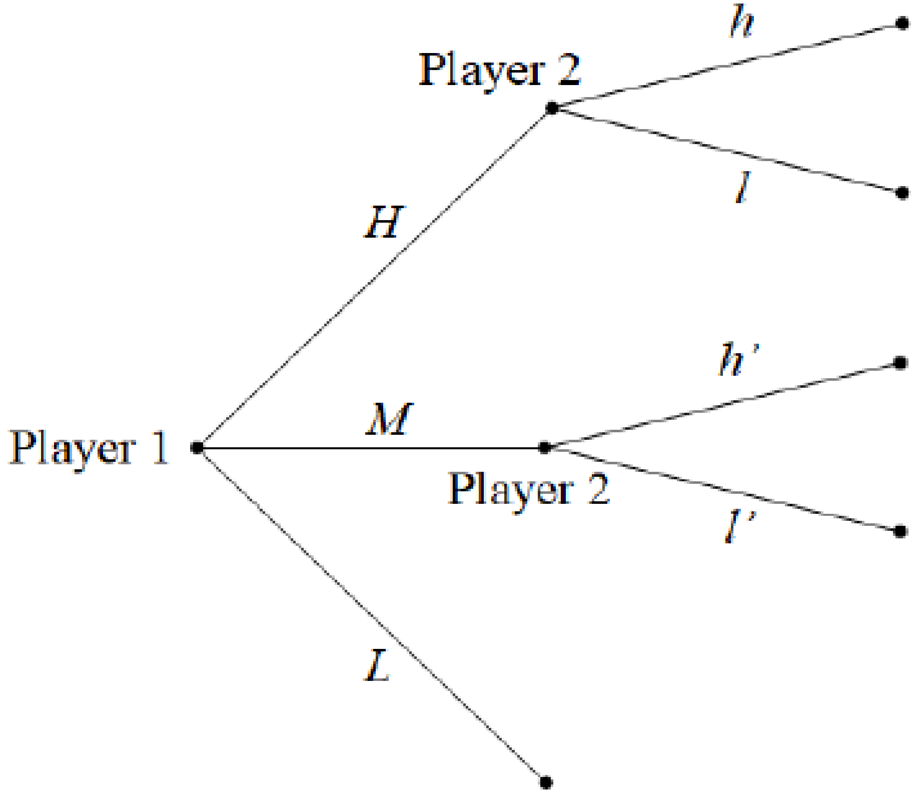

Figure 1.2. Example of a game tree - Larger version.

Strategy spaces:

$$S_1 = \{H, M, L\}$$
$$S_2 = \{hh', hl', lh', ll'\}$$

Two graphical approaches

Game tree, larger tree.

- Player 1, being called on to move at a single node (initial node), has as many strategies as action branches stemming from the initial node,

$$S_1 = \{H, M, L\}.$$

- Player 2 can respond in two different nodes (responding to H or M) , entailing four different strategies for this player (two actions in each node times two nodes) which implies

$$S_2 = \{hh', hl', lh', ll'\}$$

  - For instance, strategy hl' prescribes that player 2 responds with:
    - h after observing that player 1 chooses H, but
    - with l' when player 1 chooses M.

Two graphical approaches

Strategy spaces:
$S_1 = \{H, M, L\}$
$S_2 = \{hh', hl', lh', ll'\}$

- The above example illustrates our definition of strategies as complete contingent plans in a sequential context:
  - Each strategy in S_2 serves as an instruction manual for player 2, telling her what to do after player 1 chooses H, after M, and after L.
- But why does player 2's strategy describe how would she respond to M, if for instance player 1 chooses H?
  - Strategies must be so detailed because player 2's responses at nodes that may not be reached can make player 1 behave in one way or another in the first stage.

Introducing imperfect information in game trees

- **Information set**: An information set connects all nodes that player i cannot distinguish.
  - In addition, these nodes must have the same number of action branches, which must have the same labels.

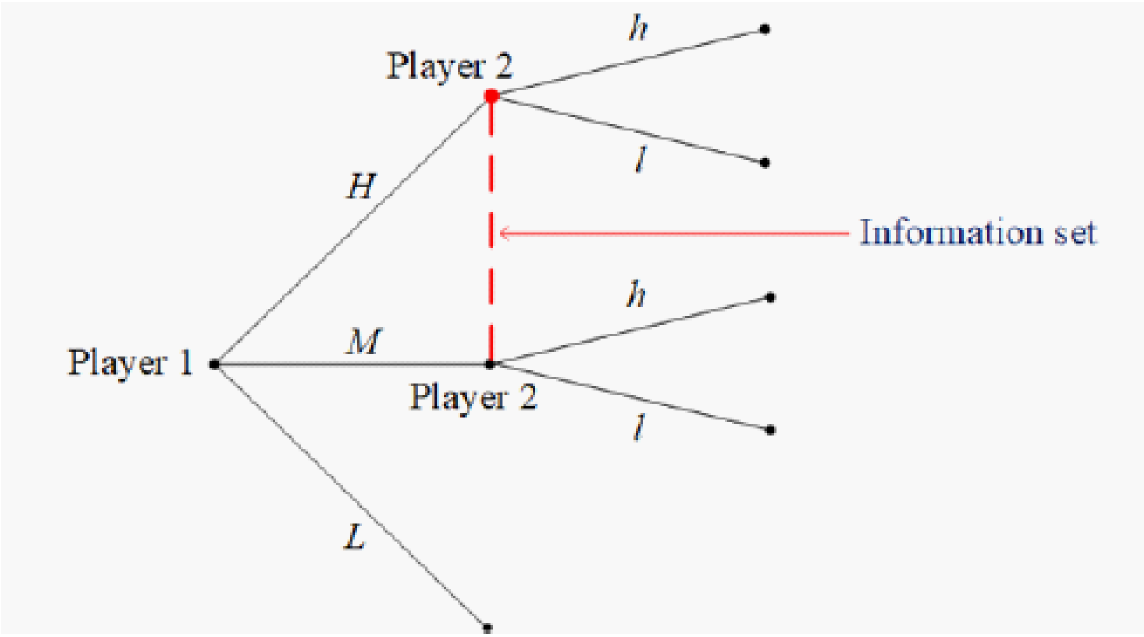

Figure 1.3. Example of a game tree - Larger version with an information set.

Introducing imperfect information in game trees

Incorrect representation of information sets

- Figure 1.4a:
  - Player 2 chooses between two actions when player 1 selects H (h and l at the top node) but…
  - a different number of actions when player 1 selects M (e.g., h,m, and l at the bottom node).
- Then she would be able to infer the node she is in by looking at the number of available actions on that node.
- In that scenario, player 2 would know what action player 1 chose, so no information set would need to be depicted.

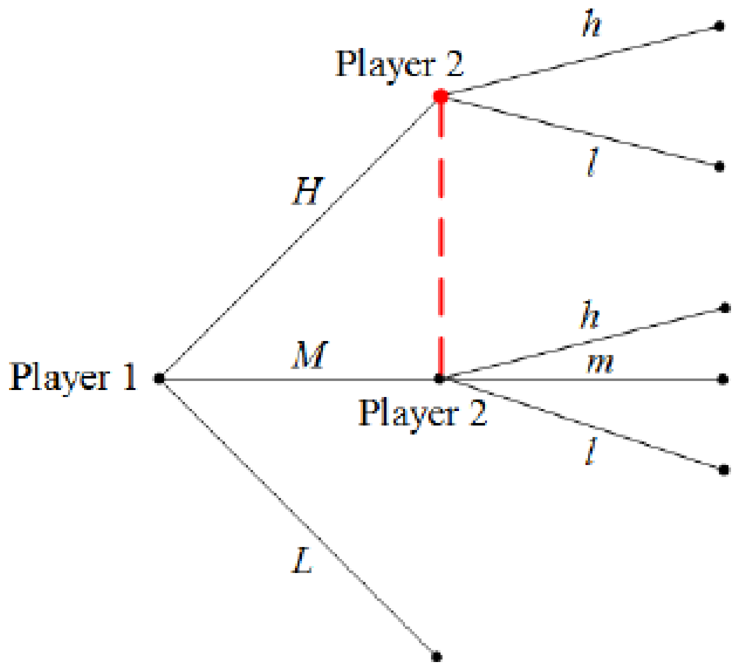

Figure 1.4a. Nodes with different number of actions

Introducing imperfect information in game trees

Incorrect representation of information sets.

- Figure 1.4b: player 2 chooses between h and l after player 1 selects H, but between h and x after player 1 chooses M.
- In this setting, player 2 would be able to infer the node she is at by looking at the specific list of actions she can choose from (either h and l, or h and x), implying that the nodes must not be connected with an information set either.

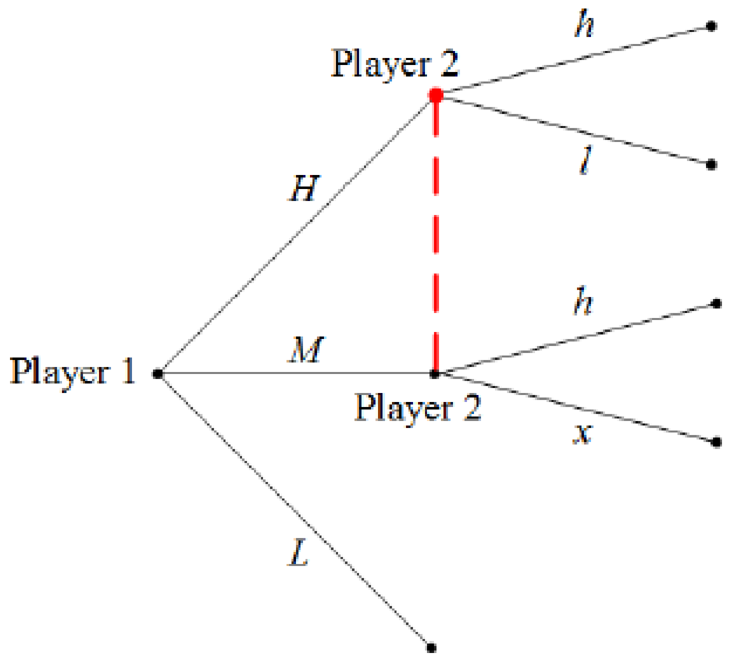

Figure 1.4b. Nodes with the same number of actions but different action labels

Introducing imperfect information in game trees

- Another interesting point of information sets is that they represent a game which, despite being sequential in nature , has every player choosing her actions "in the dark."
- In this context, a player cannot observe her rival's previous choices.
- As a consequence, the sequential-move game in Figure 1.3 becomes strategically equivalent to a simultaneous-game.
- This applies regardless of whatever player 1's and 2's choices happen simultaneously (in the same second) or sequentially (days or years apart) as ling as player 2 cannot observe whether player 1 chose H or M when called to respond.

Identifying equilibrium behavior

- We now seek to solve games, trying to predict how players will behave.
- We will use different solution concepts:
  - From deleting strictly dominated strategies, to Nash equilibrium, and to Subgame perfect equilibrium in complete-information games; and
  - From Bayesian Nash Equilibrium, to Perfect Bayesian Equilibrium or Sequential Equilibrium in incomplete-information games.

Identifying equilibrium behavior

- We evaluate solution concepts according to different criteria:
  - *Existence*:
    - Does an equilibrium exist for all games when we apply that solution concept?
    - Or are there some games where out equilibrium concept does not yield an equilibrium?
  - *Uniqueness*:
    - Does the solution concept we are using predict a unique equilibrium for *all* games?
    - Or are there games out there where our equilibrium concept predicts two or more equilibria.

Identifying equilibrium behavior

- We evaluate solution concepts according to different criteria:
  - *Robustness*:
    - Is the equilibrium (or equilibria) we found robust to small payoff changes?
    - This is, of course, assuming that the payoff *ranking* is unaltered for each player.
    - If that happens, we seek to test if the originally prescribed equilibrium "suddenly changes" when we change the payoff of one of the players slightly.
  - *Pareto optimality*:
    - Is the equilibrium (or equilibria) we found Pareto optimal, in the sense that:
      - we cannot find another strategy profile that strictly increases the payoff of at least one player without reducing the payoff of all other players.
    - If we can find such a strategy profile, the equilibrium we found is *not* Pareto optimal.
    - If we cannot find such a strategy profile, the equilibrium we found *is* Pareto optimal.

We are ready to play some games!