Chapter 10: Perfect Bayesian Equilibrium

Game Theory:

An Introduction with Step-by-Step Examples

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Introduction

- In this chapter, we extend the analysis of incomplete information, allowing for sequential interaction.
- Examples:
  - Incumbent monopolist privately observing its production costs while a potential entrant only having an estimate about the incumbent’s costs
  - Incumbent chooses its prices and the potential entrant, observing the price but not knowing the incumbent’s cost, decides whether to enter the industry or not
- An important issue in sequential-move games of incomplete information is that player’s actions can:
  - Convey information to other players acting later in the game, i.e., informative signals.
  - Conceal information from other players acting later in the game, i.e., uninformative signals.

Sequential-move games of incomplete information - Notation

1. **Nature.**
   - Reveals to player i a piece of information, $\theta_i \in \Theta_i$, which we refer to as her "type".
   - Examples:
     - A firm's production cost,
     - its product quality, or
     - a worker's ability on a certain task.
   - The realization of θ_i is
     - Privately observed by player i
     - Not observed by her rivals $j \neq i$.

Sequential-move games of incomplete information - Notation

2. **First mover (sender).** Player i privately observes θ_i and chooses an action $s_i(\theta_i)$. Afterwards, her rivals observe $s_i(\theta_i)$ but don't observe θ_i.

   a. *Separating strategy.* If action $s_i(\theta_i)$ is type-dependent,

   $$s_i(\theta_i) \neq s_i(\theta_i'), \text{ for every two types } \theta_i \neq \theta_i',$$

   we say that player i uses a "separating" strategy, i.e., different types choose different strategies.

   - *Example:* An incumbent firm uses a difference price depending on its cost.

   b. *Pooling strategy.* If action $s_i(\theta_i)$ is type-independent,

   $$s_i(\theta_i) = s_i(\theta_i'), \text{ for every two types } \theta_i \neq \theta_i',$$

   we say that player i employs a "pooling" strategy, since different types choose the same strategy (i.e., they pool into the same strategy).

   - *Example:* Following with the above example, the incumbent would use the same price regardless of its production cost.

Sequential-move games of incomplete information - Notation

2. **First mover (sender).** Continues...
   c. *Partially separating strategy.* If action $s_i(\theta_i)$ satisfies

$$s_i(\theta_i) \neq s_i(\theta_i') \text{ for at least two types } \theta_i \text{ and } \theta_i',$$

$$\text{but } s_i(\theta_i'') = s_i(\theta_i''') \text{ for other types,}$$

we say that player i uses a "partially separating" strategy.

In our ongoing example, the incumbent chooses:

- the same price p when its production cost is high and medium,

$$p = s_i(H) = s_i(M),$$

- but a different price $p' \neq p$ when its cost is low,

$$p' = s_i(L).$$

Sequential-move games of incomplete information - Notation

3. **Second mover (receiver).** Player j observes action s_i, but does not know player $i's$ type, θ_i.

We assume, however, that every player knows the prior probability distribution over types, $\mu(\theta_i)$, for every type $\theta_i \in \Theta_i$.

This probability distribution is well behaved:

a. $\mu(\theta_i) \in [0,1]$

b. $\sum_{\theta_i \in \Theta_i} \mu(\theta_i) = 1$ if player $i's$ type space is discrete, e.g., $\Theta_i = \{L, H\}$; and similarly, $\int_{\theta_i \in \Theta_i} \mu(x)\, dx = 1$ if her type space is continuous..

For example, when player i has two types, $\Theta_i = \{L, H\}$, the prior probability can be expressed as $\mu(L) = q$ and $\mu(H) = 1 - q$, where $q \in [0,1]$.

When player $i's$ types are uniformly distributed, then $\Pr\{\theta \leq \theta_i\} = F(\theta_i) = \theta_i$, with corresponding density $f(\theta_i) = 1$.

Sequential-move games of incomplete information - Notation

4. Second mover updates her beliefs.

- Upon observing s_i, player j updates her beliefs about player i's type being θ_i, which we write as
$$\mu(\theta_i|s_i).$$
- These beliefs are often known as “posterior beliefs” as opposed to the prior probability distribution, $\mu(\theta_i)$, which are referred to as “prior beliefs”.
- For compactness, we will generally use “priors” and “posteriors”, or “prior probability” and “beliefs”.
- Like priors, posteriors satisfy $\mu(\theta_i|s_i) \in [0,1]$ and :
  - $\sum_{\theta_i \in \Theta_i} \mu(\theta_i|s_i) = 1$ when types are discrete, and
  - $\int_{\theta_i \in \Theta_i} \mu_i(x|s_i)\, dx = 1$ when types are continuous.

Sequential-move games of incomplete information - Notation

4. **Second mover updates her beliefs.**
   - As we discuss below, the PBE solution concept requires players to update their beliefs according to Bayes' rule, as follows,
$$\mu_j\,(\theta_i|s_i) = \frac{\mu(\theta_i)\Pr(s_i|\theta_i)}{Pr(s_i)}$$
   - This says that the conditional probability that, upon observing action s_i, player i's type is θ_i, as captured by $\mu_j(\theta_i|s_i)$, is:
     - the (unconditional) probability that player i's type is θ_i and she chooses action s_i, $\mu(\theta_i)\Pr(s_i|\theta_i)$…
     - given that she chose s_i, $\Pr(s_i)$.
   - When $\mu_j(\theta_i|s_i)$ is higher (lower) than the prior probability $\mu(\theta_i)$, the observation of s_i increases (decreases) the chances that player i's type is θ_i, indicating that action s_i provides information that player j did not initially have in the prior probability distribution $\mu(\theta_i)$.

Sequential-move games of incomplete information - Notation

4. **Second mover updates her beliefs.**
   - In the extreme case that $\mu_j(\theta_i|s_i) = 1$, player j becomes "convinced," after observing s_i, the player i's type is θ_i
   - Similarly, if $\mu_j(\theta_i|s_i) = 0$, player j is sure that player i's type cannot be θ_i.
   - In contrast, if $\mu_j(\theta_i|s_i) = \mu_j(\theta_i)$, player j does not change her beliefs upon observing s_i, meaning that s_i did not provide her with additional information. In this case, we say that the signal was uninformative or that player j could not update her beliefs.

Sequential-move games of incomplete information - Notation

5. Second mover responds.

- After player j updates her beliefs to $\mu_j(\theta_i|s_i)$, she responds choosing an action s_j.
  - Player j may have observed her type, θ_j, before choosing her action s_j.
- Player j's type can be:
  - Publicly observed by all players, becoming common knowledge.
  - Or privately observed by player j (for instance, every player privately observes her type).

Sequential-move games of incomplete information - Notation

- After concluding the above five steps, most of the sequential-move games with incomplete information that we consider in this chapter will be over, distributing payoffs to each player.
- However, we can continue the game, allowing, for instance, player i to observe player j's action s_j in step 5, update her beliefs about j's type θ_j, $\mu_i(\theta_j|s_j)$, and respond with an action a_i, where we use a_i to distinguish it from s_i.
  - If player j does not have privately observed types, then player i could just respond with action a_i, without having to update her beliefs about θ_j.

Example 10.1. Two sequential-move games of incomplete information

- Consider this game tree.
- Only one information set.
- Player 1 privately observes whether:
  - a business opportunity is beneficial for player 2 (in the upper part of the game tree, which happens with probability p), or
  - futile (bottom part of the tree, with probability $1-p$), where $p \in [0,1]$.

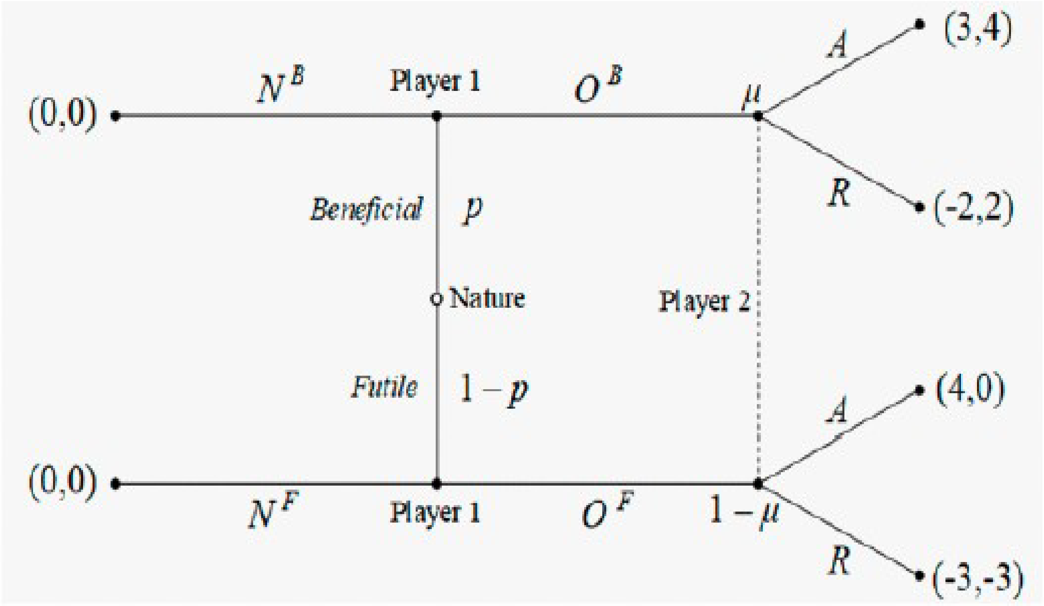


Figure 10.1. Sequential-move game with incomplete information - One information set.

Example 10.1. Two sequential-move games of incomplete information

- In this setting, we say that player 1's types are *Beneficial* or *Futile*.
- If player 1 does not make an investment offer to player 2, N (moving leftward),
  - the game is over and both players earn a payoff of zero.
- If, instead, player 1 makes an investment offer to player 2, O (moving rightward),
  - player 2 receives this offer
  - but does not observe whether it is beneficial (at the upper part of the tree) or not (at the bottom part).

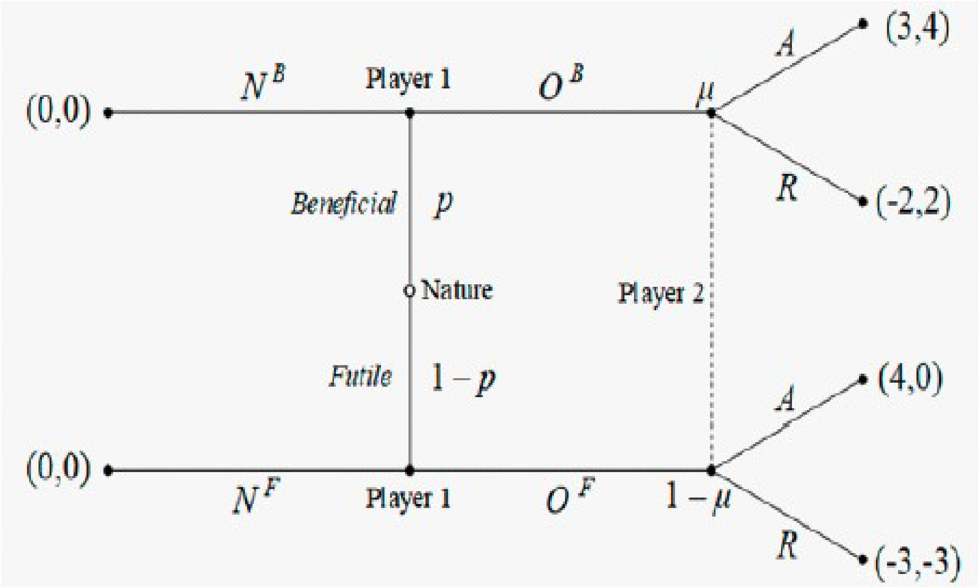


Figure 10.1. Sequential-move game with incomplete information - One information set.

Example 10.1. Two sequential-move games of incomplete information

- However, upon receiving the offer, player 2 must respond:
  - accepting (A)
  - or rejecting it (R).
- The game in Figure 10.1, then, has only one information set where the receiver (player 2) is called to move.

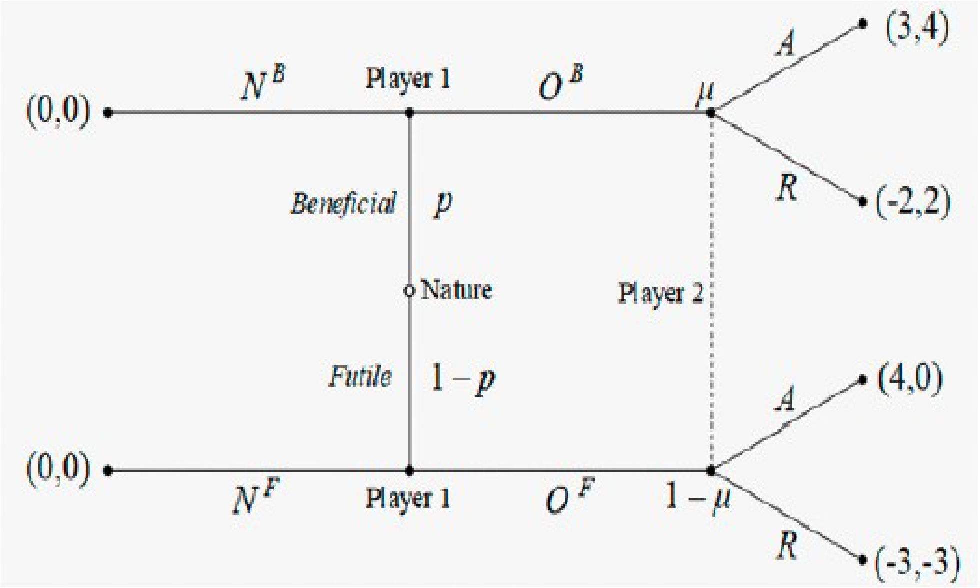


Figure 10.1. Sequential-move game with incomplete information - One information set.

Example 10.1. Two sequential-move games of incomplete information

- This game, however, depicts the case in which a receiver is called to respond after both messages from a sender,
  - yielding two information sets.
- In particular, the game represents Spence's (1973) labor-market signaling game:
  - a worker (sender) privately observed her job productivity (her type), which is either high or low,
  - then decides whether to pursue an advanced education degree (such as Masters program) that she might use as a signal about her productivity level to the employer (receiver).

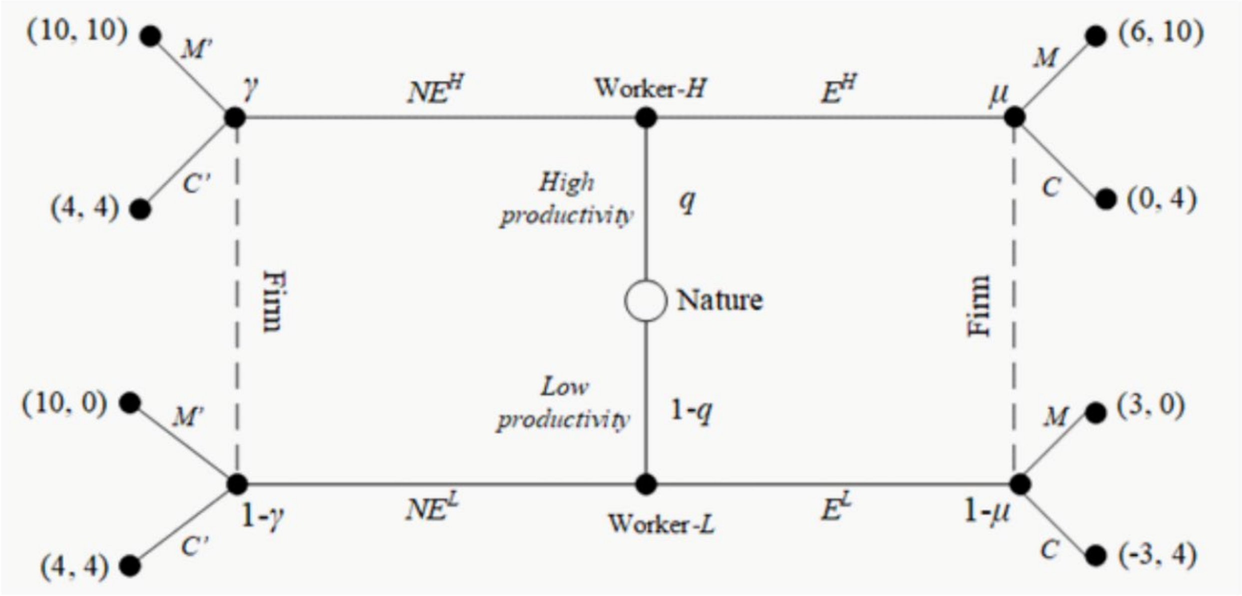


Figure 10.2. Labor market signaling game.

Example 10.1. Two sequential-move games of incomplete information

- Upon observing whether or not the applicant acquired education…
  - but without observing her true productivity,
  - the employer updates her beliefs about the worker's type,
  - and responds hiring her manager (M) or as a cashier (C).

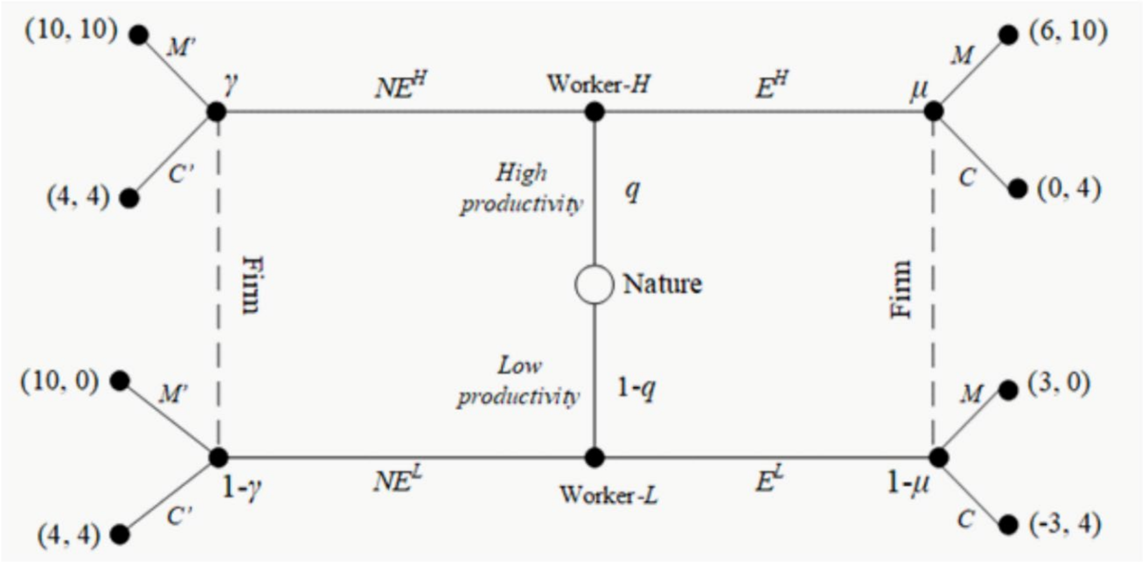


Figure 10.2. Labor market signaling game.

Example 10.1. Two sequential-move games of incomplete information

- Observing the worker's payoffs, one can infer that the cost of acquiring education is only $4 for the high-productivity worker.
- Indeed, when she is hired as a M, at the top of the figure:
    - the difference in her payoff when she avoids or acquires education, $10 - 6 = 4$.
- A similar argument applies when she is hired as a C, where the payoff difference is $4 - 0 = 4$.

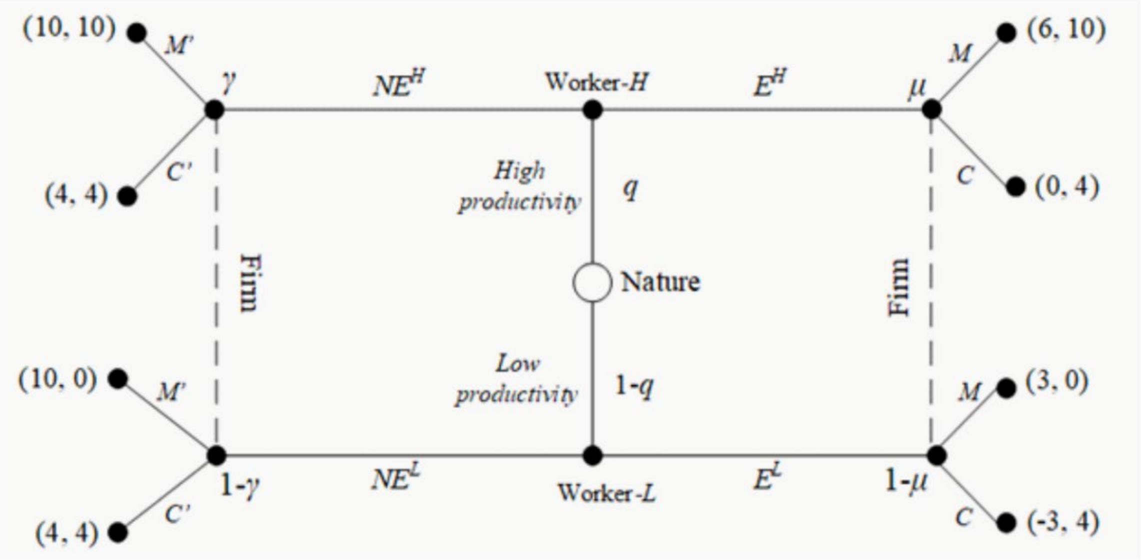


Figure 10.2. Labor market signaling game.

Example 10.1. Two sequential-move games of incomplete information

- However, the cost of acquiring education increases to \$7 for the low-productivity worker.
- In this case, when she hired as a M:
  - her payoff difference is $10 - 3 = 7$ and,
  - similarly, when she is hired as a C, the payoff difference is $4 - (-3) = 7$.

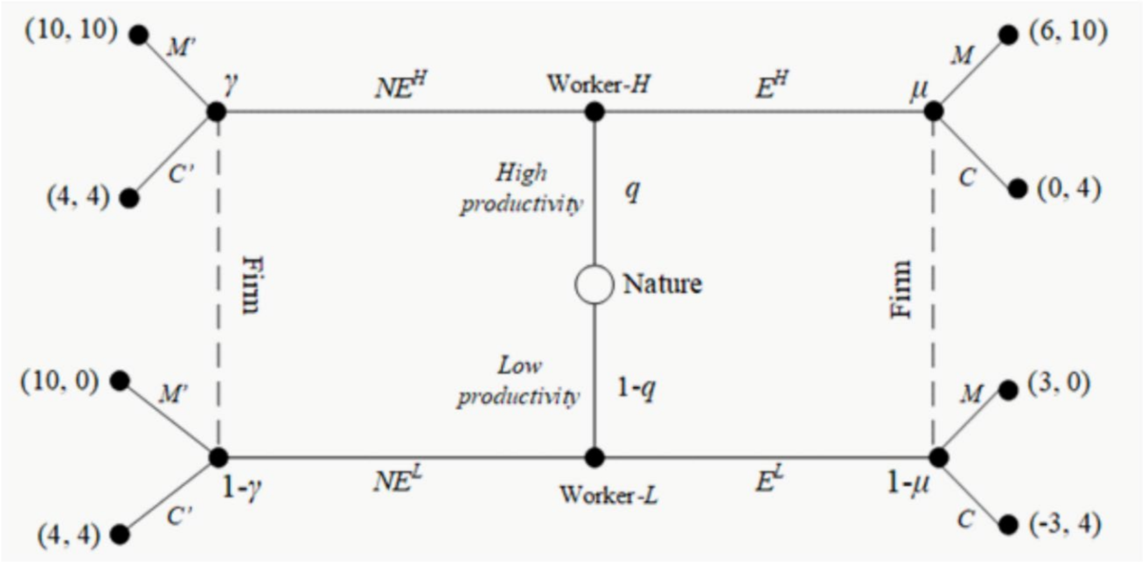


Figure 10.2. Labor market signaling game.

Example 10.1. Two sequential-move games of incomplete information

- Regarding the firm's payoffs, in the second component of each payoff pair in the terminal nodes of the figure:
  - they are unaffected by the worker's type, namely, \$4 when the worker is hired as a C, regardless of her type.
  - But are \$10 when the worker is hired as a M, and her productivity happens to be high but \$0 otherwise.

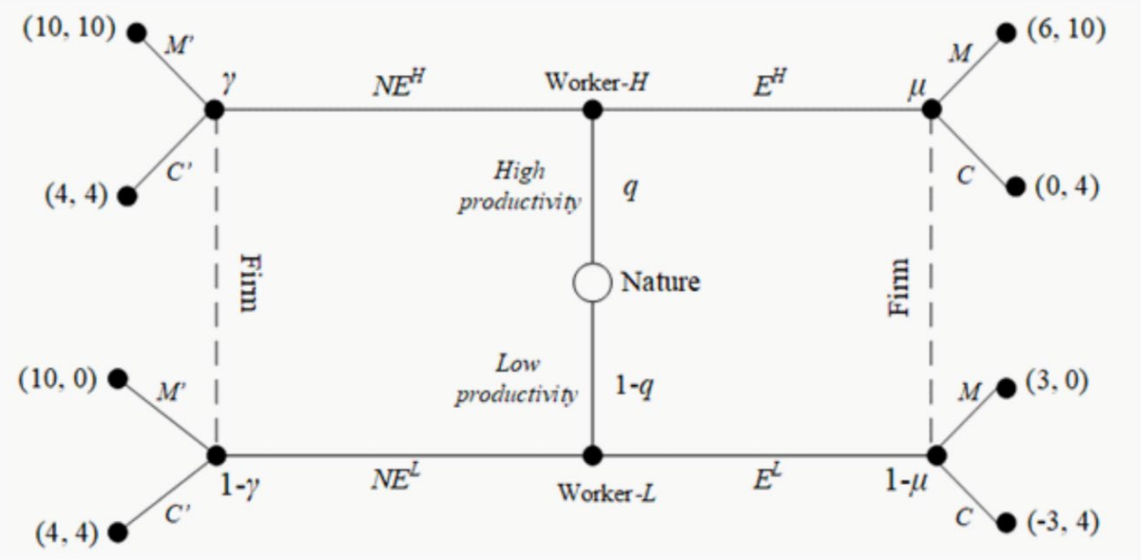


Figure 10.2. Labor market signaling game.

Example 10.1. Two sequential-move games of incomplete information

- The firm's payoffs, are, however, unaffected by whether the worker acquires education or not.
- In other words, education is *not productivity-enhancing*.
- This is, of course, a simplification!
  - Helps us focus on education working as a signal to potential employers…
  - facilitating information transmission from workers to firms,
  - even if it does not change the worker's job productivity.

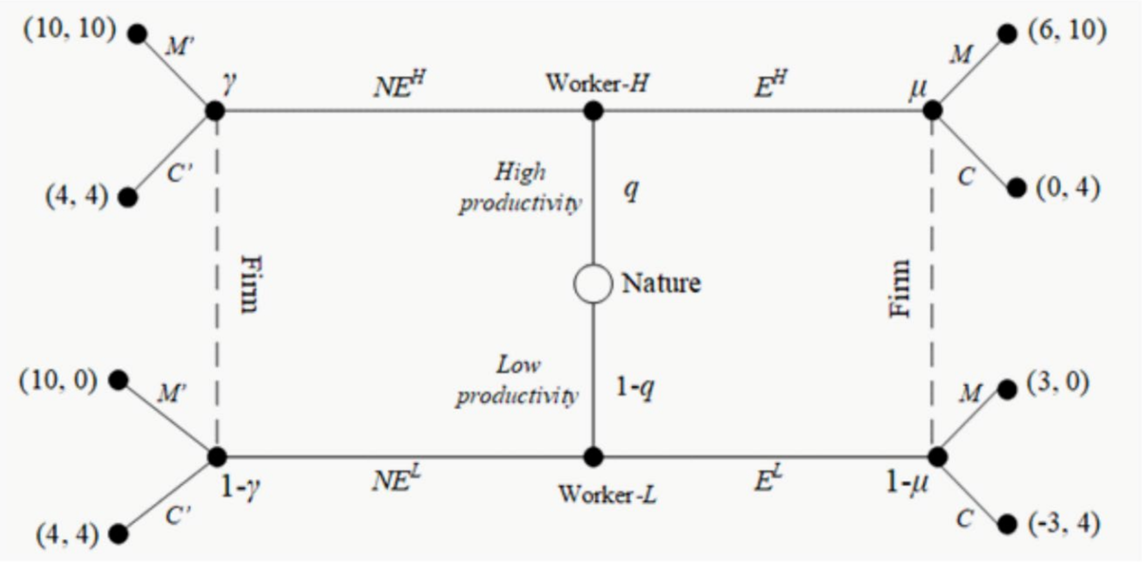


Figure 10.2. Labor market signaling game.

BNE prescribing sequentially irrational behavior

- Why not apply BNE to solve these games?
  - We can apply the BNE solution concept to find equilibrium behavior when players interact sequentially and operate under incomplete information.
- BNE may identify *sequentially irrational behavior:*
  - That is, upon reaching a node or information set, a player acts in a way that does not maximize her expected payoff.
- BNE exhibits similar issues as the NE solution concept did:
  - We can apply it to both simultaneous- and sequential-move games,
  - but when applying it to the latter we may find too many equilibria, some of them prescribing that players are sequentially irrational.
  - See Example 10.2

Example 10.2. Applying BNE to sequential-move games of incomplete information

Player 2

Player 1		A	R
	$O^B O^F$	$\underline{4-p}, \underline{4p}$	$-3+p, -3+5p$
	$O^B N^F$	$3p, \underline{4p}$	$-2p, 2p$
	$N^B O^F$	$4(1-p), \underline{0}$	$-3(1-p), -3(1-p)$
	$N^B N^F$	$0, \underline{0}$	$\underline{0}, \underline{0}$

Matrix 10.1. Bayesian normal-form representation of the game tree in Figure 10.1

- Consider the game in Figure 10.1 again (only one info. set).
- To find the BNE, we first need to represent its Bayesian normal-form.
- We use this representation in Matrix 10.1, where each cell includes expected payoffs for each player.

Example 10.2. Applying BNE to sequential-move games of incomplete information

Player 2

Player 1		A	R
	O^BO^F	$\underline{4-p}, \underline{4p}$	$-3+p, -3+5p$
	O^BN^F	$3p, \underline{4p}$	$-2p, 2p$
	N^BO^F	$4(1-p), \underline{0}$	$-3(1-p), -3(1-p)$
	N^BN^F	$0, \underline{0}$	$\underline{0}, \underline{0}$

Matrix 10.1. Bayesian normal-form representation of the game tree in Figure 10.1

- For instance, at (O^BO^F, A):
  - Player 1 makes an offer to player 2 regardless of whether the investment is beneficial or not, and
  - Player 2 accepts the offer.

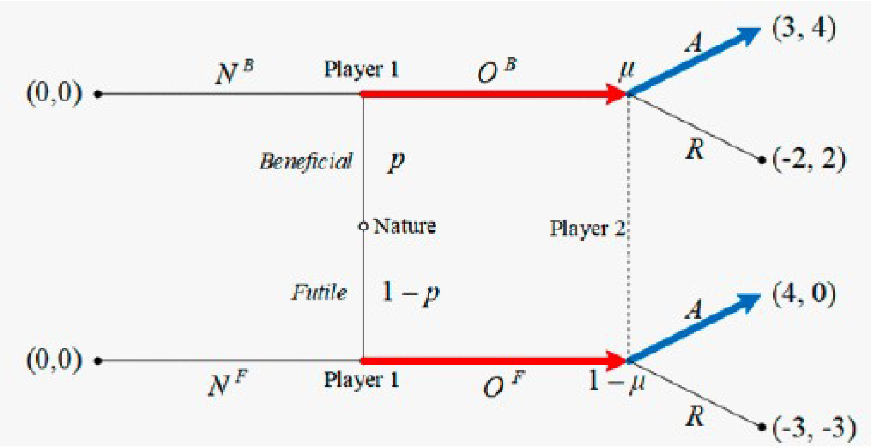


Figure 10.4a. Pooling strategy profile (O^B, O^F) - Responses.

Example 10.2. Applying BNE to sequential-move games of incomplete information

Player 2

Player 1		A	R
	$O^B O^F$	$\underline{4-p}, \underline{4p}$	$-3+p, -3+5p$
	$O^B N^F$	$3p, \underline{4p}$	$-2p, 2p$
	$N^B O^F$	$4(1-p), \underline{0}$	$-3(1-p), -3(1-p)$
	$N^B N^F$	$0, \underline{0}$	$\underline{0}, \underline{0}$

Matrix 10.1. Bayesian normal-form representation of the game tree in Figure 10.1

- Then, $(O^B O^F, A)$ yields expected payoffs of:
  - $EU_1 = p3 + (1-p)4 = 4-p$ for player 1 and
  - $EU_2 = p4 + (1-p)0 = 4p$ for player 2.
- Underlining expected payoffs, we find two BNEs in this game:
  - $(O^B O^F, A)$ and $(N^B N^F, R)$.

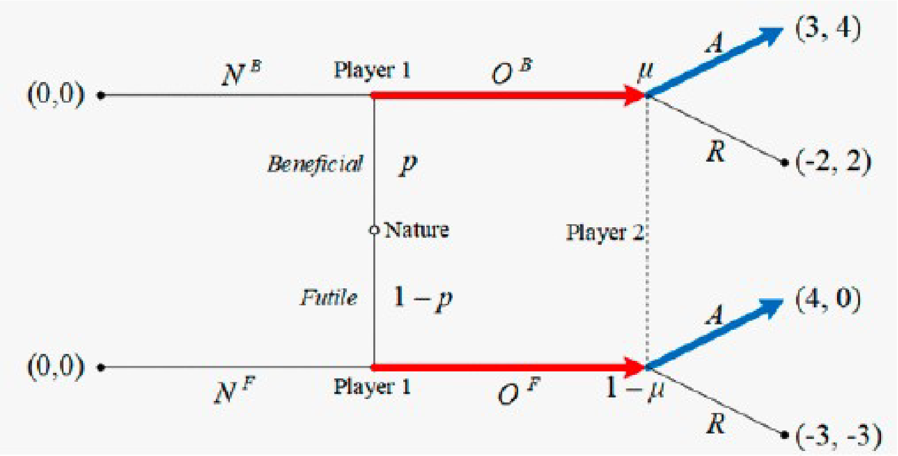


Figure 10.4a. Pooling strategy profile (O^B, O^F) - Responses.

Example 10.2. Applying BNE to sequential-move games of incomplete information

Player 2

Player 1

	A	R
O^BO^F	$\underline{4-p}, \underline{4p}$	$-3+p, -3+5p$
O^BN^F	$3p, \underline{4p}$	$-2p, 2p$
N^BO^F	$4(1-p), \underline{0}$	$-3(1-p), -3(1-p)$
N^BN^F	$0, \underline{0}$	$\underline{0}, \underline{0}$

Matrix 10.1. Bayesian normal-form representation of the game tree in Figure 10.1

- Underlining expected payoffs, we find two BNEs in this game:
  - (O^BO^F, A) and (N^BN^F, R).

Example 10.2. Applying BNE to sequential-move games of incomplete information

- **First BNE:** Intuitively, this BNE says that:
  - player 1 makes an offer regardless of whether the investment is beneficial for player 2 or not,
  - and player responds 2 accepting it.
- This occurs because, in expectation, accepting the offer yields a higher payoff than rejecting it,

$$4p > -3 + 5p,$$

which simplifies to $3 > p$; a condition that holds for all probabilities $p \in [0,1]$.

Example 10.2. Applying BNE to sequential-move games of incomplete information

- **Second BNE:** This BNE, however, predicts that:
  - player 1 never makes an offer (regardless of her type),
  - yet, player 2 would respond rejecting an offer if she ever receives one.
- This behavior is sequentially irrational:
  - if player 2 receives an offer,
  - at the only information set where she is called to move
    - (see right hand side of Figure 10.1),
  - she earns a higher expected payoff accepting than rejecting it,
  - This holds *regardless* of the belief, μ, that she sustains about whether the investment is beneficial of not.

Example 10.2. Applying BNE to sequential-move games of incomplete information

- Indeed, player 2's expected payoffs from accepting and rejecting are

$$EU_A = 4\mu + 0(1-\mu), \text{ and}$$
$$EU_R = 2\mu + (-3)(1-\mu) = -3 + 5\mu$$

which satisfy $EU_A > EU_R$ since $4\mu > -3 + 5\mu$ simplifies to $3 > \mu$, which holds for all $\mu \in [0,1]$.

- In other words, player 2 should respond accepting the offer…
  - regardless of her belief of the investment being beneficial, μ,
  - which contradicts the second BNE, $(N^B N^F, R)$, where player 2 rejects the offer with certainty.
- As a consequence, we claim that the second BNE is *sequentially irrational*:
  - Once an offer is received, it is sequentially rational to accept it.

Perfect Bayesian Equilibrium - Definition

- Then, we need a new solution concept to solve sequential-move games of incomplete information.
- To guarantee players choose sequentially rational strategies in this context.
- Just as we deployed a new solution concept, SPE, in sequential-move games of complete information.
- Before we introduce our new solution concept (Perfect Bayesian Equilibrium, PBE), let's clarify some notation.

Perfect Bayesian Equilibrium - Definition

Simplifying notation, Strategies.

- As in our discussion about equilibrium strategies in BNE:
  - Let $s_i^*(\theta_i)$ denote player i's equilibrium strategy when her type is θ_i (which could be privately or publicly observed).
- Similarly, let

$$s_{-i}^*(\theta_{-i}) \equiv \left(s_1^*(\theta_1), \dots, s_{i-1}^*(\theta_{i-1}), s_{i+1}^*(\theta_{i+1}), \dots, s_N^*(\theta_N)\right)$$

represent the strategy profile of player i's rivals, given their types,

- Then, an equilibrium strategy profile can be compactly expressed as

$$s^* \equiv \left(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\right).$$

- In most applications, where we only consider two players (the first and second mover), this strategy profile simplifies to

$$s^* = \left(s_i^*(\theta_i), s_j^*\left(\theta_j\right)\right) \text{ where } j \neq i.$$

Perfect Bayesian Equilibrium - Definition

Simplifying notation, Beliefs.

- We can also provide a compact representation of players' beliefs.
- Let μ_j denote the list of player j's updated beliefs $\mu_j(\theta_i|s_i)$:
  - for every type $\theta_i \in \Theta_i$ that player i may have,
  - and for every action $s_i \in S_i$ that she may take.

- *Example:* in the simplest scenario where player i's types and actions are binary, $\Theta_i = \{H, L\}$ and $S_i = \{s_i^A, s_i^B\}$, player j's "system of beliefs" has four components, as follows,

$$\mu_j = \left(\mu_j\left(L|s_i^A\right), \mu_j\left(H|s_i^A\right), \mu_j\left(L|s_i^B\right), \mu_j\left(H|s_i^B\right)\right)$$

which can be actually represented by the first and third components alone given that the second term can be expressed as a function of the first term, $\mu_j\left(H|s_i^A\right) = 1 - \mu_j\left(L|s_i^A\right)$, and, similarly, the fourth term is function of the third, $\mu_j\left(H|s_i^B\right) = 1 - \mu_j\left(L|s_i^B\right)$.

Perfect Bayesian Equilibrium - Definition

- **Simplifying notation, Beliefs (cont'd).**
- More generally, player j's system of beliefs has

$$(card(\Theta_i) - 1) \times card(S_i)$$

- components, where "card" denotes the cardinality of the strategy set (i.e., the number of different strategies for that player).

- For example, if player i has three types, $\Theta_i = \{L, M, H\}$, and three possible actions, $S_i = \{s_i^A, s_i^B, s_i^C\}$, player $j's$ system of beliefs has $2 \times 3 = 6$ components.
- We can apply a similar definition to the system of beliefs by each of $j's$ rivals, that is,

$$\mu_{-j} \equiv (\mu_1, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_N)$$

so a system of beliefs can be compactly expressed as

$$\mu \equiv (\mu_j, \mu_{-j}) \text{ for all players.}$$

Perfect Bayesian Equilibrium (PBE)

- **Definition.** A strategy profile s^* and a system of beliefs μ over all information sets is a Perfect Bayesian Equilibrium (PBE) if:
  a. Every player i's strategies specify optimal actions at each information set where she is called to move,
     - given the strategies of the other players,
     - and given player $i's$ system of beliefs μ_i; and

  b. Beliefs μ are consistent with Bayes' rule whenever possible.

Perfect Bayesian Equilibrium (PBE)

- Intuitively, condition (a) means that:
  - every player i chooses best responses to her rivals’ strategies in an incomplete information environment,
  - that is, given her beliefs about her rivals’ types at that point of the game tree.
- Condition (a) can, then, be interpreted as
  - the application of best responses under incomplete information that we already considered in BNEs.
- Condition (b), however, states that every player’s beliefs must be “consistent with Bayes’ rule whenever possible.”

Perfect Bayesian Equilibrium (PBE)

- To understand this requirement:
  - First, note that applying Bayes' rule is only possible along the equilibrium path, that is, at information sets that are reached in equilibrium.
- In contrast, when a player is called to move at an information set that is not reached in equilibrium,
  - she cannot use Bayes' rule to update her beliefs of her rivals' types.
- If she did, she would obtain an indeterminate result,

$$\mu_j(\theta_i|s_i) = \frac{0}{0}.$$

- (More about this in Example 10.3.)
- We refer to these beliefs as "off-the-equilibrium" beliefs.
  - Because we cannot use Bayes' rule to update them, we cannot specify any arbitrary value to them, leaving them unrestricted as follows $\mu_j(\theta_i|s_i) \in [0,1]$.

Example 10.3. Applying Bayes' rule

- For simplicity, we consider the labor-market signaling game of Figure 10.2, as that it allows for more concrete results.
- Recall that, in that context, the sender (job applicant) has:
  - one of two types describing her productivity level, $\Theta = \{L, H\}$, and
  - chooses whether to acquire education, E, or not, NE.
- For compactness, we use:
  - $E^H (NE^H)$ to represent that the high-productivity worker acquires (does not acquire) education;
  - a similar notation applies for the low-productivity worker, where $E^L (NE^L, \text{respectively})$.

Example 10.3. Applying Bayes' rule

1. *Separating strategy,* (E^H, NE^L).
   - Consider that the job applicant requires education only when her productivity is high.

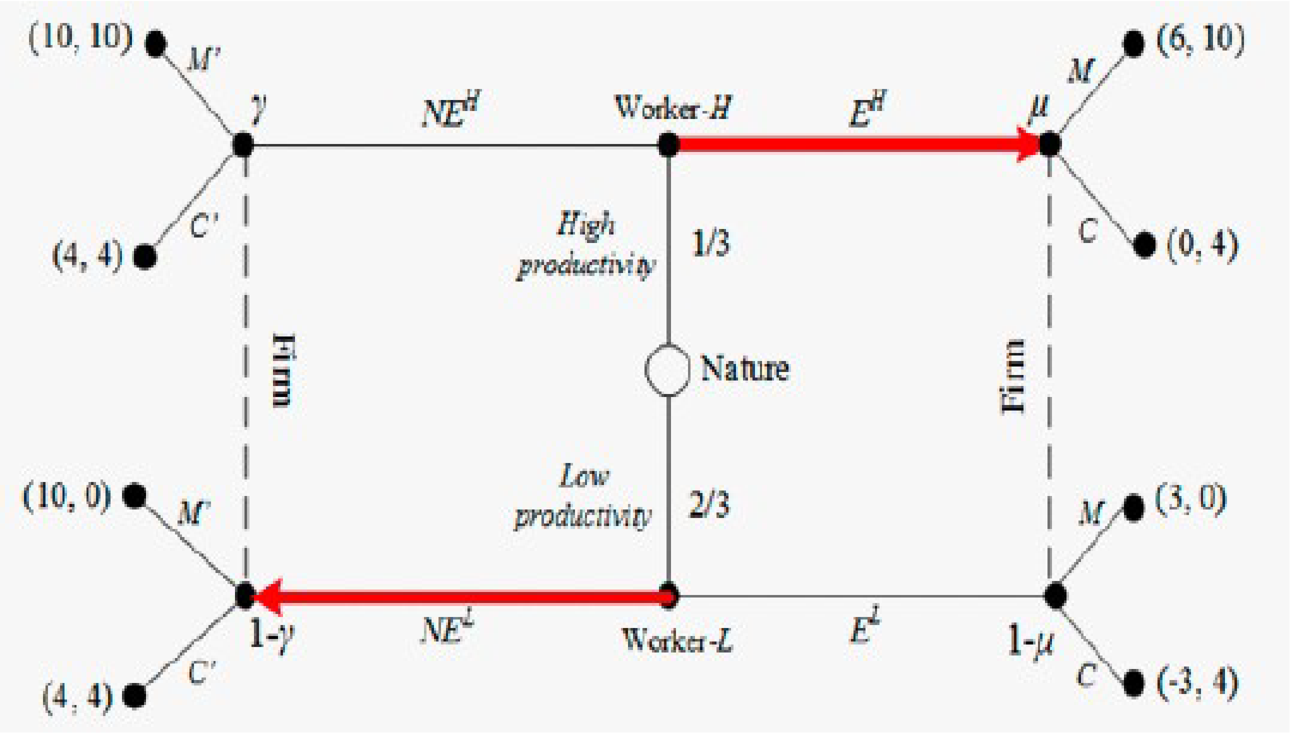


Figure 10.5. Separating strategy profile (E^H, NE^L).

Example 10.3. Applying Bayes' rule

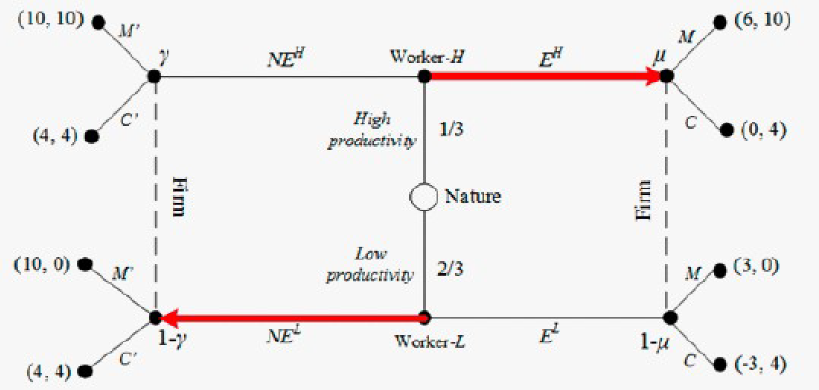


Figure 10.5. Separating strategy profile (E^H, NE^L).

1. *Separating strategy,* (E^H, NE^L).
   - Applying Bayes' rule, we have that, upon observing that the job applicant acquires education, E, the employer's updated belief about the worker's productivity being H is

$$\mu(H|E) = \frac{\mu(H)Pr(E|H)}{Pr(E)} = \frac{\mu(H)Pr(E|H)}{\mu(H)Pr(E|H) + \mu(L)Pr(E|L)} = \frac{q \times 1}{(q \times 1) + \left((1-q) \times 0\right)} = \frac{q}{q}$$

$$= 1$$

Since:

- prior probabilities are $\mu(H) = q$ and $\mu(L) = 1 - q$ by assumption, and
- in this separating strategy profile, $Pr(E|H) = 1$ and $Pr(E|L) = 0$ because the H-type chooses E with certainty while the L-type never does.

Example 10.3. Applying Bayes' rule

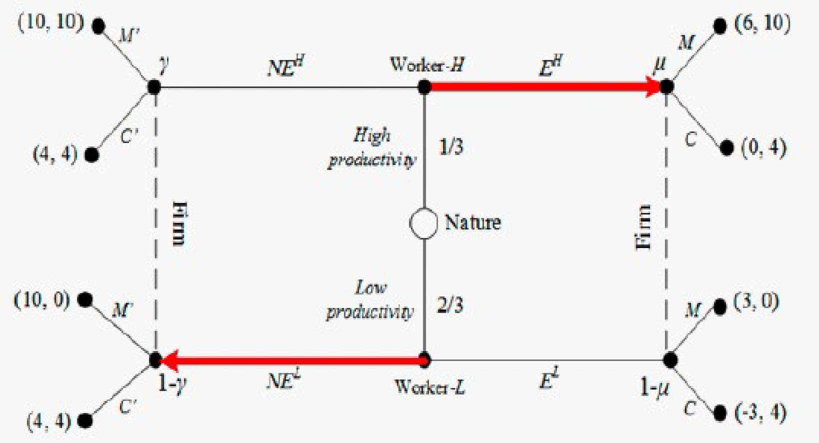


Figure 10.5. Separating strategy profile (E^H, NE^L).

1. *Separating strategy,* (E^H, NE^L).
   - Intuitively, $\mu(H|E) = 1$ says that:
     - the employer, upon observing that the job applicant acquired education,
     - becomes convinced of facing an H-type since only H-types choose E.
   - In other words, E can only originate from the H-type.
     - It is the only player type who chooses education.

Example 10.3. Applying Bayes' rule

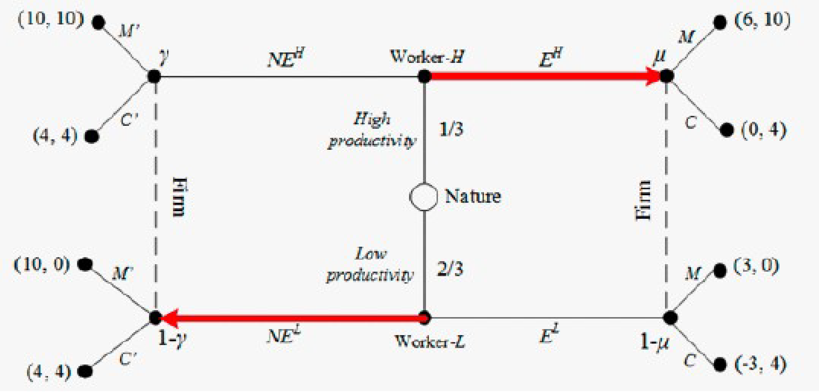


Figure 10.5. Separating strategy profile (E^H, NE^L).

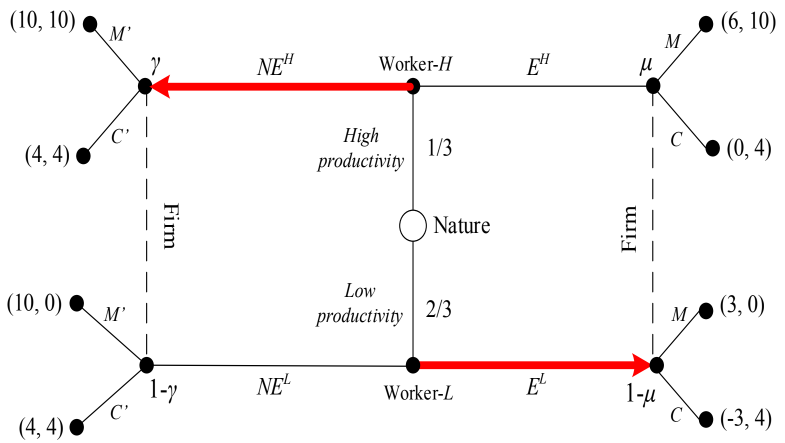
1. *Separating strategy,* (E^H, NE^L).
   - We can also evaluate the employer's posterior beliefs upon observing NE, that is,

$$\mu(H|NE) = \frac{\mu(H)Pr(NE|H)}{Pr(NE)} = \frac{\mu(H)Pr(NE|H)}{\mu(H)Pr(NE|H) + \mu(L)Pr(NE|L)} = \frac{q \times 0}{(q \times 0) + \left((1-q) \times 1\right)} = \frac{0}{1-q} = 0$$

which means that NE can only originate from the L-type of sender.

   - Observing NE helps the employer infer that she must face an L-type worker.
   - In short, $\mu(H|NE)$ =0 because $\mu(H|E)$ =1.

Example 10.3. Applying Bayes' rule



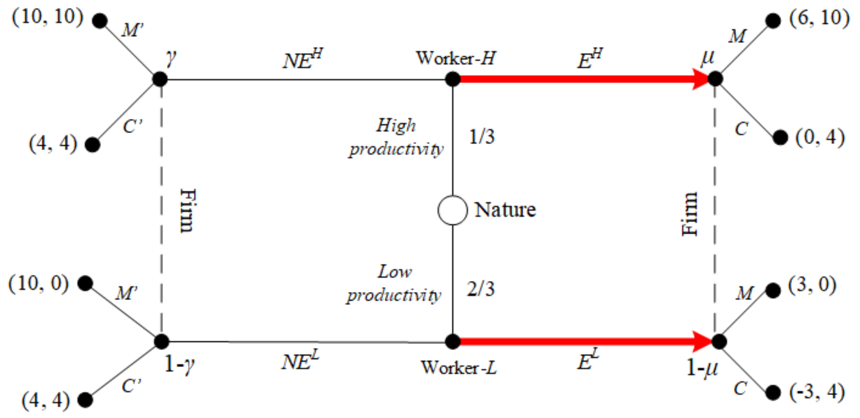
2. *Separating Strategy,* (NE^H, E^L)
   - The opposite separating strategy profile, where only the low-productivity worker acquires education, yields:
     - The opposite results, of course!
   - Upon observing that the job applicant chooses NE, the employer's updated beliefs are

$$\mu(H|E) = \frac{\mu(H)Pr(NE|H)}{Pr(NE)} = \frac{\mu(H)Pr(NE|H)}{\mu(H)Pr(NE|H) + \mu(L)Pr(NE|L)} = \frac{q \times 1}{(q \times 1) + \big((1-q) \times 0\big)} = \frac{q}{q} = 1$$

meaning that:

- NE cannot originate from the L-type of worker,
- stemming instead from the H-type worker,
- that is, $\mu(H|NE) = 1$ and $\mu(H|E) = 0$.

Example 10.3. Applying Bayes' rule



3. *Pooling strategy,* (E^H, E^L).
   - In a pooling strategy profile in which the job applicant chooses E regardless of her type, the receiver's beliefs are unaffected after applying Bayes' rule because

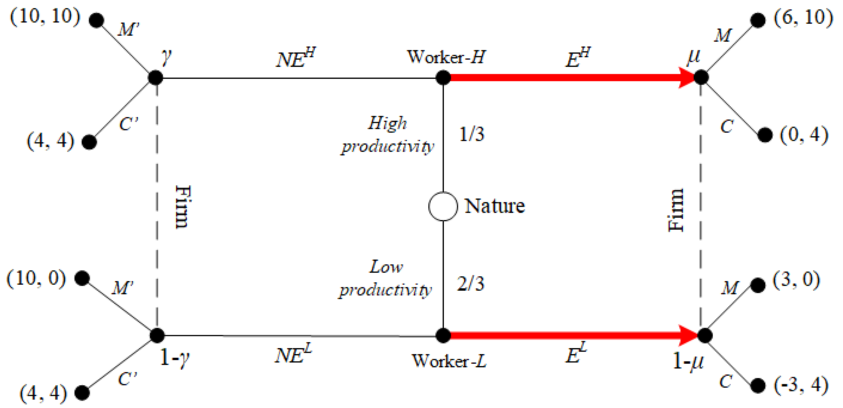
$$\mu(H|E) = \frac{\mu(H)Pr(E|H)}{Pr(E)} = \frac{\mu(H)Pr(E|H)}{\mu(H)Pr(E|H) + \mu(L)Pr(E|L)} = \frac{q \times 1}{(q \times 1) + \left((1-q) \times 1\right)} = \frac{q}{q + (1-q)} = q$$

and since $\mu(H) = q$, we obtain that the employer's posterior and prior beliefs coincide, that is,

$$\mu(H|E) = \mu(H) = q \quad \text{and} \quad \mu(L|E) = \mu(L) = 1 - q$$

- In other words, because all types of sender (job applicant in this example) choose the same message (E in this strategy profile):
  - the receiver (employer) cannot infer or extract any additional information she did not originally had.
- This is a useful property to remember when checking whether a pooling strategy profile can be sustained as a PBE.
  - No need to compute Bayes' rule.

Example 10.3. Applying Bayes' rule



3. *Pooling strategy, (E^H, E^L).*
   - What if, instead, the employer observes the job applicant choosing NE.
   - In this strategy profile, this should not happen in equilibrium.
     - We refer to this type of message as "off the equilibrium"
     - to emphasize it should not occur in the strategy profile we analyze.

   - If in this setting, the receiver (employer) tries to use Bayes' rule to update her beliefs, she finds an indetermination because

$$\mu(H|NE) = \frac{\mu(H)Pr(NE|H)}{Pr(NE)} = \frac{\mu(H)Pr(NE|H)}{\mu(H)Pr(NE|H) + \mu(L)Pr(NE|L)} = \frac{q \times 0}{(q \times 0) + \left((1-q) \times 0\right)} = \frac{0}{0}$$

   - In this case, the off-the-equilibrium beliefs (since NE is an off-the-equilibrium strategy) can take any arbitrary number.
     - That is, $\mu(H|NE) \in [0,1]$.
     - A similar argument applies to $\mu(L|NE)$ since, again, NE should have not been observed in this strategy profile.

Example 10.3. Applying Bayes' rule

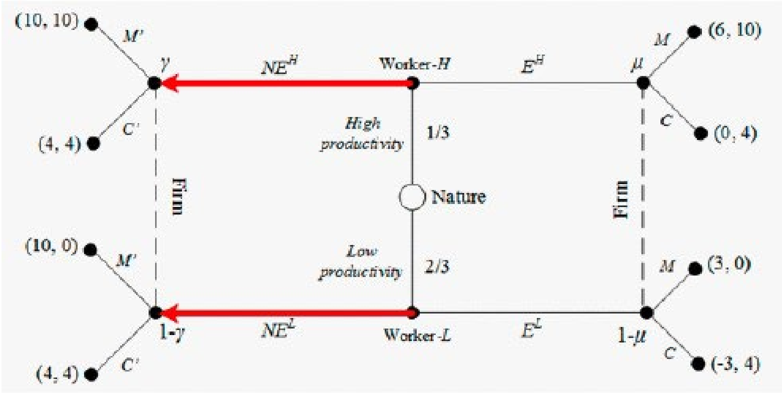


Figure 10.6. Pooling strategy profile (NE^H, NE^L).

4. *Pooling strategy,* (NE^H, NE^L).
   - A similar reasoning applies to the pooling strategy profile where the job applicant does not acquire education, NE, regardless of her type, that is,

$$\mu(H|NE) = \frac{\mu(H)Pr(NE|H)}{Pr(NE)} = \frac{\mu(H)Pr(NE|H)}{\mu(H)Pr(NE|H) + \mu(L)Pr(NE|L)} = \frac{q \times 1}{(q \times 1) + \big((1-q) \times 1\big)} = \frac{q}{q + (1-q)} = q$$

meaning that posteriors and priors coincide:

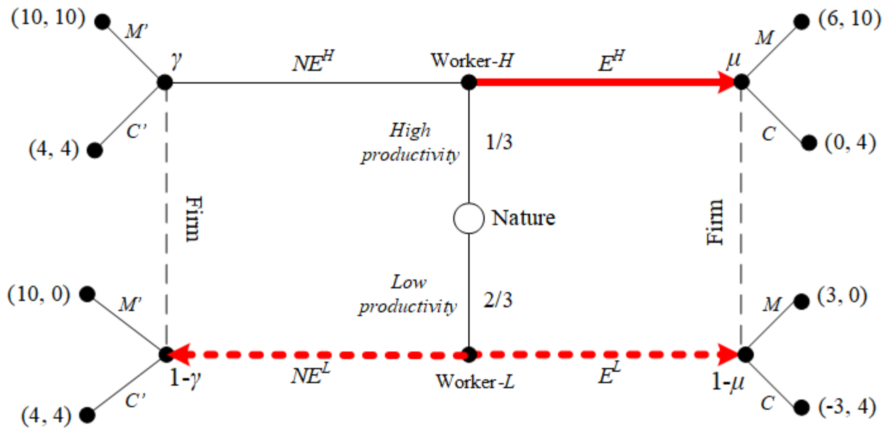
- $\mu(H|NE) = \mu(H) = q,$
- which implies that $\mu(L|NE) = \mu(L) = 1 - q.$

- Following the same argument as in the pooling strategy profile (E^H, E^L), we can confirm that Bayes' rule does not help us pin down off-the-equilibrium beliefs in this case either since

$$\mu\,(H|E) = \frac{\mu(H)Pr(E|H)}{Pr(E)} = \frac{\mu(H)Pr(E|H)}{\mu(H)Pr(E|H) + \mu(L)Pr(E|L)} = \frac{q \times 0}{(q \times 0) + \big((1-q) \times 0\big)} = \frac{0}{0}$$

- We say that off-the-equilibrium beliefs can take any arbitrary number, $\mu\,(H|E) \in [0,1]$.

Example 10.3. Applying Bayes' rule

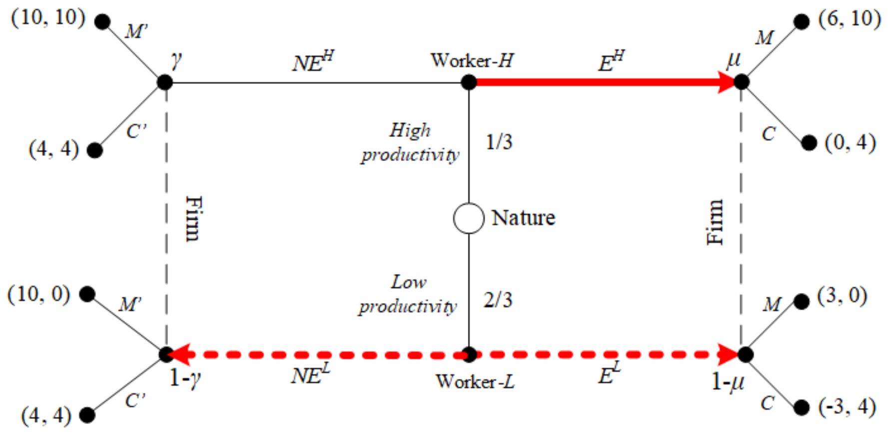


5. *Semi-separating strategy.*
   - If the sender (job applicant) uses a semi-separating strategy such as:
     - E^H with probability 1 and
     - E^L with probability $\sigma \in [0,1]$.
   - Then player $j's$ updated beliefs are

$$\mu(H|E) = \frac{\mu(H)Pr(E|H)}{Pr(E)} = \frac{\mu(H)Pr(E|H)}{\mu(H)Pr(E|H) + \mu(L)Pr(E|L)} = \frac{q \times 1}{(q \times 1) + \left((1-q) \times \sigma\right)}$$
$$= \frac{q}{q + (1-q)\sigma}$$

and $\mu(H|E) = 1 - \mu(L|E)$.

Example 10.3. Applying Bayes' rule



5. *Semi-separating strategy.*
   - When the L-type never plays E, $\sigma = 0$, the above expression simplifies to:
     - $\mu(H|E) = 1$, as in the separating strategy profile (E^H, NE^L).
   - In contrast, when she plays E with certainty, $\sigma = 1$, the employer's posterior belief becomes:
     - $\mu(H|E) = q$, as in the pooling strategy profile (E^H, E^L).
   - More generally:
     - $\mu(H|E) = \frac{q}{q+(1-q)\sigma}$ decreases in the probability with which the L-type plays E, as captured by σ.
     - Starting at $\mu(H|E) = 1$ when $\sigma = 0$ and reaching $\mu(H|E) = q$ when $\sigma = 1$.
     - Figure.
   - Informally, the employer is more convinced of facing an H-type worker when the L-type rarely acquires education than when she often does.

Tool 10.1 Finding PBEs in signaling games

1. Specify a strategy profile for the sender.
   - Such as (E^H, NE^L) in the labor-market signaling game.
   - This is just our "candidate" for a PBE.
2. Update the beliefs of the receiver using Bayes' rule, whenever possible.
   - This means that you can update beliefs in equilibrium.
   - Not off-the-equilibrium, where beliefs are left unrestricted.
3. Given the receiver's updated beliefs in Step 2, find her optimal response.
4. Given the receiver's response in Step 3, find the sender's optimal message.
5. If the sender's optimal message found in Step 4:
   a. Coincide with the message initially postulated in Step 1, we just confirmed that the strategy profile listed in Step 1, along with the receiver's updated beliefs in Step 2 and response in Step 3, can be sustained as a PBE.
   b. Does not coincide with that initially postulated in Step 1, we can claim that the strategy profile in Step 1 cannot be supported as a PBE.

Finding PBEs in games with one information set

Separating strategy profile (O^B, N^F)

1. Specifying a strategy profile.

- Step 1 is given by specifying a "candidate" of strategy profile for the sender that we seek to test as a PBE, (O^B, N^F).
- Figure 10.3 highlights the tree branch corresponding to:
  - O^B, at the top right, and
  - N^F , at the bottom left.

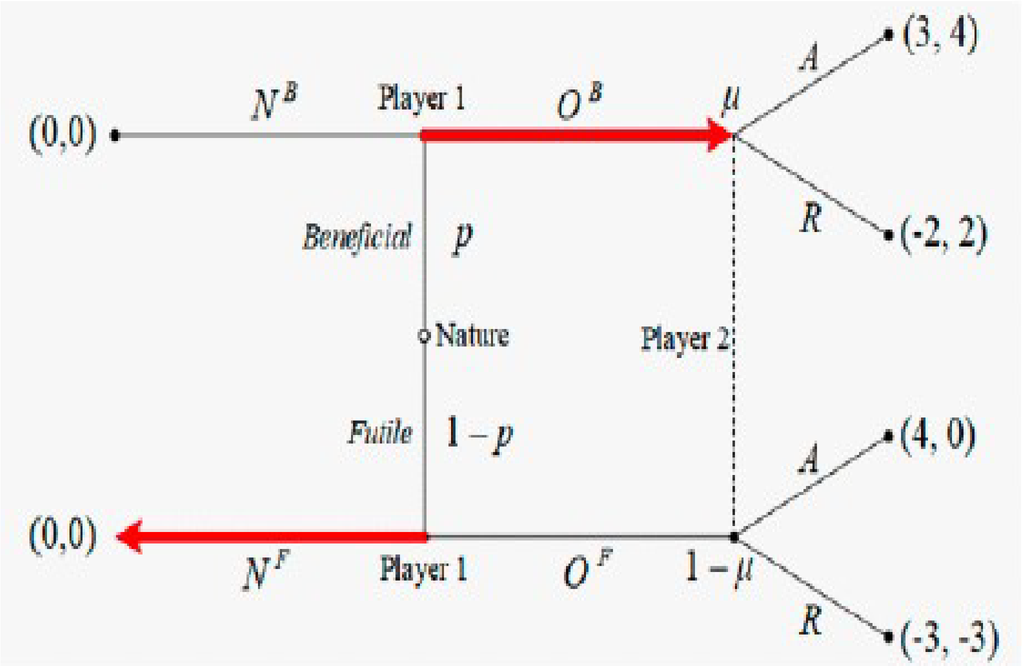


Figure 10.3. Separating strategy profile (O^B, N^F).

Finding PBEs in games with one information set

Separating strategy profile (O^B, N^F)

2. *Bayes' rule.*
   - Step 2 is straightforward in this case.
     - As described in Example 10.3, $\mu = 1$,
     - If the receiver receives an offer in this separating strategy profile, he infers that the investment must be beneficial.
   - Graphically, if he is called to move at the only information set of the game tree…
     - he puts full probability weight on being at the top node of this information set.
   - Informally, the receiver focuses his attention on the top right corner of the tree.

Finding PBEs in games with one information set

Separating strategy profile (O^B, N^F)

3. Optimal Response

- Given our result from Step 2, $\mu = 1$, the receiver responds accepting the offer, A, since $4 > 2$.
  - (Note: we only compare his payoffs at the top right corner of the tree, as he puts full probability weight on the top node.).
- To keep track of our results, we shade the branch corresponding to A.
- Note that A is shaded both at the top and bottom nodes:
  - since the receiver cannot condition his response to the sender's true type, which he does not observe.

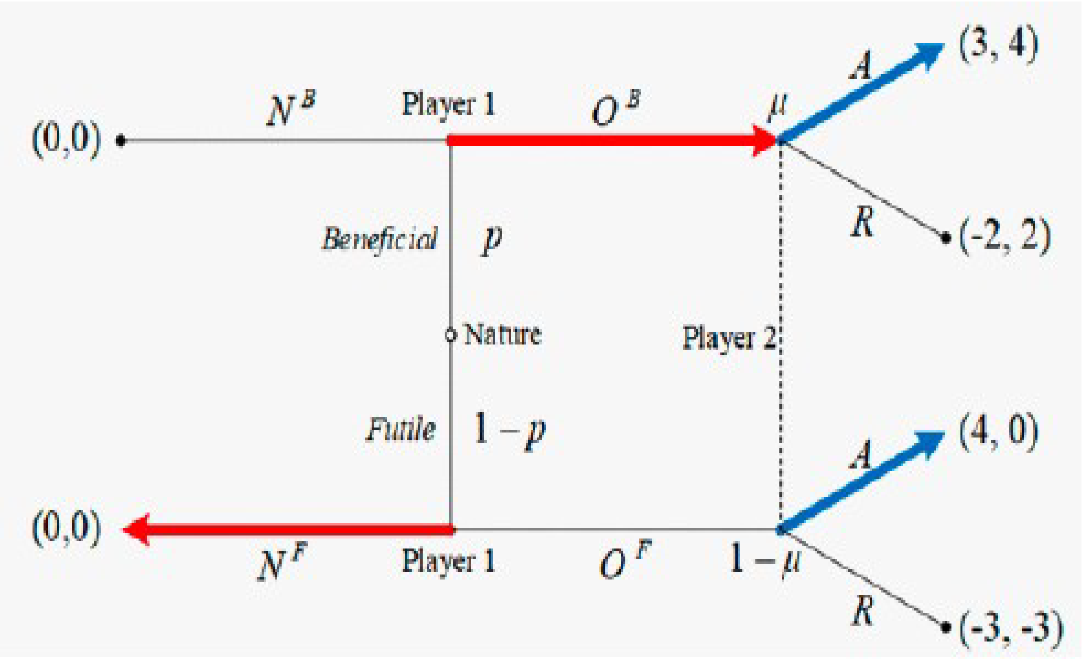


Figure 10.3a. Separating strategy profile (O^B, N^F) - Responses.

Finding PBEs in games with one information set

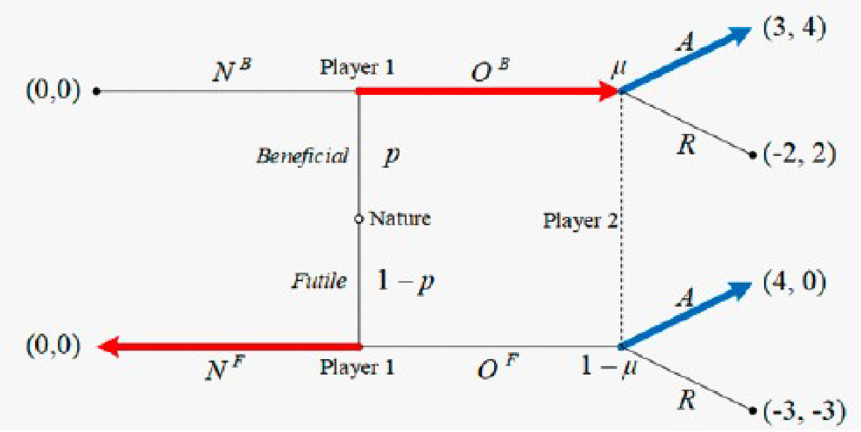


Figure 10.3a. Separating strategy profile (O^B, N^F) - Responses.

Separating strategy profile (O^B, N^F)

4. *Optimal messages*
   - From our results in Step 3, we now need to identify the sender's optimal message, which needs to be separately done for each of the sender's types.

a. When the investment is beneficial (at the top of the figure), choosing O^B, as prescribed by this strategy profile, yields 3, since the sender anticipates that the offer will be accepted by the receiver, as found in step 3.
b. Graphically, we only need to follow the shaded branches.
c. If, instead, the sender deviates towards N^B (top left in the figure), her payoff decreases to 0.
d. Therefore, the sender chooses O^B when the investment is beneficial.

Finding PBEs in games with one information set

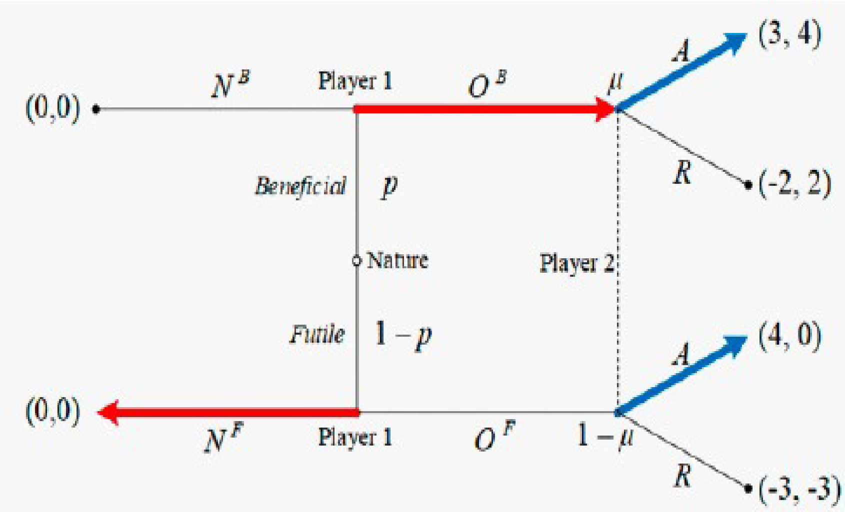


Figure 10.3a. Separating strategy profile (O^B, N^F) - Responses.

Separating strategy profile (O^B, N^F)

4. Optimal messages

- From our results in Step 3, we now need to identify the sender's optimal message, which needs to be separately done for each of the sender's types.

b. When the investment is futile (at the bottom of the figure), choosing N^F, as prescribed by this strategy profile, yields 0.
Deviating towards O^F (bottom right hand of the figure) increases her payoff to 4, since she anticipate the offer is accepted.

Finding PBEs in games with one information set

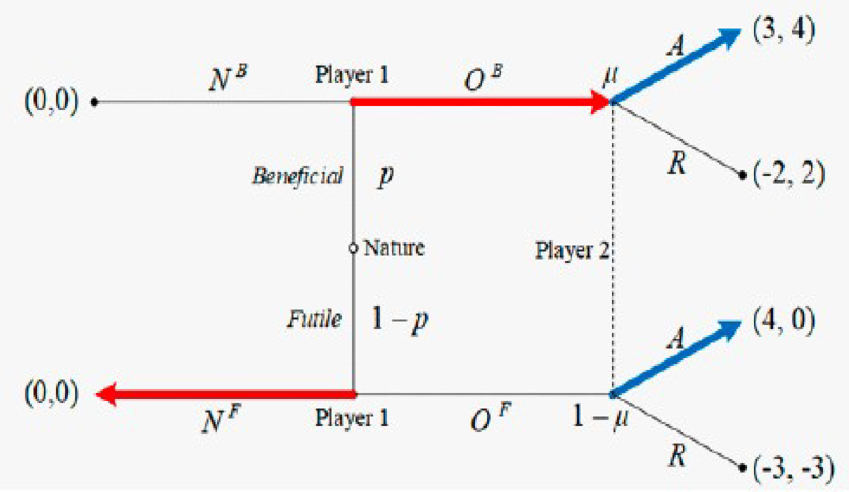


Figure 10.3a. Separating strategy profile (O^B, N^F) - Responses.

Separating strategy profile (O^B, N^F)

5. Summary

- From step 4b, we found that at least one of the sender types has incentives to deviate from (O^B, N^F),
  - namely, when the business opportunity is futile, N^F cannot be supported as optimal for the sender.
- In summary, the separating strategy profile (O^B, N^F) *cannot be sustained* as PBE.
- Had we found that all sender types had incentives to behave as prescribed in Step 1
  - making an offer when the test is beneficial, O^B, but not making it otherwise, N^F
  - we would be able to claim that (O^B, N^F) can be sustained as PBE.

Finding PBEs in games with one information set

Pooling strategy profile (O^B, O^F)

1. *Specifying a strategy profile.*

- We start specifying the strategy profile "candidate" that we test as a PBE, (O^B, O^F).
- Figure 10.4 shades the branches corresponding to:
  - O^B, at the top right, and
  - O^F at the bottom right.

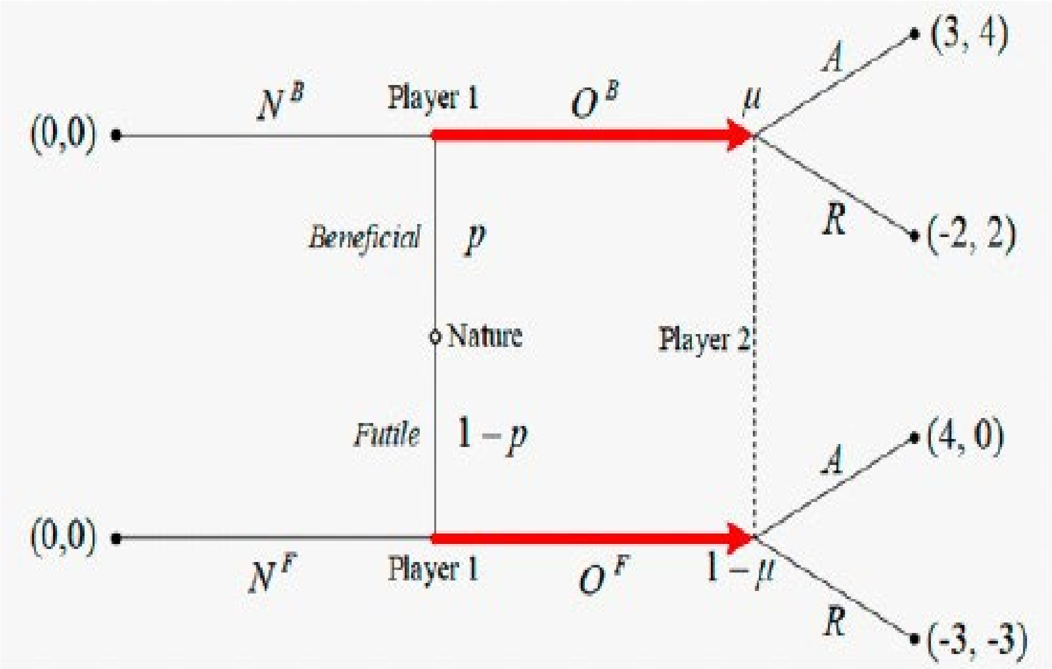


Figure 10.4. Pooling strategy profile (O^B, O^F).

Finding PBEs in games with one information set

Pooling strategy profile (O^B, O^F)

2. Bayes' rule.

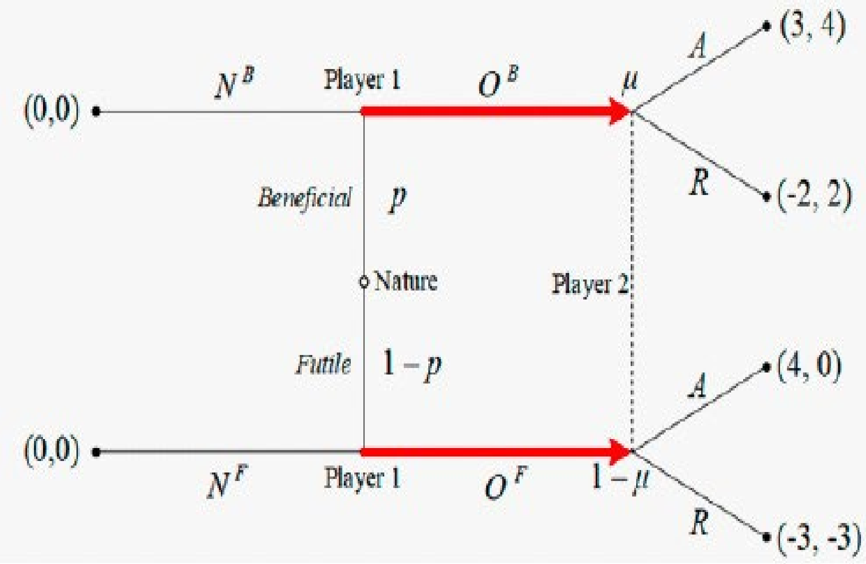


Figure 10.4. Pooling strategy profile (O^B, O^F).

- In Step 2, we update the receiver's beliefs.
- As described in section 10.4, posterior and prior beliefs coincide in this strategy profile,
  - entailing that $\mu = p$.
- Intuitively, upon receiving an offer,
  - the receiver cannot infer any information from this offer,
  - as all sender types make offers,
  - being left as uninformed as he was at the beginning of the game.

Finding PBEs in games with one information set

Pooling strategy profile (O^B, O^F)

3. *Optimal Response*
   - Given our result from Step 2, $\mu = p$, the receiver responds accepting the offer, A, since his expected payoff satisfies

$$EU_A = 4p + 0(1-p) > 2p + (-3)(1-p) = EU_R,$$

which simplifies to

$$EU_A = 4p > -3 + 5p = EU_R$$

which ultimately reduces to $3 > p$, which holds for all $p \in [0,1]$.

   - In other words, player 2 responds accepting the offer regardless of the probability that the investment is beneficial, p.
   - To keep track of our results, we shade the branch corresponding to A in Figure 10.4a.

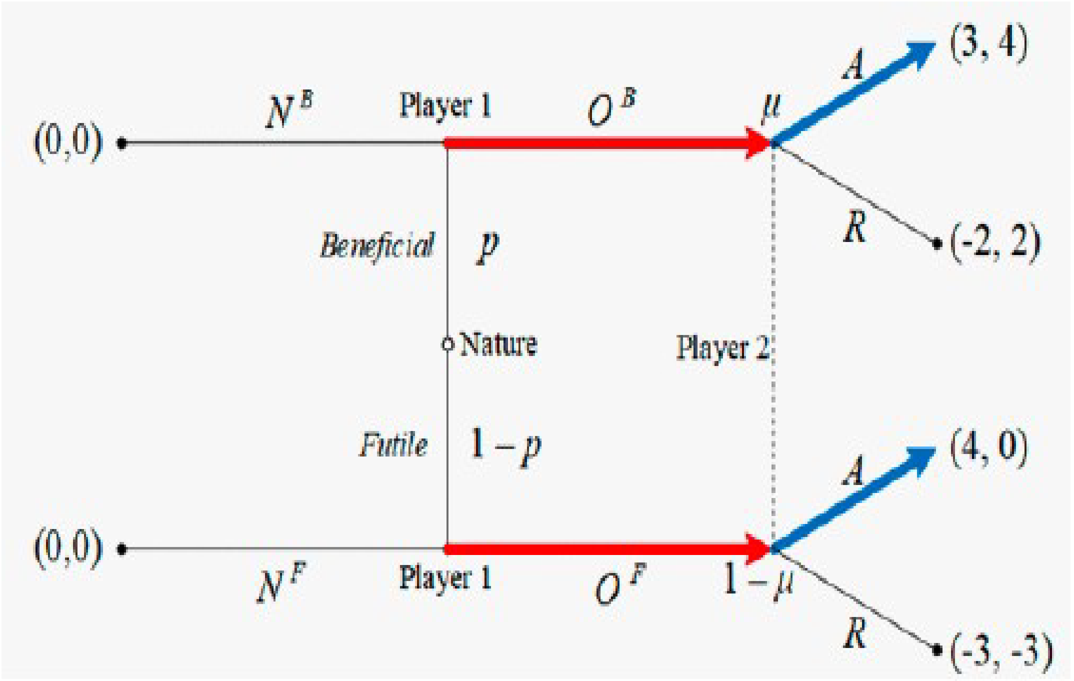


Figure 10.4a. Pooling strategy profile (O^B, O^F) - Responses.

Finding PBEs in games with one information set

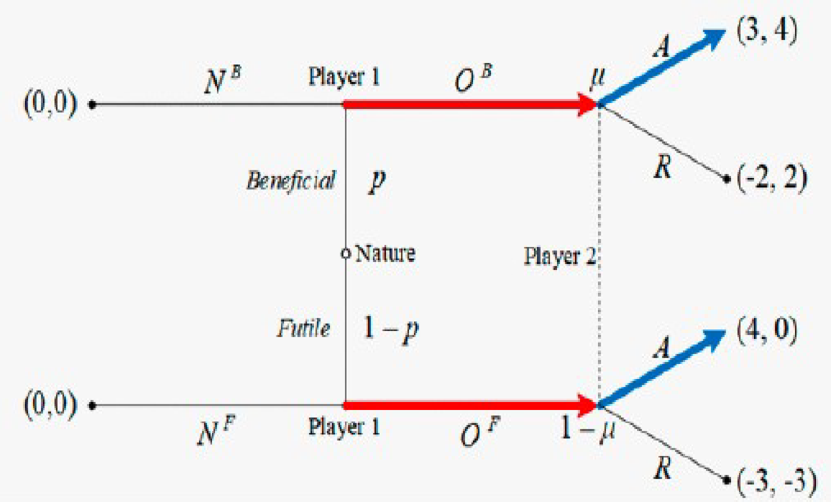


Figure 10.4a. Pooling strategy profile (O^B, O^F) - Responses.

Pooling strategy profile (O^B, O^F)

4. *Optimal messages*
   - From our results in Step 3, we now need to identify the sender's optimal message, separately analyzing each sender's types.

a. When the investment is beneficial (at the top of figure 10.4a), choosing O^B, as prescribed by this strategy profile, yields 3, whereas deviating to N^B(top left in the figure), decreases her payoff to 0. Therefore, the sender chooses O^B.

b. When the investment is not beneficial (at the bottom of the figure), choosing O^F, as prescribed by this strategy profile, yields 4.
   Deviating towards N^F would decrease her payoff to 0. Then, sender chooses O^F.

Finding PBEs in games with one information set

Pooling strategy profile (O^B, O^F)

5. Summary

- From Step 4b, we found that all sender types prefer to behave as prescribed in Step 1, (O^B, O^F) , implying that:
- this pooling strategy profile can be supported as a PBE,
  - with the receiver holding beliefs $\mu = p$ and
  - responding accepting the offer.

Finding PBEs in games with two information sets

Separating strategy profile (E^H, NE^L)

1. *Specifying a strategy profile.*

- We first specify the separating strategy profile that we seek to test as a PBE,(E^H, NE^L).
- Figure 10.5 shades the branches corresponding to E^H, at the top right, and to NE^L at the bottom left.

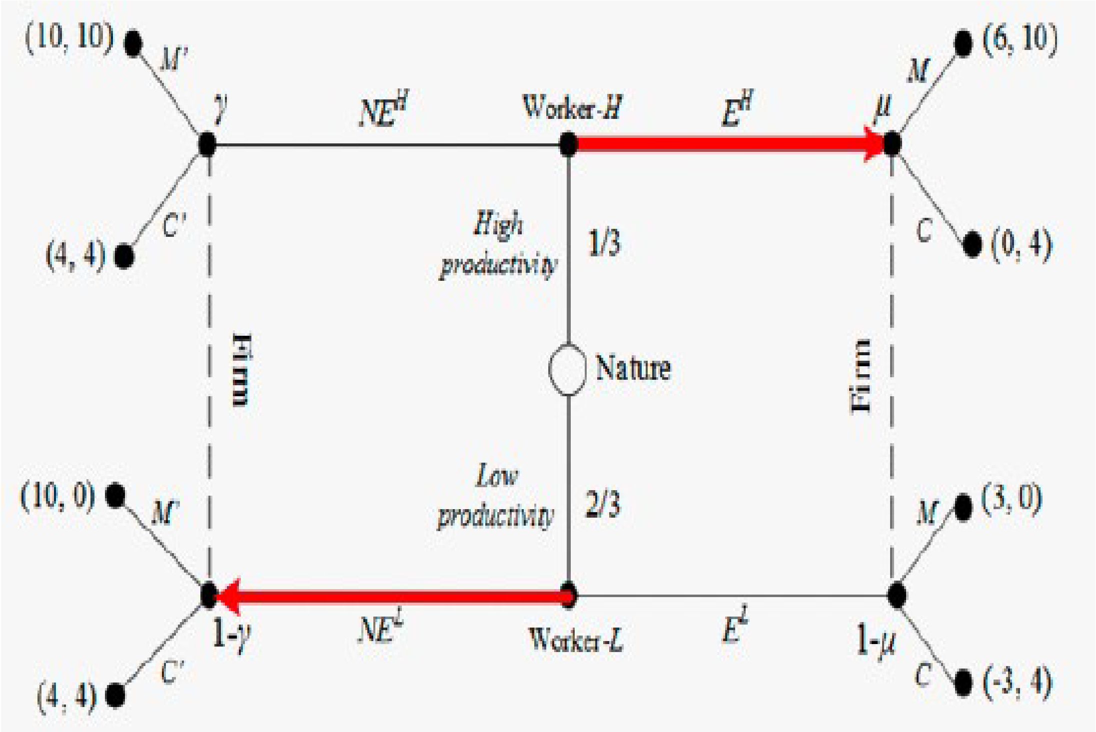


Figure 10.5. Separating strategy profile (E^H, NE^L).

Finding PBEs in games with two information sets

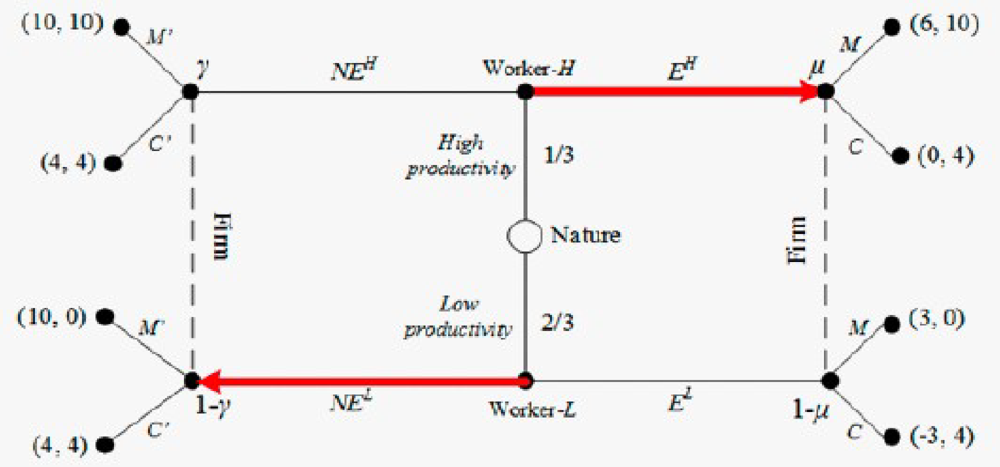


Figure 10.5. Separating strategy profile (E^H, NE^L).

Separating strategy profile (E^H, NE^L)

2. Bayes' rule.

- We can now update the firm's beliefs. Beliefs in this strategy profile satisfy

$$\mu = \frac{\frac{1}{3}\alpha^H}{\frac{1}{3}\alpha^H + \frac{2}{3}\alpha^L} = \frac{\frac{1}{3}1}{\frac{1}{3}1 + \frac{2}{3}0} = 1$$

where $\alpha^H(\alpha^L)$ denotes the probability that a high-productivity (lo-productivity) worker acquires education.

- Intuitively, upon observing education, the firm believes it must only originate from a high-productivity worker, which corresponds to the top right node of the game tree.

Finding PBEs in games with two information sets

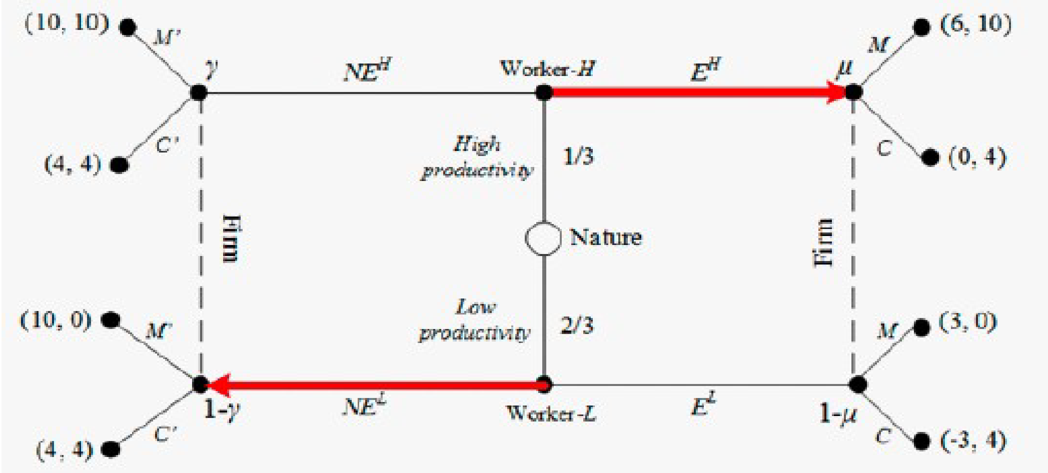


Figure 10.5. Separating strategy profile (E^H, NE^L).

Separating strategy profile (E^H, NE^L)

2. Bayes' rule.

- Upon observing no education, however, the firm's beliefs are

$$\gamma = \frac{\frac{1}{3}(1-\alpha^H)}{\frac{1}{3}(1-\alpha^H)+\frac{2}{3}(1-\alpha^L)} = \frac{\frac{1}{3}0}{\frac{1}{3}0+\frac{2}{3}1} = 0$$

or, alternatively, the probability of being at the bottom left node, $1-\gamma$, is 100 percent.

- Hence, if the firm observes no education, it assigns full probability of facing a low-productivity worker.

Finding PBEs in games with two information sets

Separating strategy profile (E^H, NE^L)

3. Optimal Response

- Given our results from Step 2, we now analyze the firm's responses upon observing each of the two possible messages (education or no education).

a. Upon observing education, the firm responds hiring the worker as a manager, M, since $10 > 4$ at the top right side of the game tree. (Recall that the firm believes after observing education, that it deals with a high-productivity worker.)

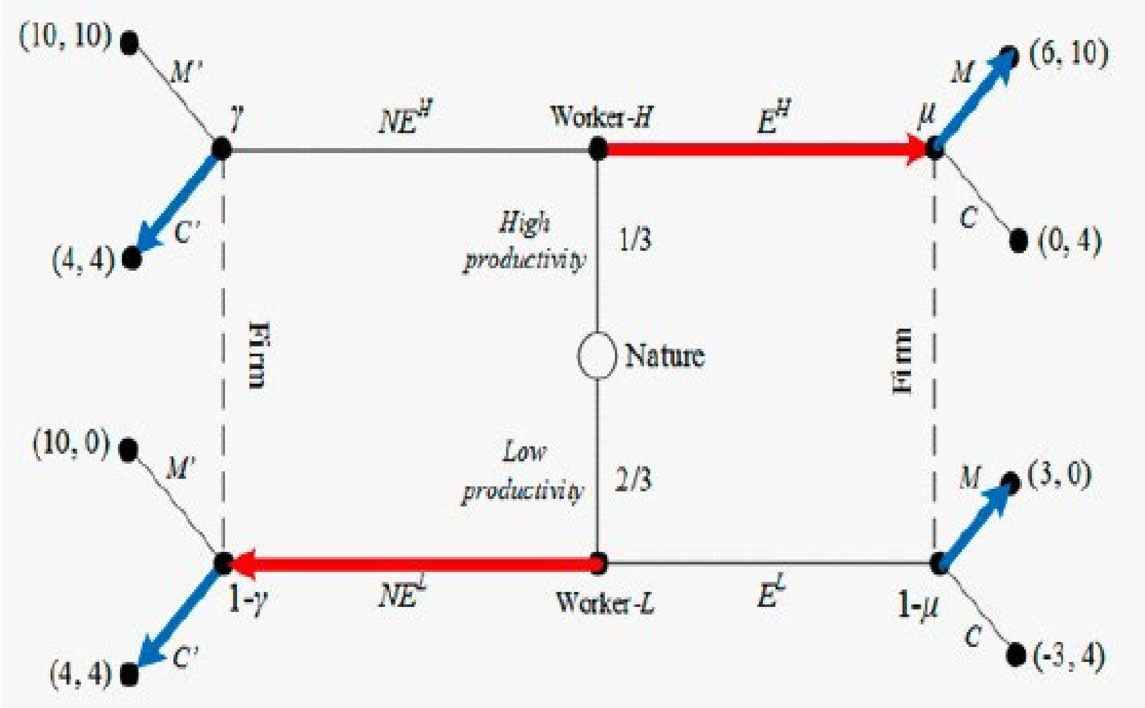


Figure 10.5a. Separating strategy profile E^H, NE^L - Responses.

Finding PBEs in games with two information sets

Separating strategy profile (E^H, NE^L)

3. Optimal Response

b. If, instead, the firm observes no education, it responds hiring the worker as cashier, C', because $4 > 0$ at the bottom left corner of the tree.

To keep track of our results, Figure 10.5a shades the branches corresponding to M in the right side of the tree, and those corresponding to C' in the left side.

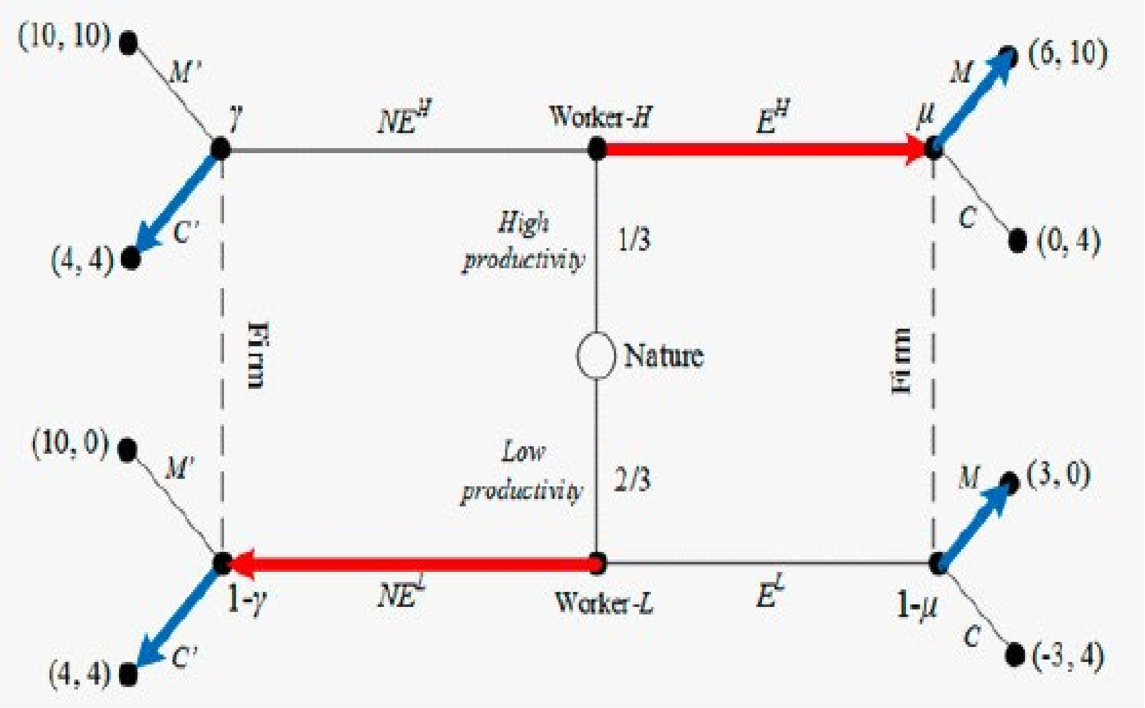


Figure 10.5a. Separating strategy profile E^H, NE^L - Responses.

Finding PBEs in games with two information sets

Separating strategy profile (E^H, NE^L)

4. *Optimal Messages.* From our results in Step 3, we now identify the worker's optimal message, separately analyzing each type.
   a. *High productivity.*
      - At the top of the game tree, the high-productivity worker chooses E^H, moving rightward, instead of deviating to NE^H on the left side of the tree, since $6 > 4$.
      - Intuitively, this worker type anticipates that education will be recognized as a signal of her high productivity, inducing the firm to respond hiring her as a manager, as indicated by the shaded branches originating from the central node at the top of the figure and moving rightward.

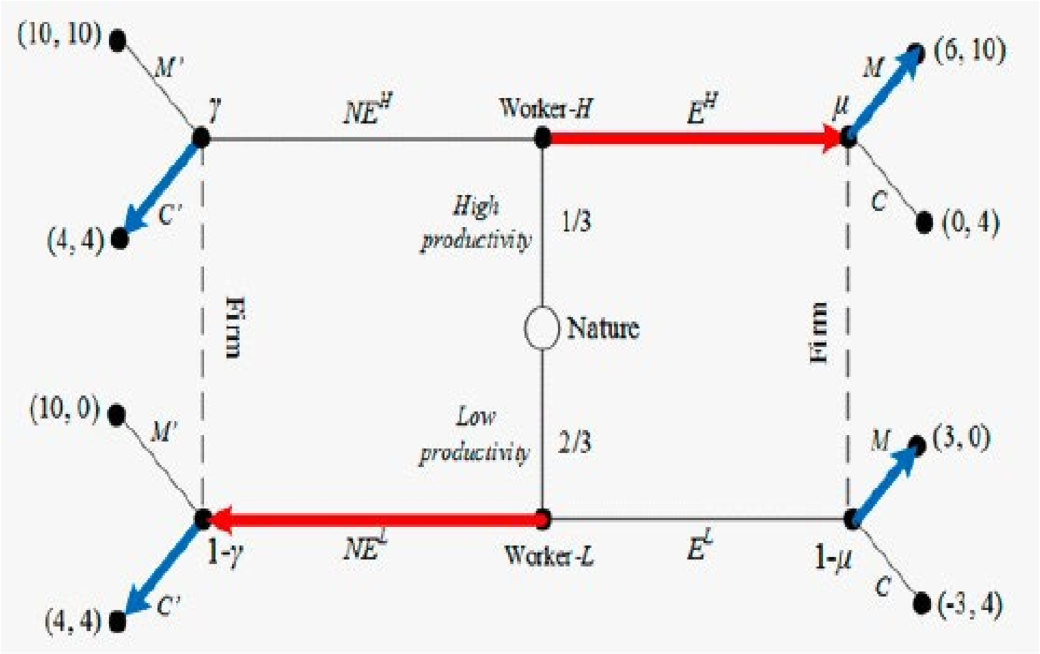


Figure 10.5a. Separating strategy profile E^H, NE^L - Responses.

Finding PBEs in games with two information set

Separating strategy profile (E^H, NE^L)

4. *Optimal Messages.*

a. *High productivity.*

- If this worker deviates to NE^H, she would save the education costs, but is identified as a low-productivity type, hired as a cashier, and earning only 4.
- As a consequence, the cost of acquiring education (relatively low for this worker type) is offset by the wage gain that she experiences when she is hired as a manager.

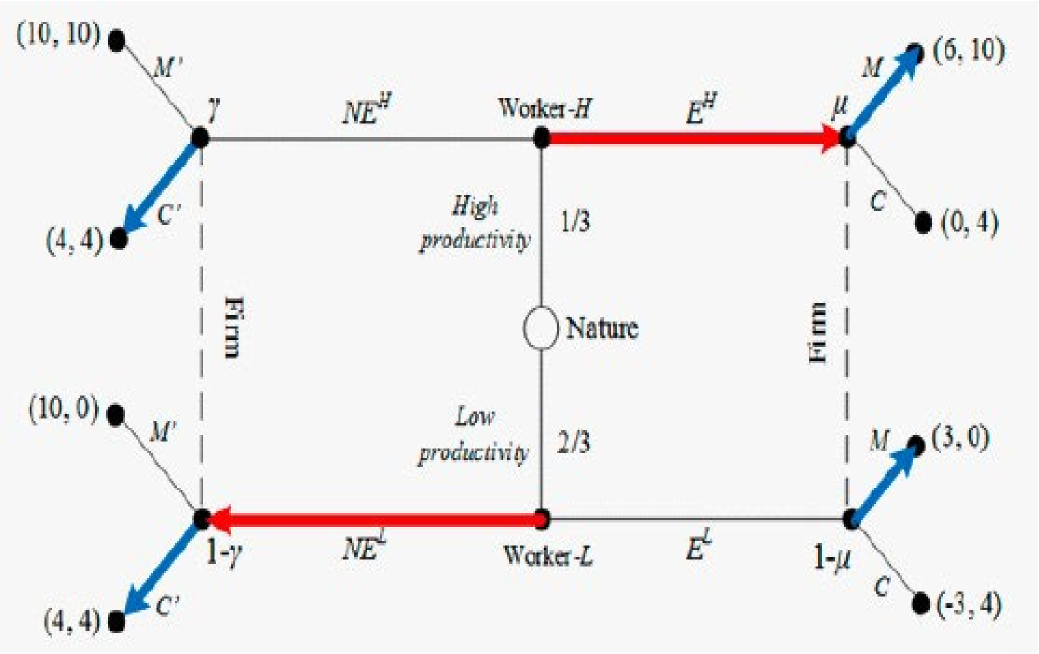


Figure 10.5a. Separating strategy profile E^H, NE^L - Responses.

Finding PBEs in games with two information set

Separating strategy profile (E^H, NE^L)

4. Optimal Messages.

b. Low productivity.

- At the bottom of the game tree, the low-productivity worker chooses NE^L, moving leftward, which yields 4, instead of deviating to acquire education, on the right side of the tree, as that would only yield 3.
- Intuitively, acquiring education helps her "fool" the firm into believing that she is a high-productivity worker and hiring her as a manager.

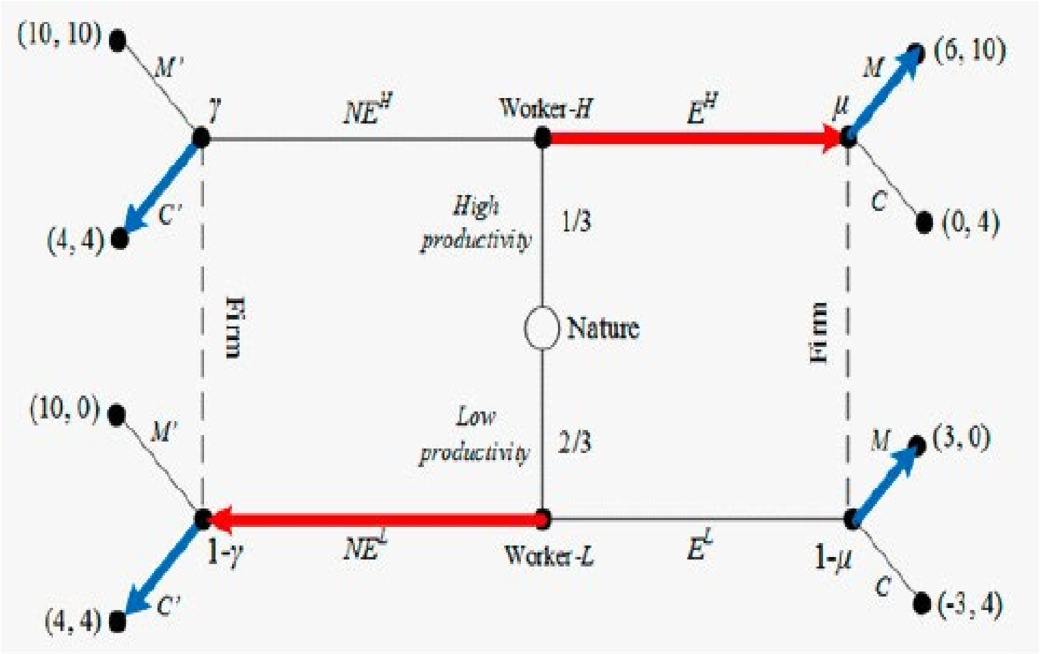


Figure 10.5a. Separating strategy profile E^H, NE^L - Responses.

Finding PBEs in games with two information set

Separating strategy profile (E^H, NE^L)

4. *Optimal Messages.*

b. *Low productivity.*

- Her wage gain, however, does not offset her cost of acquiring education, which is larger than that of the high-type worker.
- As a result, the low productivity worker does not have incentives to mimic the high-productivity type acquiring education.

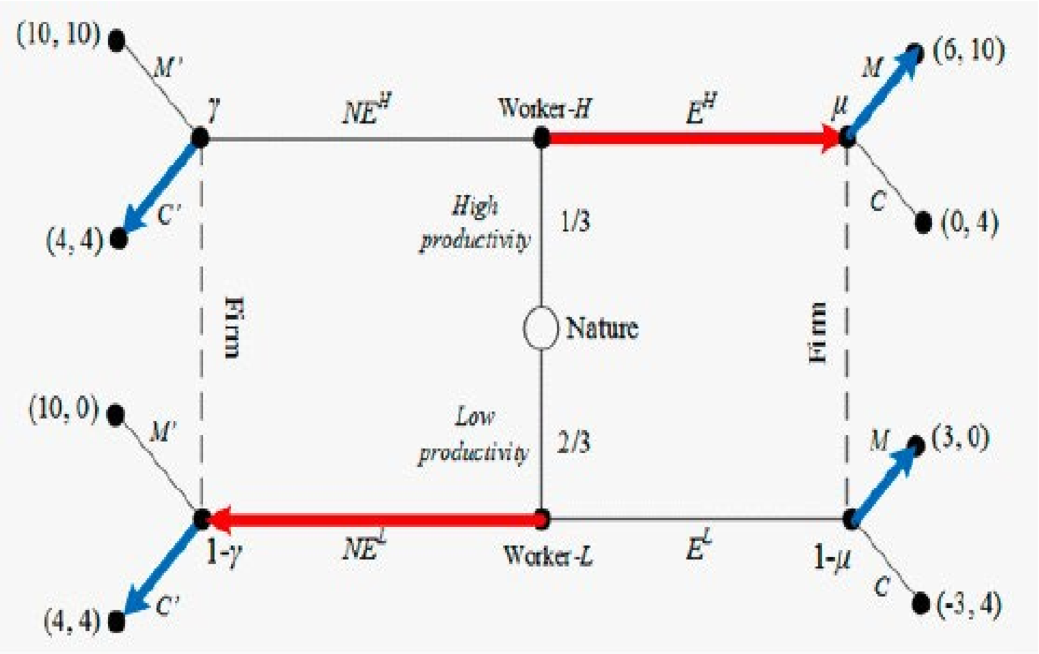


Figure 10.5a. Separating strategy profile E^H, NE^L - Responses.

Finding PBEs in games with two information set

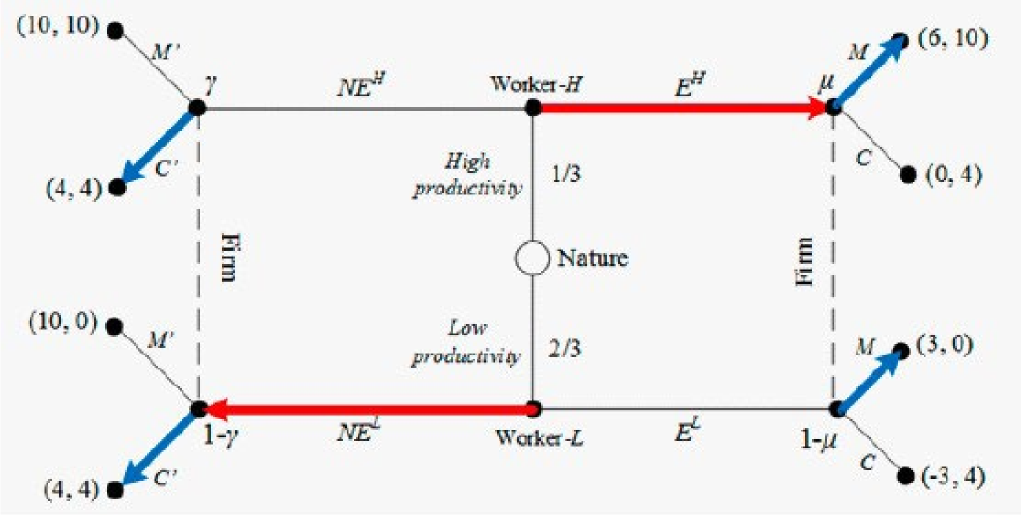


Figure 10.5a. Separating strategy profile E^H, NE^L - Responses.

Separating strategy profile (E^H, NE^L)

5. Summary.

- From Steps 4a-4b, we found that all sender types prefer to behave as prescribed in Step 1, (E^H, NE^L),
- implying that this separating strategy profile can be supported as a PBE:
- with the firm holding beliefs $\mu = 1$ and $\gamma = 0$, and
- responding with (M, C') i.e., hiring the worker as a manager upon observing education but as a cashier otherwise.

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

1. *Specifying a strategy profile.*

- We start by specifying the pooling strategy profile, (NE^H, NE^L) in which no worker type acquires education

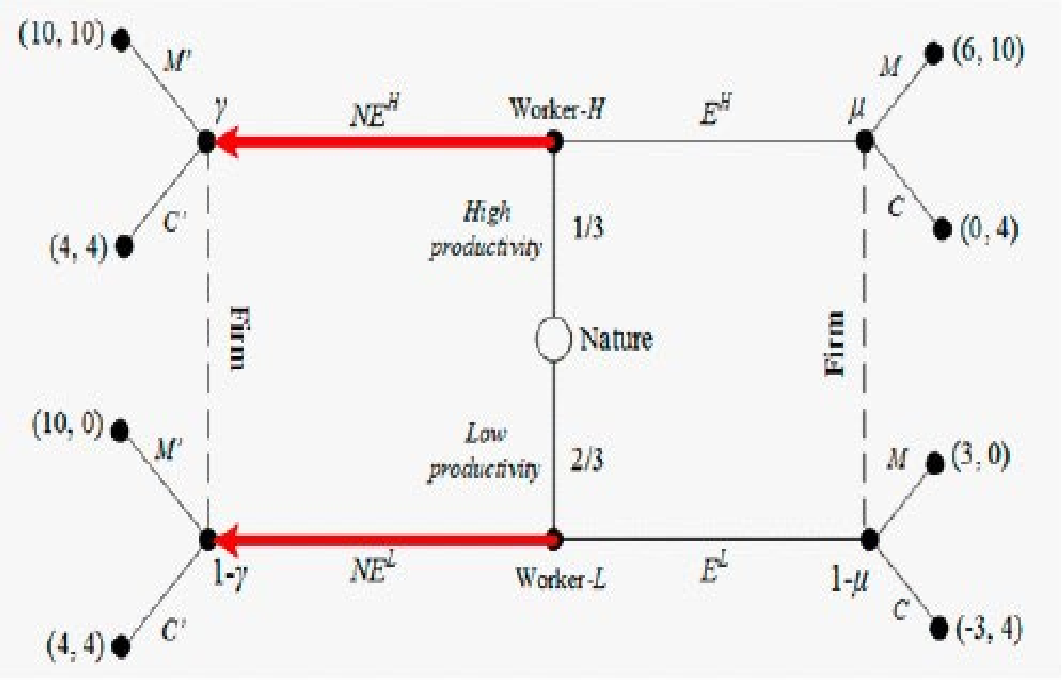


Figure 10.6. Pooling strategy profile (NE^H, NE^L).

Finding PBEs in games with two information sets

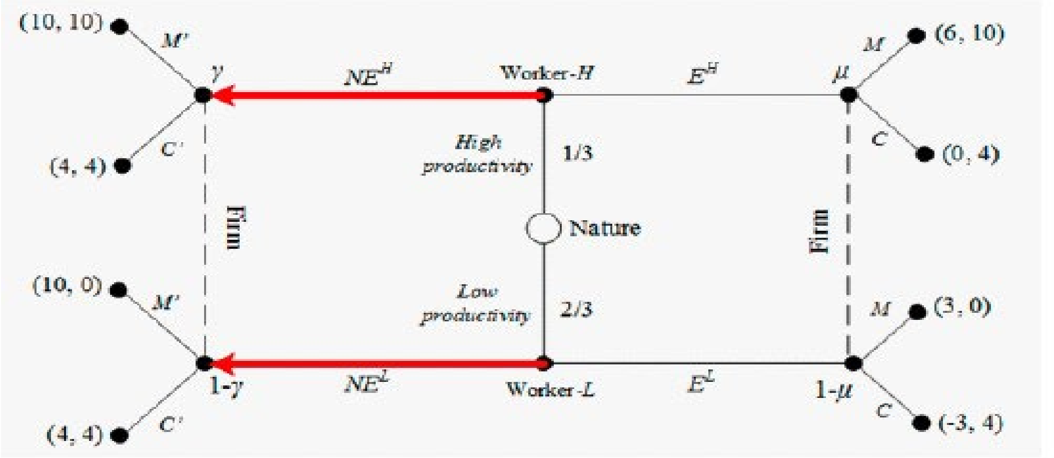


Figure 10.6. Pooling strategy profile (NE^H, NE^L).

Pooling strategy profile (NE^H, NE^L)

2. Bayes' rule.

- Upon observing no education, the firm's beliefs are

$$\gamma = \frac{\frac{1}{3}(1-\alpha^H)}{\frac{1}{3}(1-\alpha^H)+\frac{2}{3}(1-\alpha^L)} = \frac{\frac{1}{3}1}{\frac{1}{3}1+\frac{2}{3}1} = \frac{1}{3}$$

Implying that posterior beliefs (γ) coincide with prior beliefs $(1/3)$.

- In other words, observing that the worker did not acquire education,
- provides no information about her type to the firm,
- since in this strategy profile, all worker types do not acquire education,
- i.e., $1-\alpha^H = 1-\alpha^L = 1$.

Finding PBEs in games with two information sets

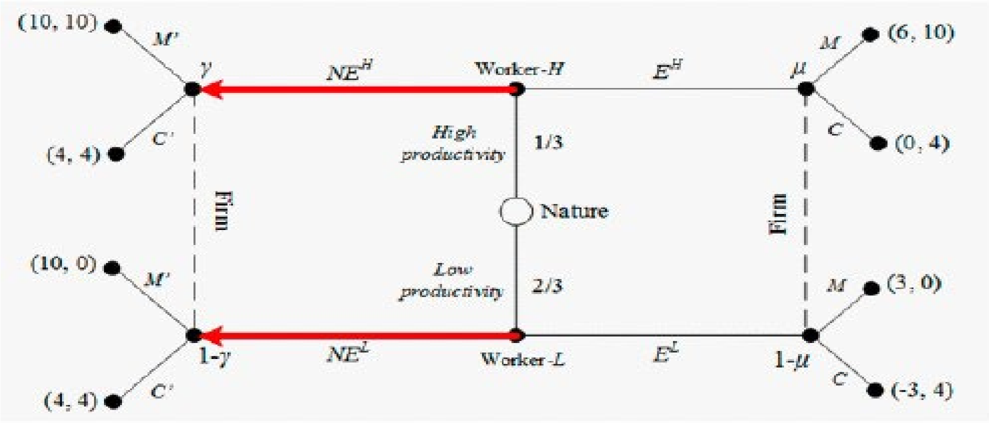


Figure 10.6. Pooling strategy profile (NE^H, NE^L).

Pooling strategy profile (NE^H, NE^L)

2. Bayes' rule.

- If, instead, the firm observes that the worker has education, which should not occur in this strategy profile, its updated beliefs are

$$\mu = \frac{\frac{1}{3}\alpha^H}{\frac{1}{3}\alpha^H + \frac{2}{3}\alpha^L} = \frac{\frac{1}{3}0}{\frac{1}{3}0 + \frac{2}{3}0} = \frac{0}{0}$$

thus being undefined.

- Graphically, the information set on the right side of the game tree (when the firm observes education) happens off-the-equilibrium path and the firm's beliefs in this information set is known as "off-the-equilibrium beliefs."
- These beliefs cannot be pinned down using Bayes' rule, as we only found an indetermination.
- For generality, these beliefs are left unrestricted, so that they can take any admissible value, i.e., $\mu \in [0,1]$.

Finding PBEs in games with two information sets

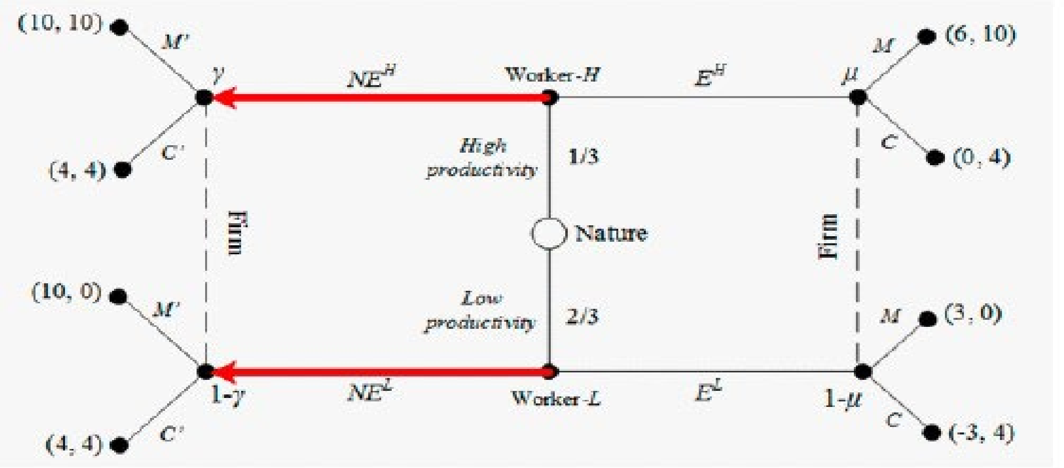


Figure 10.6. Pooling strategy profile (NE^H, NE^L).

Pooling strategy profile (NE^H, NE^L)

3. Optimal Responses.

- Given our results from Step 2, we now analyze the firm's responses upon observing each of the two possible messages (education or no education).

a. Upon observing an educated applicant, the firm responds hiring the worker as a manager, M', or as a cashier, C', based on its expected profit on the left side of the tree, as follows

$$E\pi_{Firm}(M') = 10\frac{1}{3} + 0\frac{2}{3} = \frac{10}{3} \simeq 3.33 \quad \text{and}$$

$$E\pi_{Firm}(C') = 4\frac{1}{3} + 4\frac{2}{3} = 4.$$

Therefore, the firm responds hiring the worker as a cashier, C', when she did not acquire education, since $4 > 3.33$.

Finding PBEs in games with two information sets

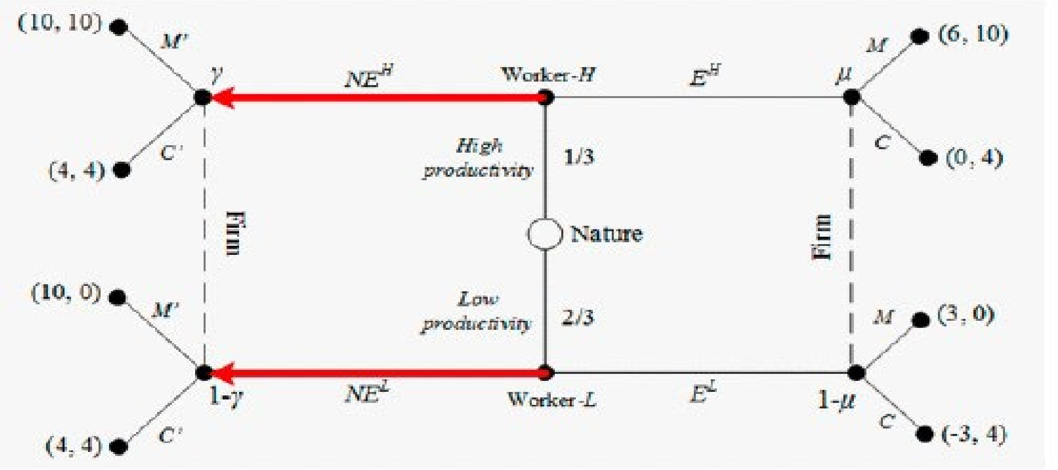


Figure 10.6. Pooling strategy profile (NE^H, NE^L).

Pooling strategy profile (NE^H, NE^L)

3. *Optimal Responses.*

b. If, instead, the firm observes education, it responds hiring the worker as a manager, M, if its expected profit at the right side of the game tree satisfies

$$E\pi_{Firm}(M) = 10\mu + 0(1-\mu) = 10\mu \text{ and}$$

$$E\pi_{Firm}(C) = 4\mu + 4(1-\mu) = 4$$

and, comparing these expected profits, we obtain that $10\mu > 4$ holds if $\mu > \frac{2}{5}$.

- Then, the firm responds to an educated applicant hiring her as a manager, M, when its off-the-equilibrium beliefs satisfy $\mu > \frac{2}{5}$ (when it assigns a sufficiently high probability weight on facing a high-productivity worker),
- but responds hiring the applicant as a cashier, C, otherwise.

Finding PBEs in games with two information sets

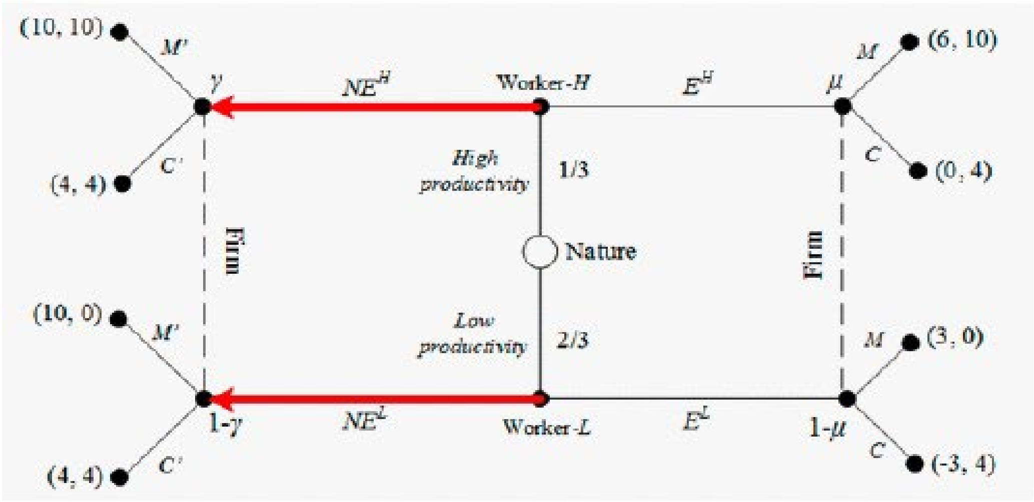


Figure 10.6. Pooling strategy profile (NE^H, NE^L).

Pooling strategy profile (NE^H, NE^L)

3. Optimal Responses.

- we then need to divide our following step, where we examine the worker's decisions, into two cases:
  1. $\mu > \frac{2}{5}$, where the firm responds hiring the worker as a manager, M, after observing education;
  2. $\mu \leq \frac{2}{5}$, where the firm responds hiring her as a cashier, C, after observing education.

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

4. *Optimal Messages.* From our results in Step 3, we now identify the worker's optimal message, separately analyzing cases (1)-(2)

   Case 1: $\mu > \frac{2}{5}$, illustrated in Figure 10.6a, where the firm responds with C' after observing no education (left side of the tree) and with M, after observing education (right side, given that $\mu > \frac{2}{5}$)

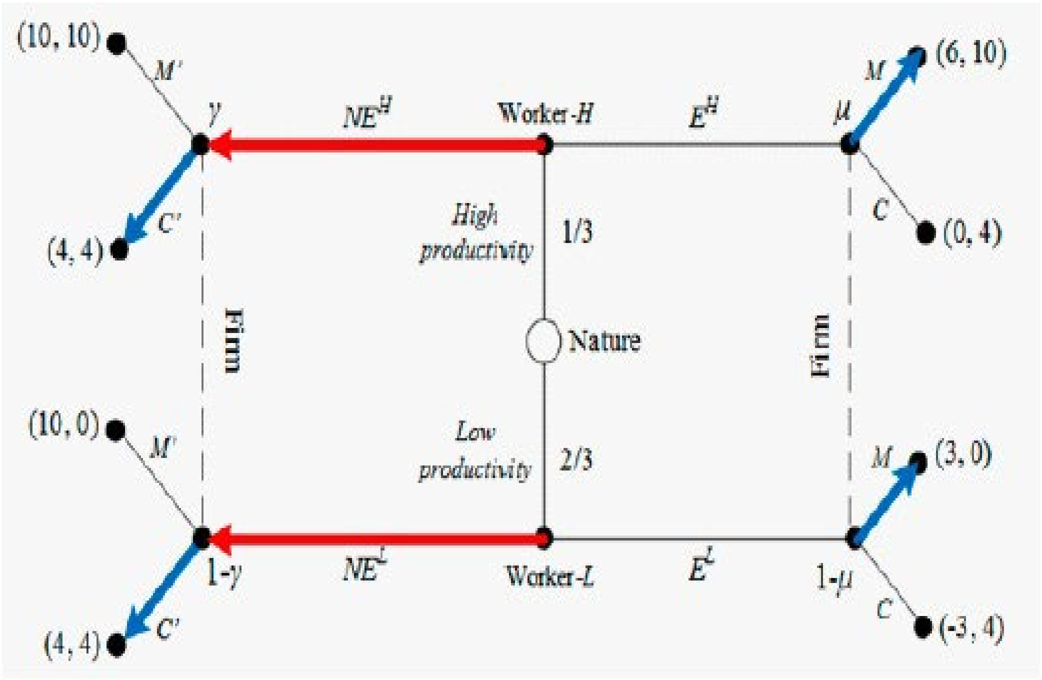


Figure 10.6a. Pooling strategy profile (NE^H, NE^L) - Responses (C', M).

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

4. *Optimal Messages.* Case 1: $\mu > \frac{2}{5}$

a. *High Productivity.*

- At the top of the game tree, if the high-productivity worker chooses NE^H, moving leftward, she earns 4, as she anticipates that the firm responds with C'.
- If, instead, the worker deviates to E^H, she earns 6, implying that she does not have incentives to behave as prescribed by this strategy profile.
- At this point, we do not need to check if the low-productivity worker has incentives to choose NE^L, since we already found that a worker's type (high productivity) would deviate.

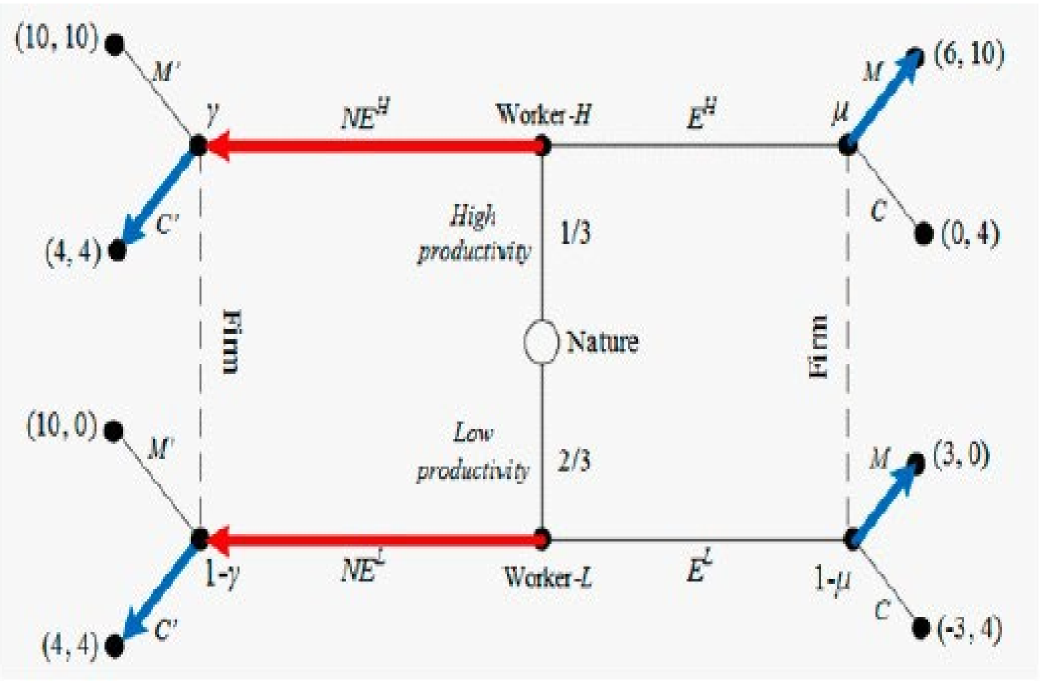


Figure 10.6a. Pooling strategy profile (NE^H, NE^L) - Responses (C', M).

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

4. *Optimal Messages.* Case 1: $\mu > \frac{2}{5}$

b. *Low Productivity.*

- At the bottom of the game tree, if the low-productivity worker chooses NE^L, moving leftward, she earns 4, which exceeds her payoff from deviating to E^L, on the right side, 3.
- Intuitively, the low-productivity worker anticipates that she will be recognized as such when she does not acquire education, and hired as a cashier.
- Even if she could fool the firm into believing that it deals with a high-productivity worker when she acquires education, and be hired as a manager…
  - The cost of investing in education is too high for this type of worker to undergo such deviation.

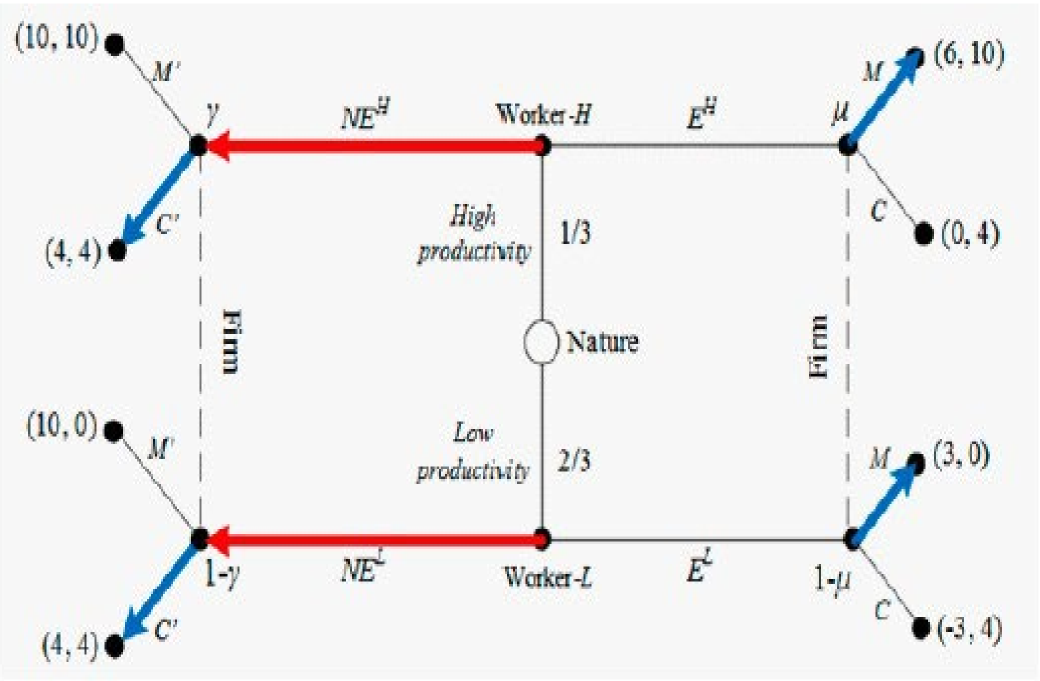


Figure 10.6a. Pooling strategy profile (NE^H, NE^L) - Responses (C', M).

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

4. *Optimal Messages.* Case 2: $\mu \leq \frac{2}{5}$, illustrated in Figure 10.6b, where the firm responds with C' after observing no education (left side of the tree)
5. and with C, after observing education (right side, given that $\mu \leq \frac{2}{5}$)

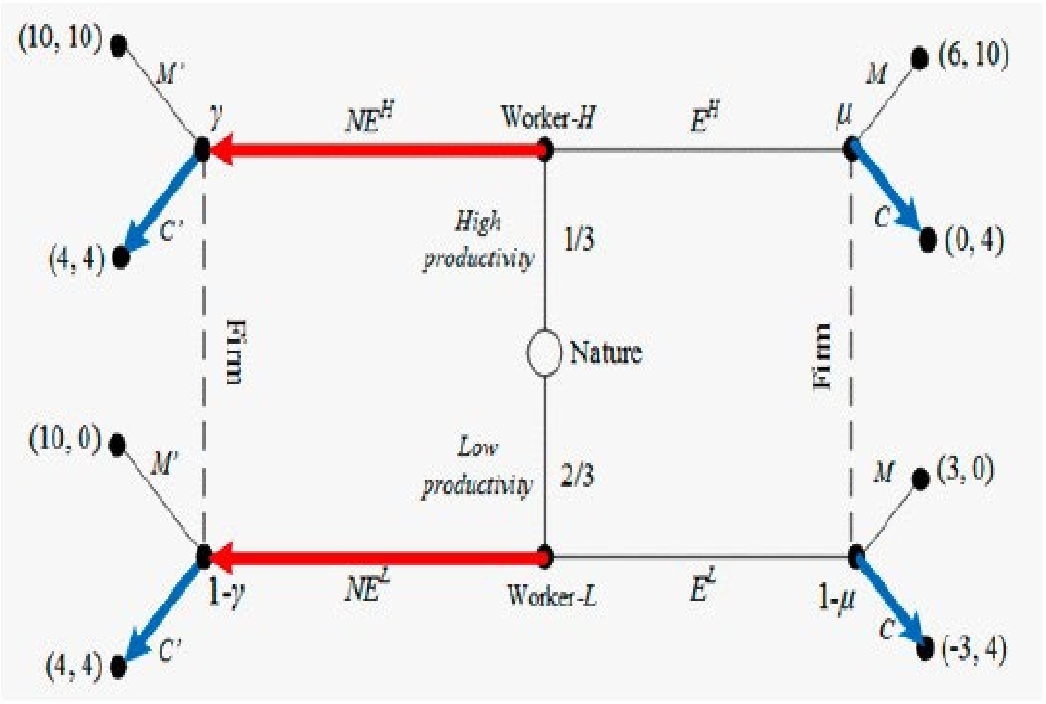


Figure 10.6b. Pooling strategy profile (NE^H, NE^L) - Responses (C', C).

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

4. *Optimal Messages.* Case 2: $\mu \leq \frac{2}{5}$

a. *High Productivity.*
   - At the top of the game tree, if the high-productivity worker chooses NE^H, moving leftward, she earns 4, as she anticipates that the firm responds with C'.
   - If, instead, the worker deviates to E^H, she only earns 0, implying that she does not have incentives to deviate.
   - Intuitively, the high productivity worker is hired as a cashier regardless of whether she acquires education,
     - entailing that it is optimal for her to not undergo the investment in education as it does not affect the firm's response.

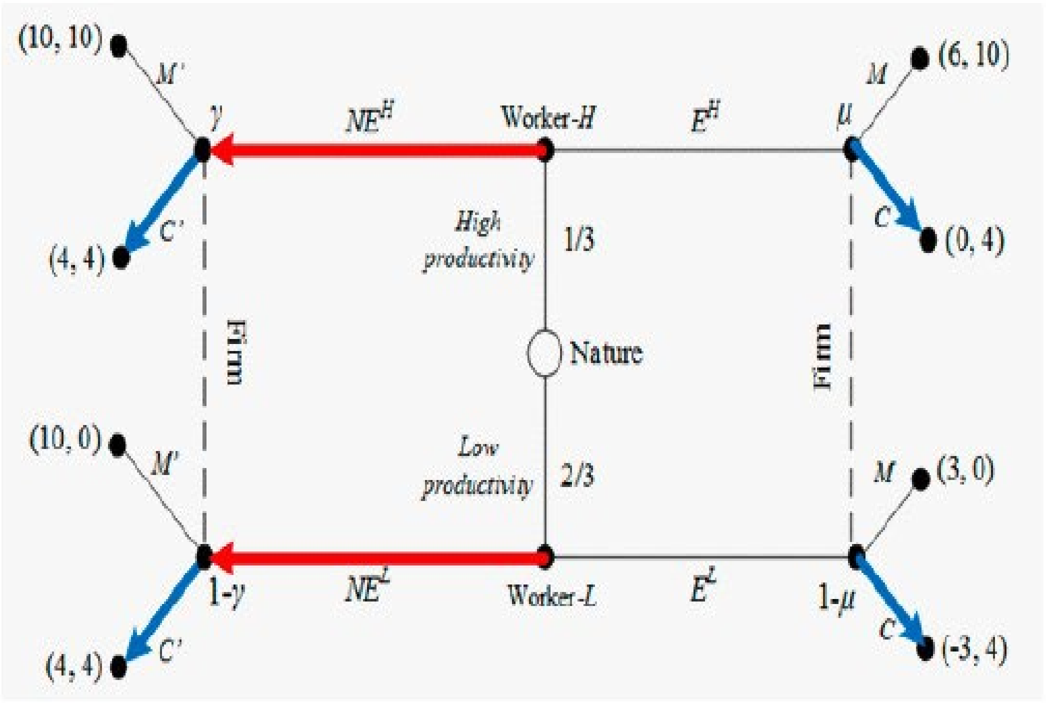


Figure 10.6b. Pooling strategy profile (NE^H, NE^L) - Responses (C', C).

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

4. *Optimal Messages.* Case 2: $\mu \leq \frac{2}{5}$

b. *Low Productivity.*

- At the bottom of the game tree, if the low-productivity worker chooses NE^L, moving leftward, she earns 4, which exceeds her payoff from deviating to E^L, on the right side, -3.
- In this case, this type of worker faces similar incentives as the high-type above, as her educational level does not affect the firm's response,
  - leading her to not invest in education.

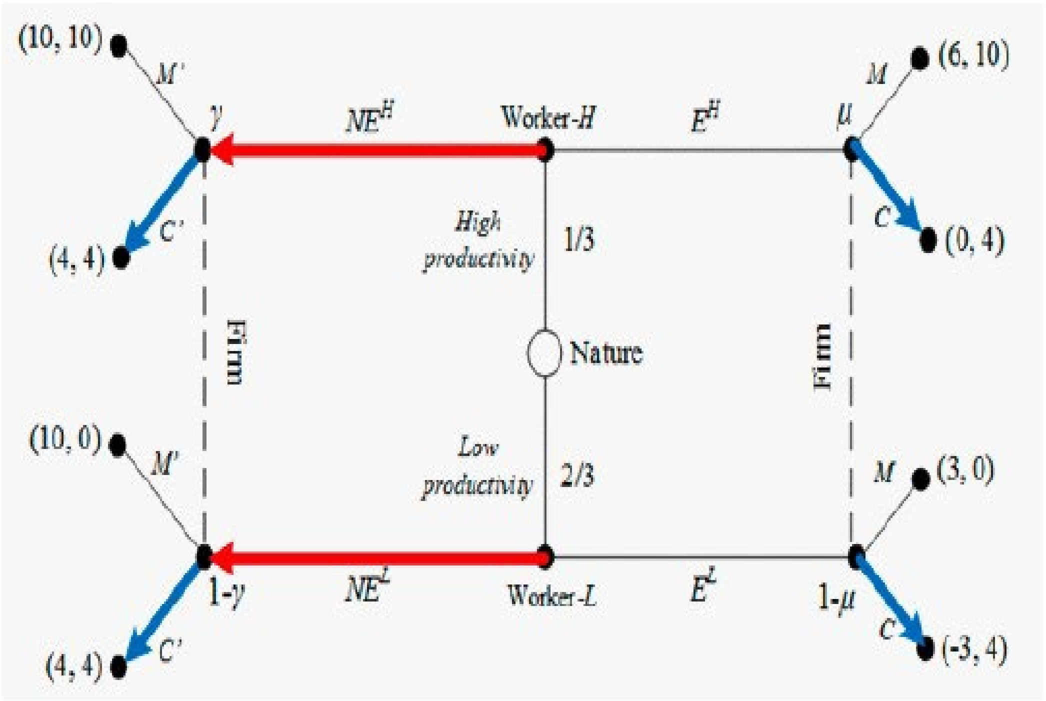


Figure 10.6b. Pooling strategy profile (NE^H, NE^L) - Responses (C', C).

Finding PBEs in games with two information sets

Pooling strategy profile (NE^H, NE^L)

5. Summary.

- When $\mu > \frac{2}{5}$ holds (Case 1), we found that one sender type (the high productivity worker) prefers to deviate from the pooling strategy profile, implying that (NE^H, NE^L) cannot be supported as PBE (see step 4a).
- In contrast, when $\mu \leq \frac{2}{5}$ holds, both worker types have incentives to behave as prescribed by pooling strategy profile (NE^H, NE^L), implying that it is sustained as a PBE with the firm responding with (C, C') and with equilibrium beliefs $\gamma = 1/3$ and off-the-equilibrium beliefs $\mu \leq \frac{2}{5}$.

Insensible off-the-equilibrium beliefs

- The pooling strategy profile (NE^H, NE^L), where no worker type acquires education, requires that $\mu \leq \frac{2}{5}$
- Informally, this is like saying that in a world where no worker type goes to college, if a firm observes a worker with a college degree (surprise!), it is likely to face a low-productivity worker.
- You may feel that this off-the-equilibrium belief is a bit insensible
- Recall that, from our above discussion in Case 1 (Step 4b):
  - the low-productivity worker would not like to deviate, from $NE^L\ to\ E^L$, even if it could fool the firm into believing that such a message stems from the high-productivity worker, and respond hiring her as a manager.
- In other words, the cost of acquiring education for the low-productivity worker is so high that it prevents her from deviating from NE^L.

Evaluating PBE as a solution concept

1. **Existence? Yes.**

- When we apply PBE to any game, we find that at least one equilibrium exists.
- Intuitively, this result is equivalent to the existence of SPE in sequential-move games of complete information,
  - but extended to an incomplete information setting.
- For this finding to hold, however, we may need to allow for mixed strategies; a
  - lthough all applications in this chapter produce a pure-strategy PBE.

Evaluating PBE as a solution concept

2. Uniqueness? No.

- This point is illustrated by the labor market signaling game in section 10.7 where we found <u>two PBEs</u>:
  1. The separating strategy profile where only the high-type worker acquired education (E^H, NE^L), and
  2. The pooling strategy profile where no types of worker does, (NE^H, NE^L)
- Other games, however, such as that in section 10.6, have a unique PBE:
  - (O^B, N^F), where player 1 only makes an investment offer, when such investment is beneficial for player 2.
- Therefore, we cannot guarantee that the PBE solution concept provides a unique equilibrium prediction in *all* games,
  - entailing that uniqueness does not hold for PBE.

Evaluating PBE as a solution concept

3. Robust to small payoff perturbations? Yes.

- PBE yields the same equilibrium predictions if we were to change the payoff of one of the players by a small amount (e.g., 0.001 or, generally, any ε that approaches zero).
- This occurs because, if a strategy s_i is sequentially optimal for player i in our original game, it must still be optimal after we apply a small payoff perturbation.

Evaluating PBE as a solution concept

4. Socially optimal? No.

- As described in the labor market signaling game (section 10.7), the presence of incomplete information gives rise to inefficiencies.
- In particular, in the separating PBE (E^H, NE^L), the high-type worker invests in a costly education (which does not improve her productivity) just to signal her type to the firm, and thus, be hired as a manager.
- In a complete information setting, instead, the firm would observe the worker's types, hiring the high (low) type as a manager (cashier), leaving the worker with no incentives to acquire education to convey their types to the firm.
- In that setting, their payoffs would be $(10,10)$ when the worker's productivity is high, and $(4,4)$ when it is low.

Evaluating PBE as a solution concept

4. Socially optimal? No.

- In contrast, in the separating PBE, equilibrium payoffs are $(6,10)$ and $(4,4)$. As a a consequence, if players behaved as under complete information, the high-type worker would improve her payoff from 6 to 10 (savings in education acquisition) while the payoffs of the low-type worker and the firm would remain unaffected.
- A similar argument applies to the pooling PBE (NE^H, NE^L), where equilibrium payoffs are $(4,4)$ regardless of the worker's type.
- In this context, the high-type worker and firm would improve their payoffs if they could behave as under complete information (increasing from 4 to 10 for both of them); while the low-type worker and firm would see their payoffs unchanged.

Semi-separating PBE

- What if, after checking for all the separating and pooling strategy profiles, as candidates for PBEs (our “usual suspects”)…
  - we find that none of them can be supported as PBE?
- Does it mean that the signaling game has no PBE?
  - No, it just means that we need to allow at least on sender type to randomize her messages.
- This is analogous as our inability to find psNEs in a simultaneous-move games of complete information,
  - where we could identify msNE where at least one player randomizes.

Semi-separating PBE

- Figure 10.7 depicts a simple poker game where, first, player 1 (sender) privately observes whether she has a high or low hand; then player 1 chooses to bet (B) or resign (R).
- If she resigns, the game is over and player 2 earns the initial pot.
- However, if she bets, player 2 (receiver) must respond calling or folding without observing player 1's hand but knowing that high and low hands are equally likely.

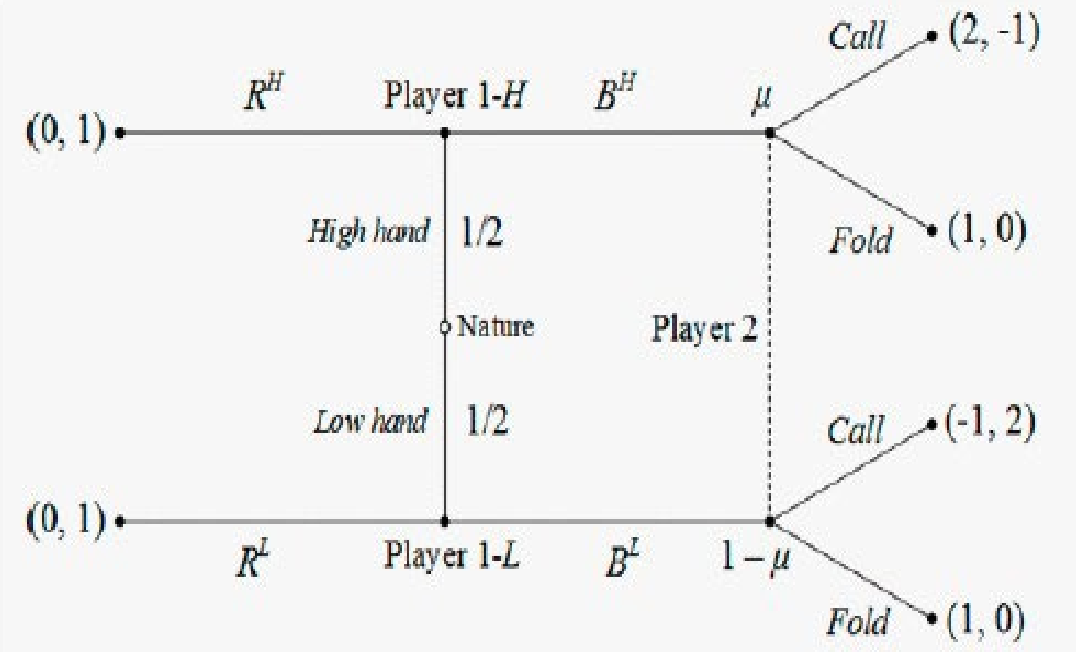


Figure 10.7. Poker game.

Semi-separating PBE

- In this setting, none of the separating strategy profiles, (B^H, R^L) and (R^H, B^L), or the pooling strategy profiles (B^H, B^L) or (R^H, R^L) can be sustained as a PBE.
- Unlike in previous signaling games, this occurs because player 1, when holding a high hand, would like to induce player 2 to respond calling it; but when holding a low hand, she would prefer player 2 to respond folding (informally, to not call her bluff).

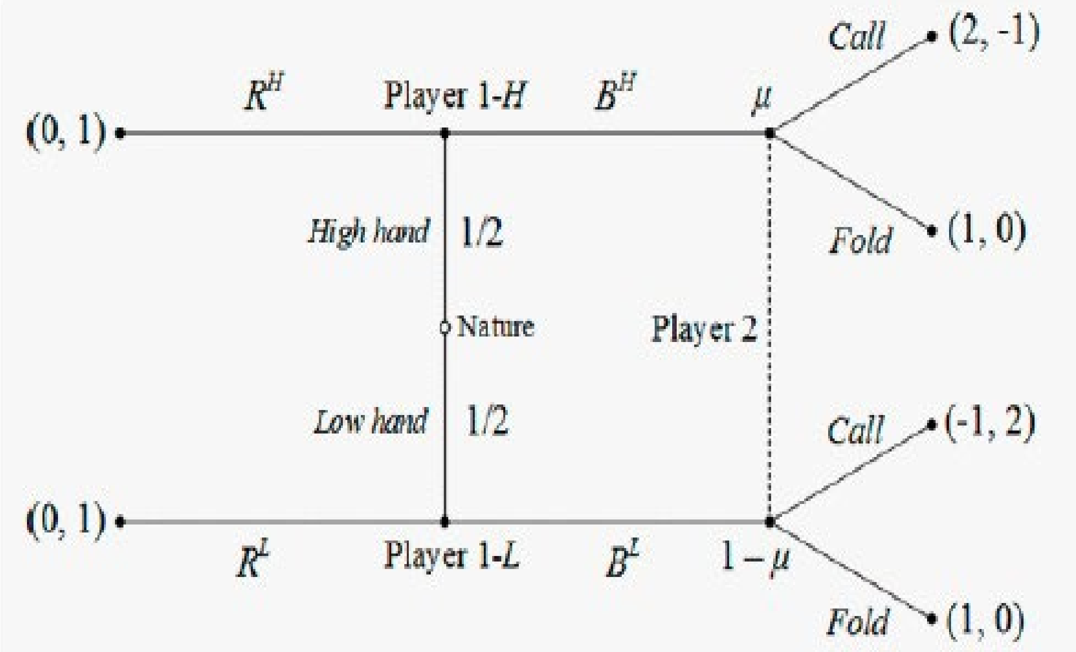


Figure 10.7. Poker game.

Semi-separating PBE

- In other words, each sender type would like the receiver (player 2) to *incorrectly infer* her type:
  i. thinking that player 1's hand is low, calling it, when it is actually high (at the top right side of the tree);
  ii. Thinking that her hand is high, folding, when it is in fact low (bottom right side).
- To prevent her type from being recognized, player 1 can create some "noise" in the signal, by betting with a positive probability.
- This randomization reduces player 2's ability to infer her type:
  - betting can now originate from the high- or low-hand players, with potentially different probabilities,
  - having a similar effect as in msNEs, where players mix to keep their rivals guessing about what their next moves are.

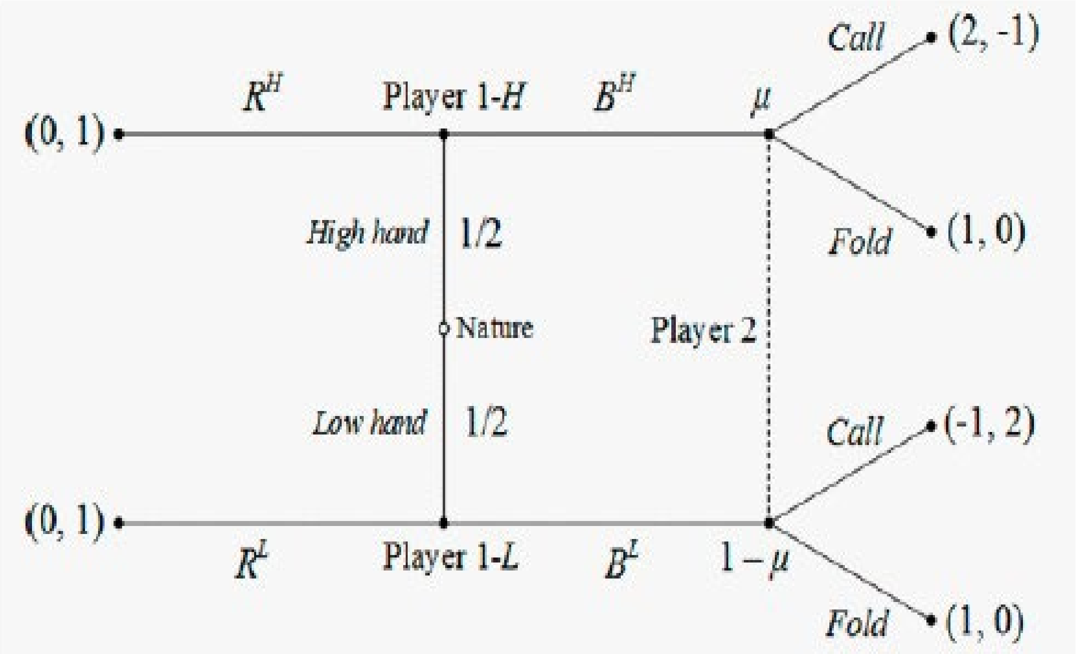


Figure 10.7. Poker game.

Semi-separating PBE

- Figure 10.7a reproduces 10.7, but highlighting the branch corresponding to betting for the high-hand player 1, B^H, who finds bet to be a strictly dominant strategy, i.e., B^H yields a payoff of 2, if player 2 responds calling, or 1, if she responds folding, both of them exceeding her payoff from resigning, R^H, where she earns 0 with certainty.
- Therefore, the high-hand player 1 play B^H in pure strategies.
- In contrast, the low-hand player 1 assigns probability p on B^L and $1 - p$ on R^L, where $p \in (0,1)$.

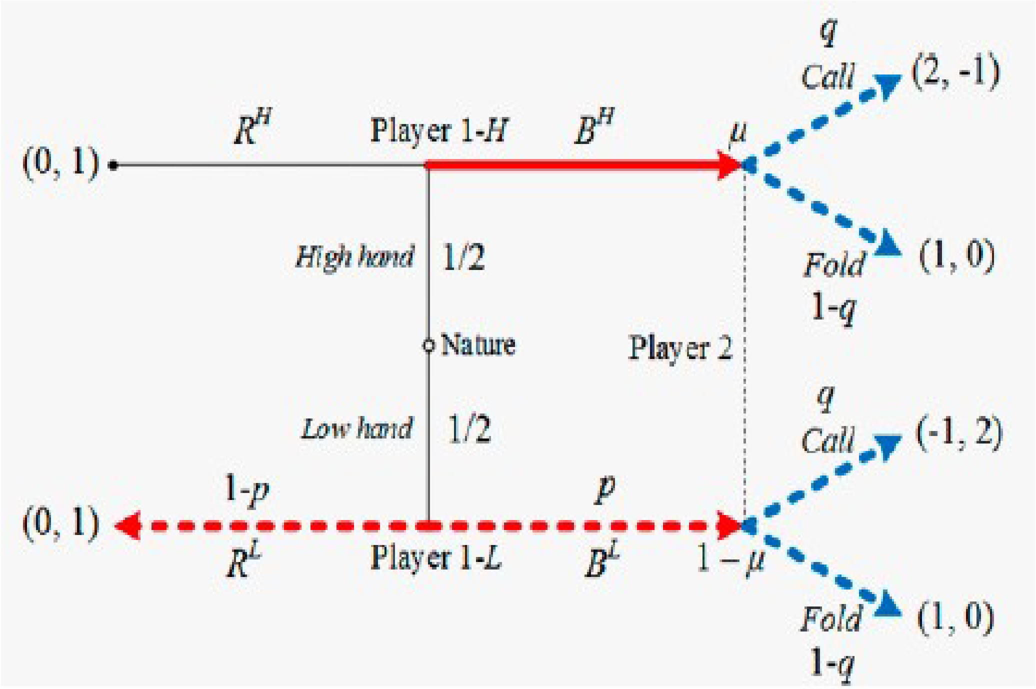


Figure 10.7a. Poker game - Semiseparating strategy profile.

Semi-separating PBE

- Finally, we labelled player 2's probability of calling as q, where $q \in (0,1)$, and that we seek to identify.
- Note that if, instead, we allowed for $q = 0$ $(q = 1)$, and player 2 folded (called) in pure strategies, the player 1 would like to bet (resign) with certainty when her hand is low, implying that $p = 1$ $(p = 0)$, implying that a pooling (separating) could be supported as a PBE, which we know cannot hold.
- Therefore, player 2 must be randomizing with (non-degenerate) strategies, such that $q \in (0,1)$.

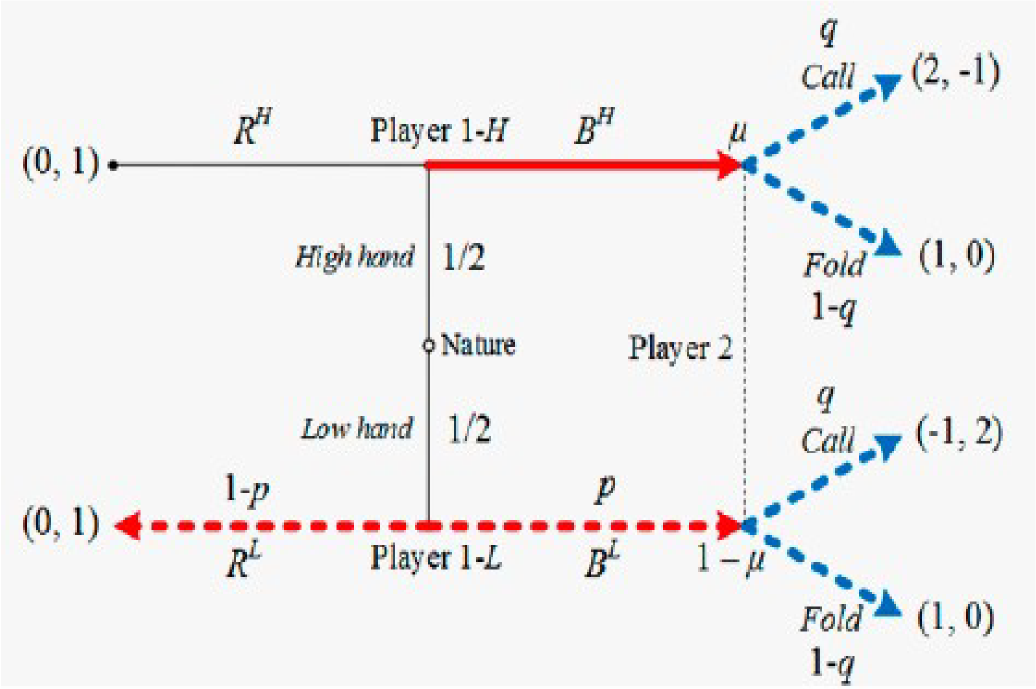


Figure 10.7a. Poker game - Semiseparating strategy profile.

Semi-separating PBE: Identifying PBEs

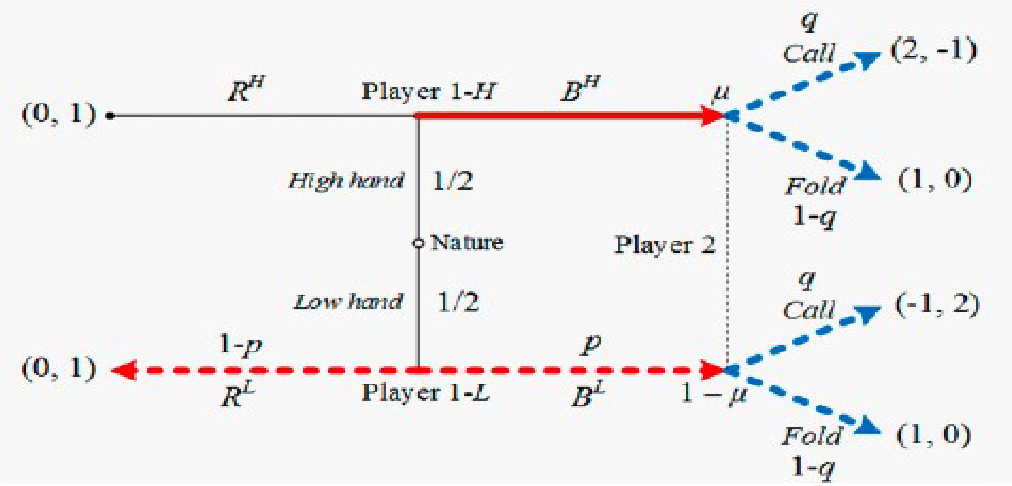


Figure 10.7a. Poker game - Semiseparating strategy profile.

1. *Specifying a strategy profile.*
   - We first specify the semi-separating strategy profile that we seek to test as a PBE:
     - player 1 chooses B^H with certainty, B^L with probability p, and
     - player 2 responds calling with probability $q \in (0,1)$.
2. *Bayes' Rule.* We can now update player 2's beliefs. Upon observing player 1 betting, beliefs satisfy

$$\mu = \frac{\frac{1}{2}1}{\frac{1}{2}1 + \frac{1}{2}p} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}p} = \frac{1}{1+p}$$

since player 1 bets with certainty when her hand is high, but bets with probability p otherwise.

Semi-separating PBE: Identifying PBEs

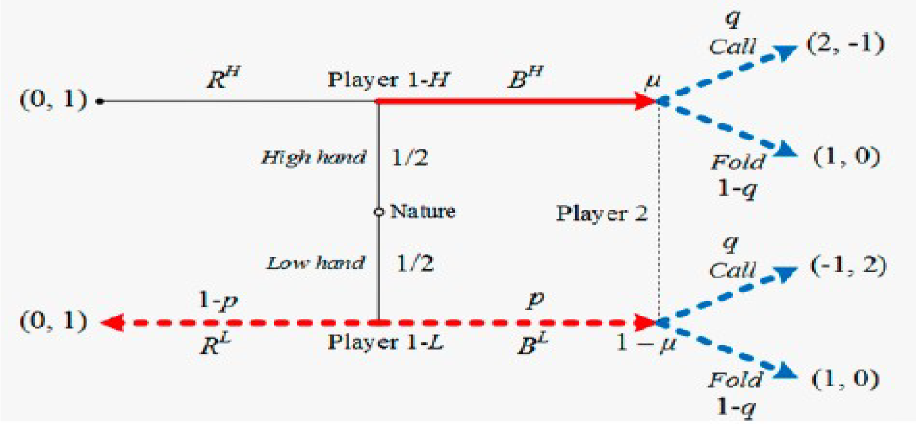


Figure 10.7a. Poker game - Semiseparating strategy profile.

3. *Optimal responses.* Given our results from Step 2, we now analyze player 2's response.

- Recall that player 2 must be mixing, $q \in (0,1)$. Otherwise, as discussed above, the low-hand player 1 would have incentives to use pure strategies, which are not PBEs of this game.
- In addition, if player 2 is mixing, she must be indifferent between C and F, entailing that

$$\mu(-1) + (1-\mu)2 = \mu 0 + (1-\mu)0$$

Where:

- the left side represents player 2's expected utility of calling (which yields a payoff of -1 when player 1's hand is high, but 2 otherwise);
- while the right side denotes her expected utility from folding (which is 0 regardless of player 1's hand. Solving for μ, yields $\mu = \frac{2}{3}$.

Semi-separating PBE: Identifying PBEs

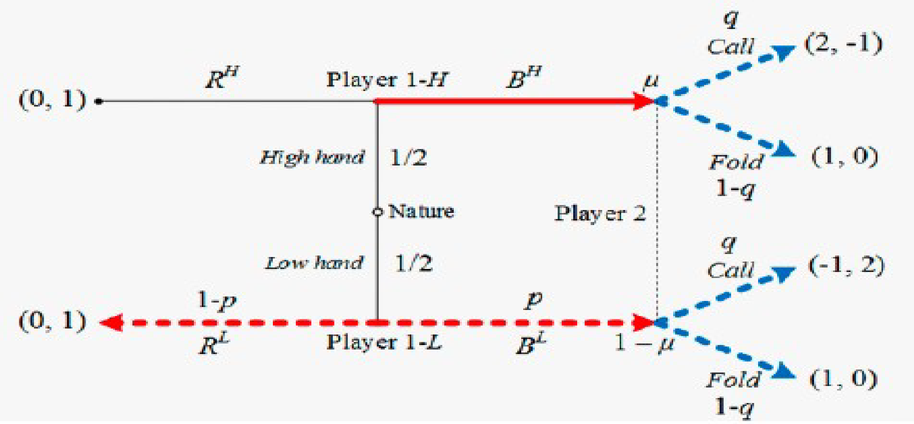


Figure 10.7a. Poker game - Semiseparating strategy profile.

3. *Optimal responses.*

- But, with which probability q does player 2 call?
  - We know that, as in msNEs, player 2 must be mixing with probability q that makes the low-hand player 1 indifferent between betting and resigning.
- This implies that $EU_1(B^L) = EU_1(R^L)$ or

$$q(-1) + (1-q)1 = 0$$

- since the low-hand player earns -1 when player 2 responds calling but 1 when player 2 folds.
- Solving for q, we find that $q = \frac{1}{2}$, implying that player 2 calls with 50 percent probability.

Semi-separating PBE: Identifyir

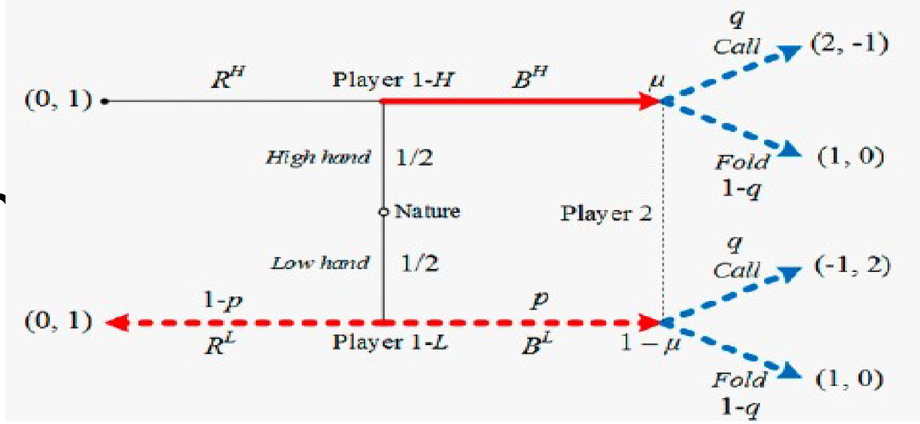


Figure 10.7a. Poker game - Semiseparating strategy profile.

4. Optimal messages.

- From our results in Step 3, we now identify player 1's optimal message.
- The high-hand player 1 finds betting, B^H, to be strictly dominant,
  - so we focus on the low-hand player 1, who calls with probability p.
- From Bayes' rule, we know that $\frac{2}{3} = \frac{1}{1+p}$, where $\mu = \frac{2}{3}$ from Step 3.
  - Solving for p, we obtain that $p = \frac{1}{2}$, so the low-hand player 1 bets with 50 percent probability.

Semi-separating PBE: Identifying PBEs

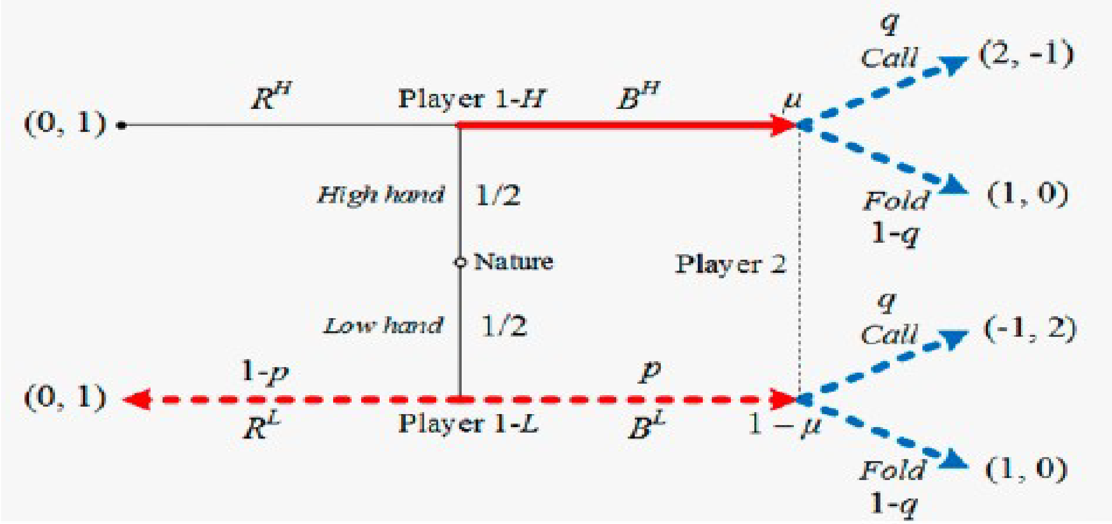


Figure 10.7a. Poker game - Semiseparating strategy profile.

5. Summary.

- From the above steps, we found a PBE where:
  - the high-hand player 1 bets, B^H, with certainty;
  - the low-hand player 1 bets, B^L, with probability $p = \frac{1}{2}$; and
  - player 2 responds calling with probability $q = \frac{1}{2}$, sustaining beliefs $\mu = \frac{2}{3}$.

Extensions: What if…

- For presentation purposes, the labor market signaling game only had:
  - two types, two messages, and two responses.
- But, what if…
  - The sender had three or more possible types.
  - The sender had three or more possible messages.
  - The receiver had three or more possible responses.
- We next analyze each of these extensions at a time.
  - First, finding how to depict them in the game tree.
  - And, then, how it affects our application of Tool 10.1 to find PBEs.

Extensions: What if the receiver has more than two available responses?

- We allow for the firm to have 3 available responses:
  - hiring the worker as the company director, manager, or cashier as depicted in Figure 10.8.
- Relative to figure 10.2, only the number of branches stemming from nodes connected by each information sets increases,
  - from two to three branches originating from each node (director, manager, or cashier hire).

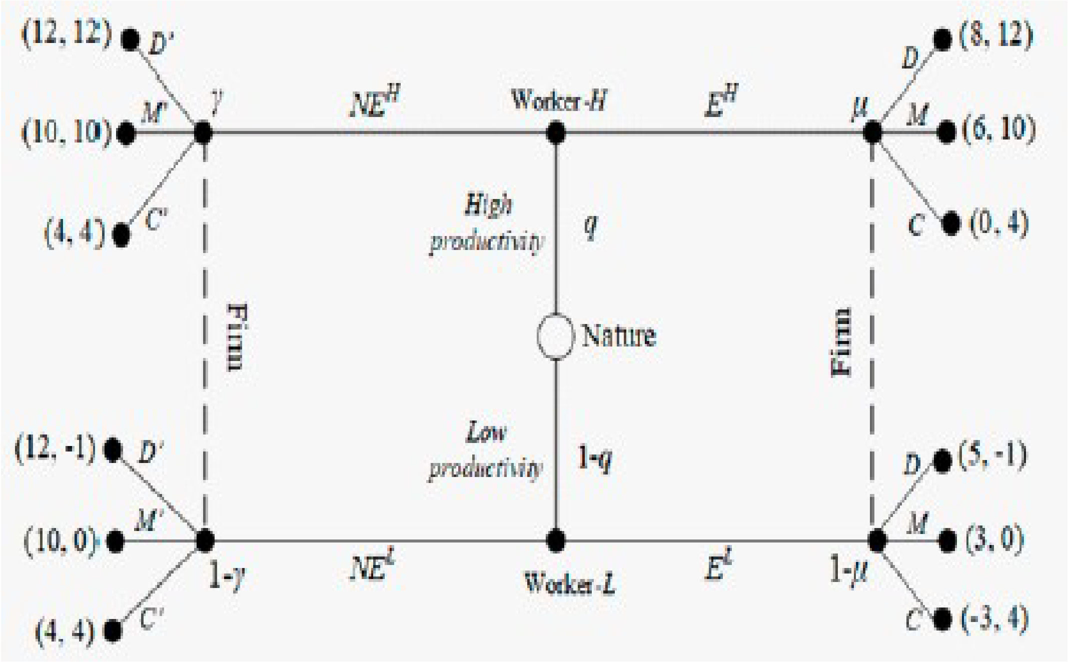


Figure 10.8. Labor-market signaling game with three responses.

How is Tool 10.1 affected by more responses?

- Our analysis in section 10.7 would be mostly unaffected.
- In particular, the list of strategy profiles to test in Step 1 remains unchanged,

  $(E^H, NE^L), (NE^H, E^L), (E^H, E^L)$ and (NE^H, NE^L).
- The firm's updated beliefs in Step 2 are also unaffected,
  - because the number of sender types and the number of available messages are unchanged,
  - therefore, not modifying the firm's information sets.
- However, the firm's optimal response (Step 3) is affected,
  - as the firm would choose the response that yields the highest payoff, by comparing its three possible responses.
  - Exercise 10.9 asks you to test which strategy profiles can be sustained as PBEs.
- This argument extends to other signaling games where we allow for $k \geq 2$ available responses, in which the firm would choose the one that yields the highest payoff.

What if the sender has more than two available messages?

- Alternatively, the labor market signaling game could be extended by allowing the worker to choose between more than two available messages, such as:
  - acquiring an advanced graduate degree (A),
  - an undergraduate degree (E), or
  - no college education (NE).
- Figure 10.9 illustrates this setting
- Relative to Figure 10.2, each worker type (either high or low productivity) chooses now among three, instead of two, possible branches $(A, E, or\ NE)$.

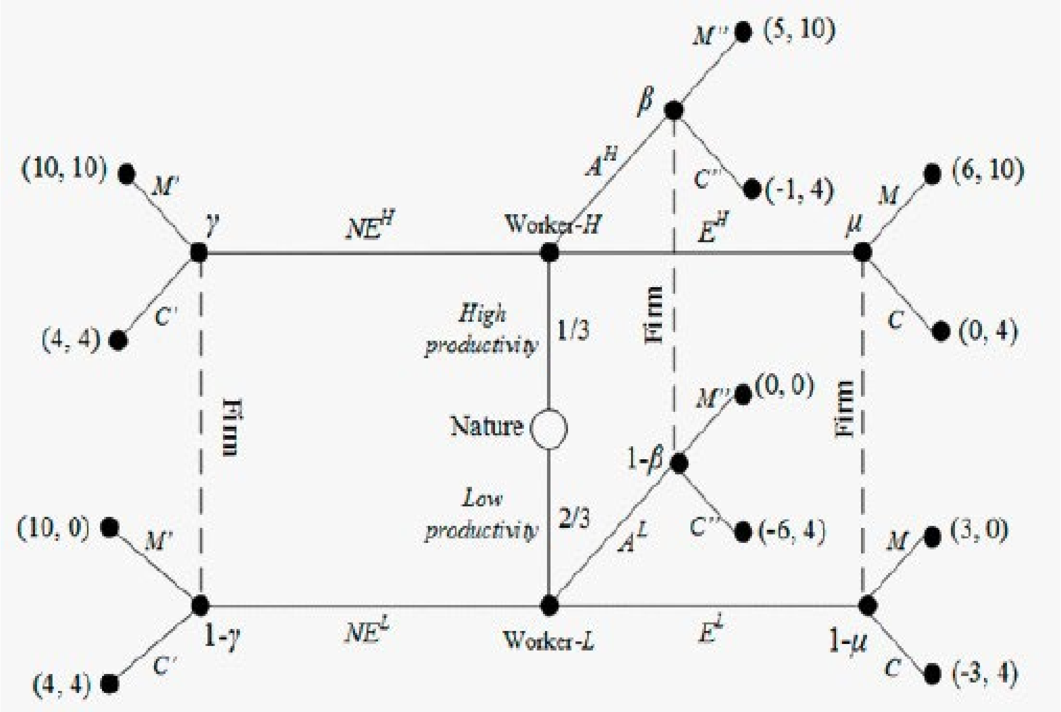


Figure 10.9. Labor-market signaling game with three messages

What if the sender has more than two available messages?

- Each branch, in turn, gives rise to a different information set:
  - one after the firm observes A,
  - another after observing E, and
  - another after observing NE.
- In each information set, however, there are still two nodes:
  - upon observing a given message, the firm faces the same type of uncertainty as in Figure 10.5,
  - namely, not knowing whether the message originates from the high- or low-type.

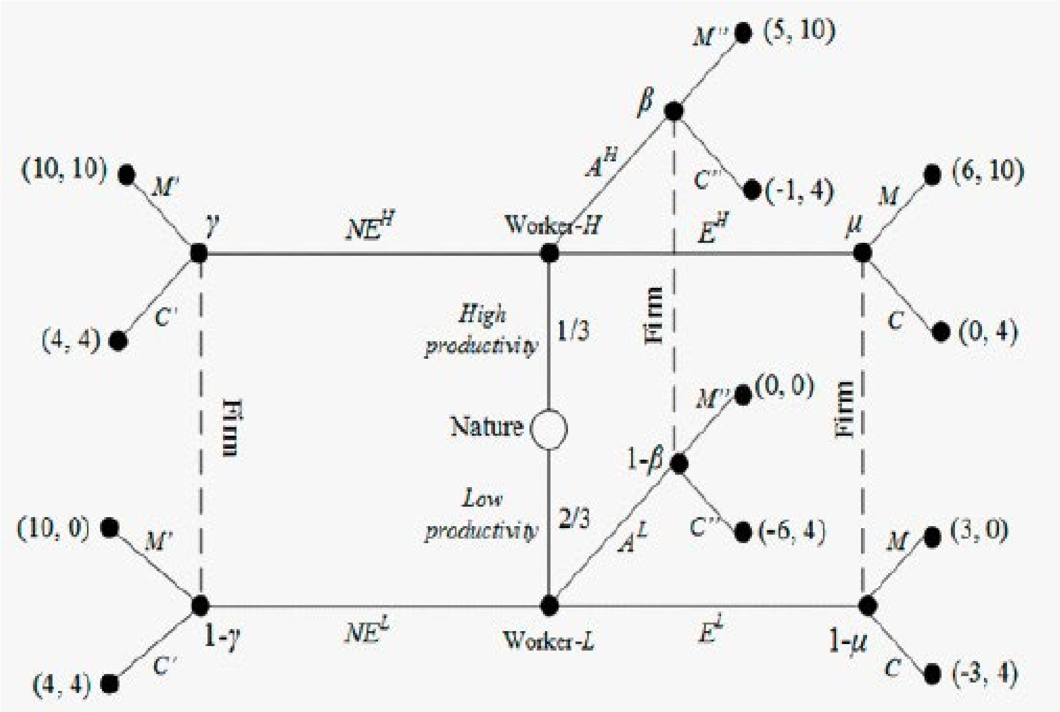


Figure 10.9. Labor-market signaling game with three messages

What if the sender has more than two available messages?

- While Figure 10.9 is relatively similar to Figure 10.2 (except for the branches where the worker chooses A), its tree representation may be difficult to interpret.
- For clarity, games with more than two messages are often graphically represented with a game tree like that in Figure 10.9a.
- Figure 10.9a can be understood as the "vertical" representation of the game, as nature chooses first, followed by each worker type, and the firm after observing each possible message.

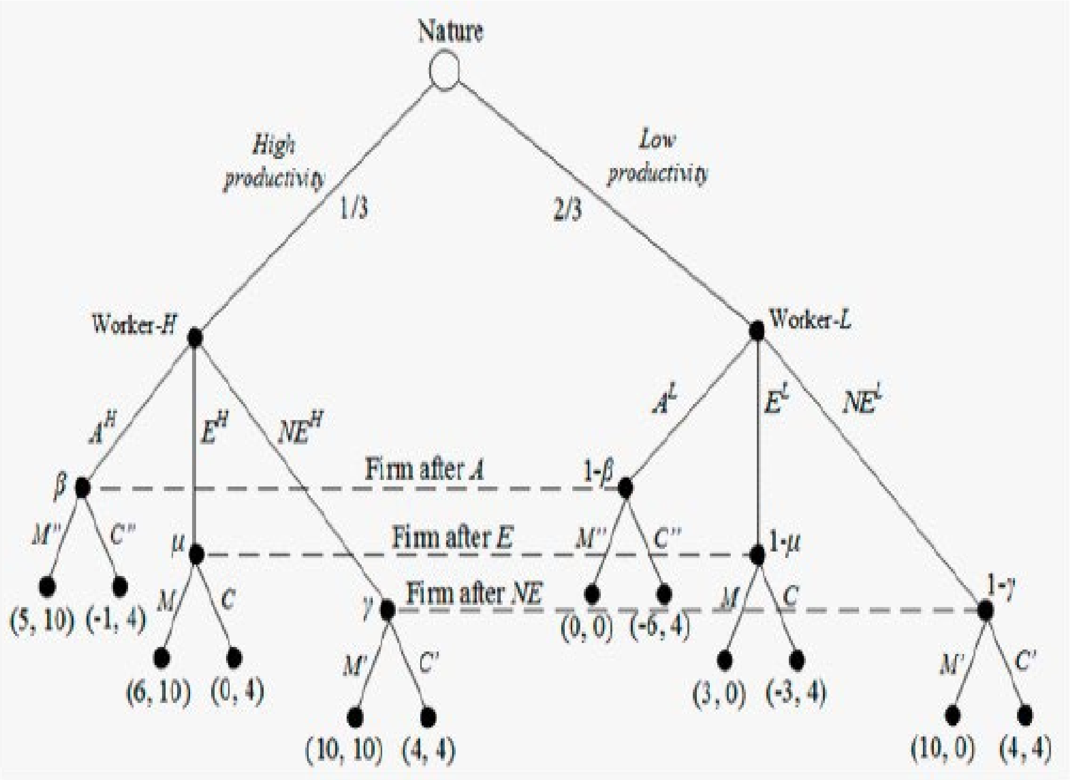


Figure 10.9a. Labor-market signaling game with three messages - Alternative representation

What if the sender has more than two available messages?

- Regarding payoffs, note that the high-productivity worker incurs an additional cost of 1 when choosing A^H, relative to when he chooses E^H;
  - whereas the low-productivity worker's additional cost is 3.
- The firm's payoffs are, of course, unaffected since education (either E or A) is not productivity-enhancing.

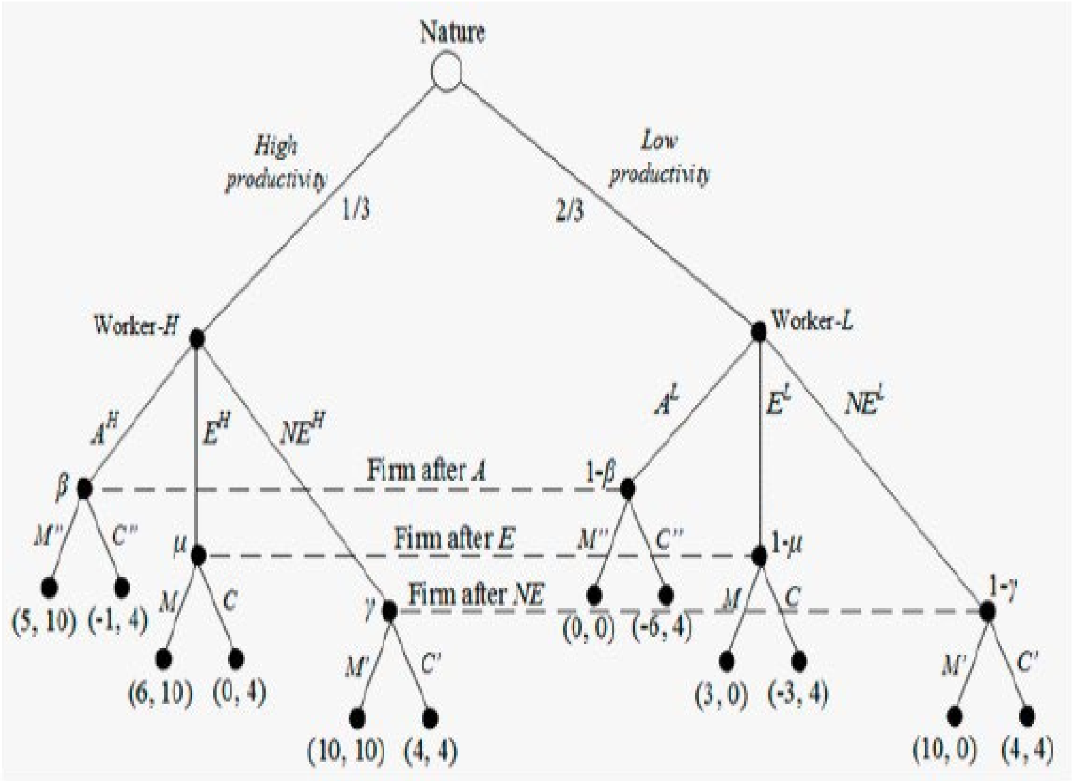


Figure 10.9a. Labor-market signaling game with three messages - Alternative representation

How is Tool 10.1 affected by more available messages?

- We now examine how the five-step tool to identify PBE is affected when the sender has more available messages.
- We start at Step 1, where we consider a specific strategy profile as a candidate for PBE.
- In this setting, we have the following strategy profiles to check as PBEs:
  - six separating,
  $$(A^H, E^L), (A^H, N^L), (E^H, N^L), (E^H, A^L), (N^H, A^L), (N^H, E^L),$$
  - three pooling,
  $$(A^H, A^L), (E^H, E^L), \text{ and } (N^H, N^L).$$
- More generally:
  - If there are x sender types and each sender has y available messages, then there are a total of y^x different strategy profiles.
  - In our example, $x = 2$ types and $y = 3$ messages, entailing $3^2 = 9$ different profiles (6 separating and 3 pooling).

How is Tool 10.1 affected by more available messages?

1. *Specifying a strategy profile.*

- Consider the strategy profile (A^H, E^L), depicted in Figure 10.9b.
- Intuitively, both worker types acquire education,
  - but the high type invests in an advanced graduate degree while the low type completes an undergraduate degree.

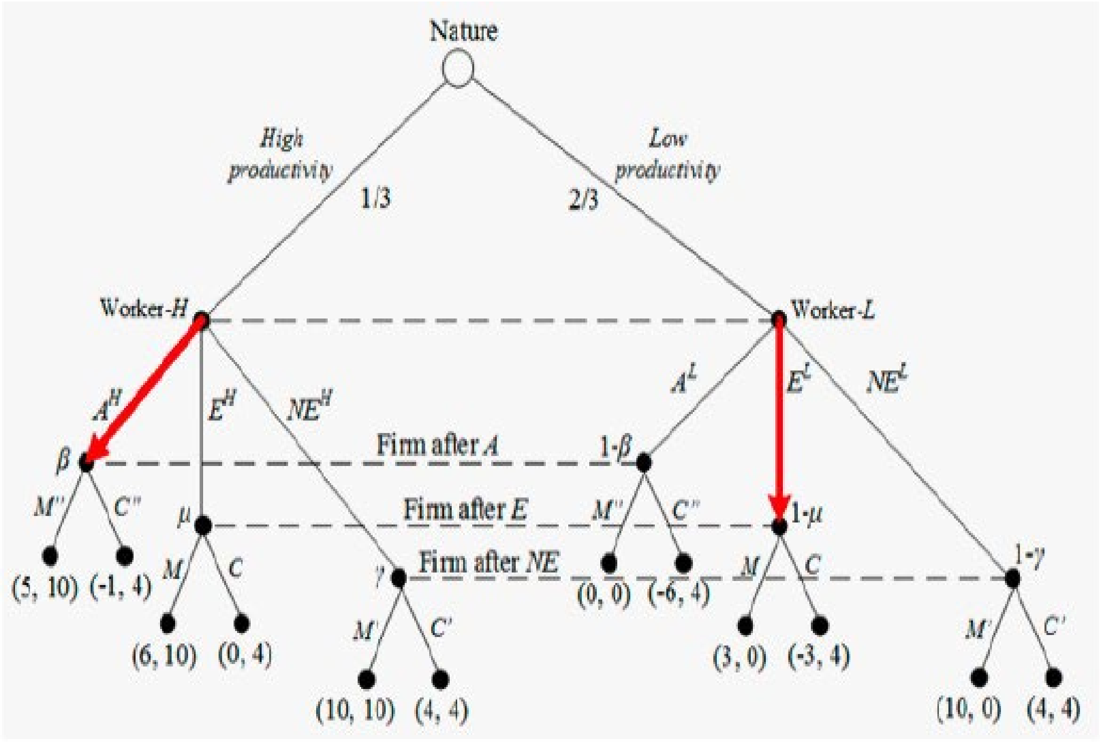


Figure 10.9b. Labor-market signaling game with three messages - Strategy profile (A^H, E^L).

How is Tool 10.1 affected by more available messages?

2. *Bayes' rule.* In this context, the firm's updated beliefs are

$$\mu(H|j) = \frac{\frac{1}{3}\alpha_j^H}{\frac{1}{3}\alpha_j^H + \frac{2}{3}\alpha_j^L}$$

where $j = \{A, E, NE\}$ denotes the message that the firm observes, and $\alpha_j^H\left(\alpha_j^L\right)$ represents the probability that the high (low) type sends message j.

- In the strategy profile (A^H, E^L), this belief becomes
  - $\mu(H|A) = \frac{\frac{1}{3}1}{\frac{1}{3}1+\frac{2}{3}0} = 1$ after observing A,
  - $\mu(H|E) = \frac{\frac{1}{3}0}{\frac{1}{3}0+\frac{2}{3}1} = 0$ after observing E, and
  - $\mu(H|NE) = \frac{\frac{1}{3}0}{\frac{1}{3}0+\frac{2}{3}0} = \frac{0}{0}$ after observing NE (off-the-equilibrium).

How is Tool 10.1 affected by more available messages?

2. Bayes' rule.

- In summary, upon observing $A(E)$, the firm is convinced of facing a high (low) productivity worker, as only this worker type uses this message;
  - but upon observing NE, which is off-the-equilibrium path, the firm cannot update its beliefs according to Bayes' rule.
  - We must leave them unrestricted as $\mu(H|NE) \in [0,1]$.
- In most applications, however, researchers assume that,
  - if firm's beliefs are $\mu(H|E) = 0$ after E,
  - they must be $\mu(H|NE) = 0$ after NE.
  - Intuitively, if an undergraduate degree signals that the worker's productivity is low, observing that the worker did not even complete college (a lower signal) must provide a similar information about the worker's type.
- From a practical approach, this assumption simplifies our analysis in the subsequent steps of Tool 10.1, and we consider it here.

How is Tool 10.1 affected by more available messages?

3. *Optimal Responses.*
   a. Upon observing, A, the firm responds hiring the worker as a manager, M'', since $10 > 4$ at the left of the game tree, $\beta = 1$.
   b. Upon observing, E, the firm responds hiring the worker as a cashier, C, since $4 > 0$ at the right side of the game tree, $\mu = 0$.
   c. Upon observing, E, the firm responds hiring the worker as a cashier, C', since $4 > 0$ at the right side of the game tree, $\gamma = 0$.

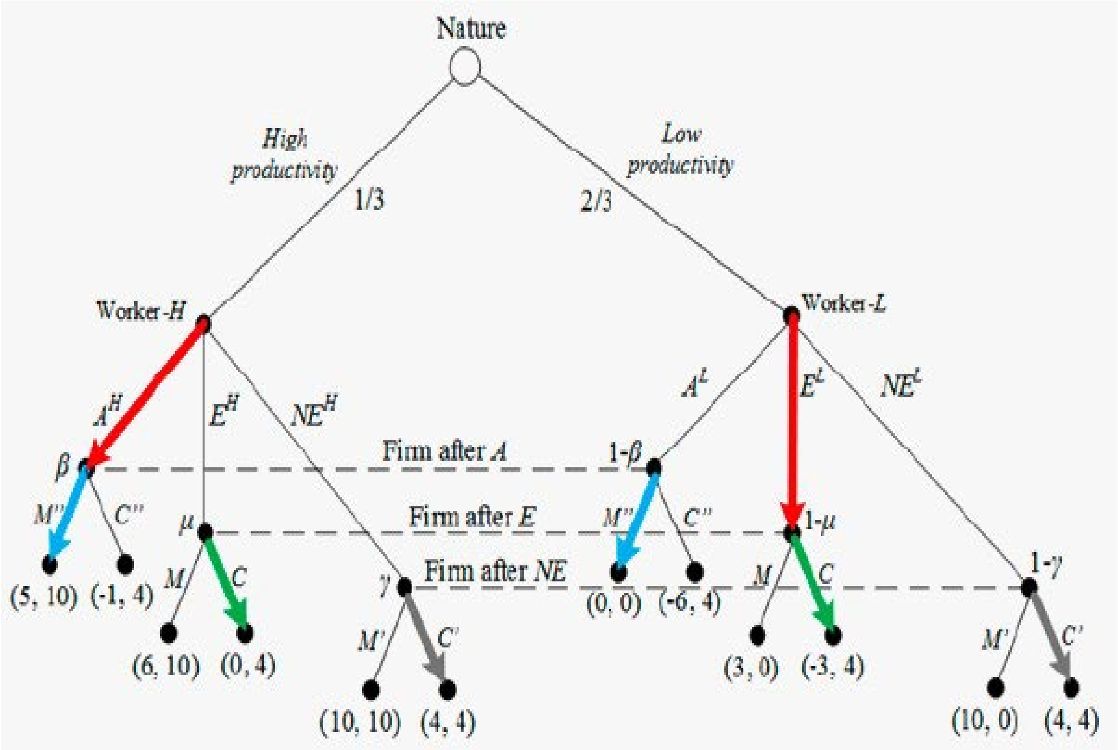


Figure 10.9c. Labor-market signaling game with three messages – Optimal messages

How is Tool 10.1 affected by more available messages?

3. *Optimal Messages.* From our results in Step 3, we now identify the worker's optimal message, separately analyzing each type.

a. *High productivity.*

- At the right side of the game tree, if the high-productivity worker chooses A^H, moving upward, she earns 5, as she anticipates the firm responding with M''.
- If instead, she deviates to E^H on the left side of the tree, she would earn 0 (as she is hired as a cashier, C).
- A similar argument applies, if she does not acquire education, NE^H, at the left side of the tree, where she is also hired as a cashier, C', and her payoff is 4.
- Therefore, she does not have incentives to deviate from A^H.

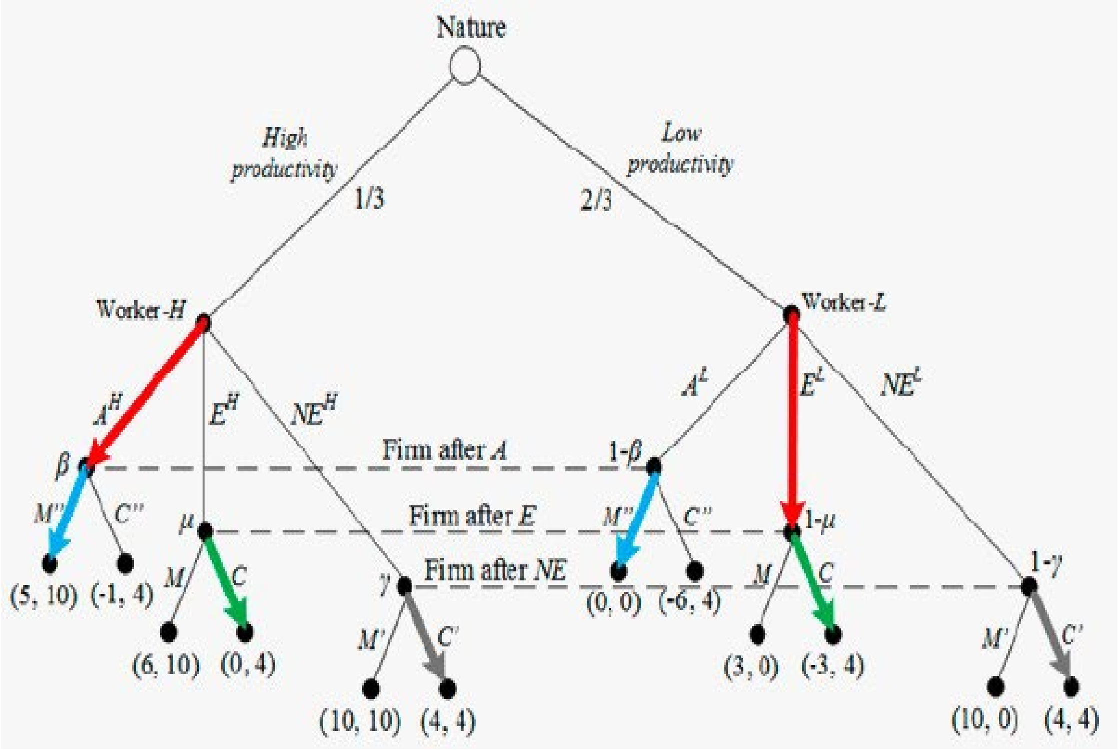


Figure 10.9c. Labor-market signaling game with three messages – Optimal messages

How is Tool 10.1 affected by more available messages?

3. *Optimal Messages.*

b. Low productivity.

- At the left side of the game tree:
  - if the low-productivity worker chooses E^L, on the center, she earns -3.
  - Then, she has incentives to deviate to NE^L, on the right side of the tree, which yields 4.
  - (This is her best deviation, as deviating to A^L only entails a payoff of 0.)

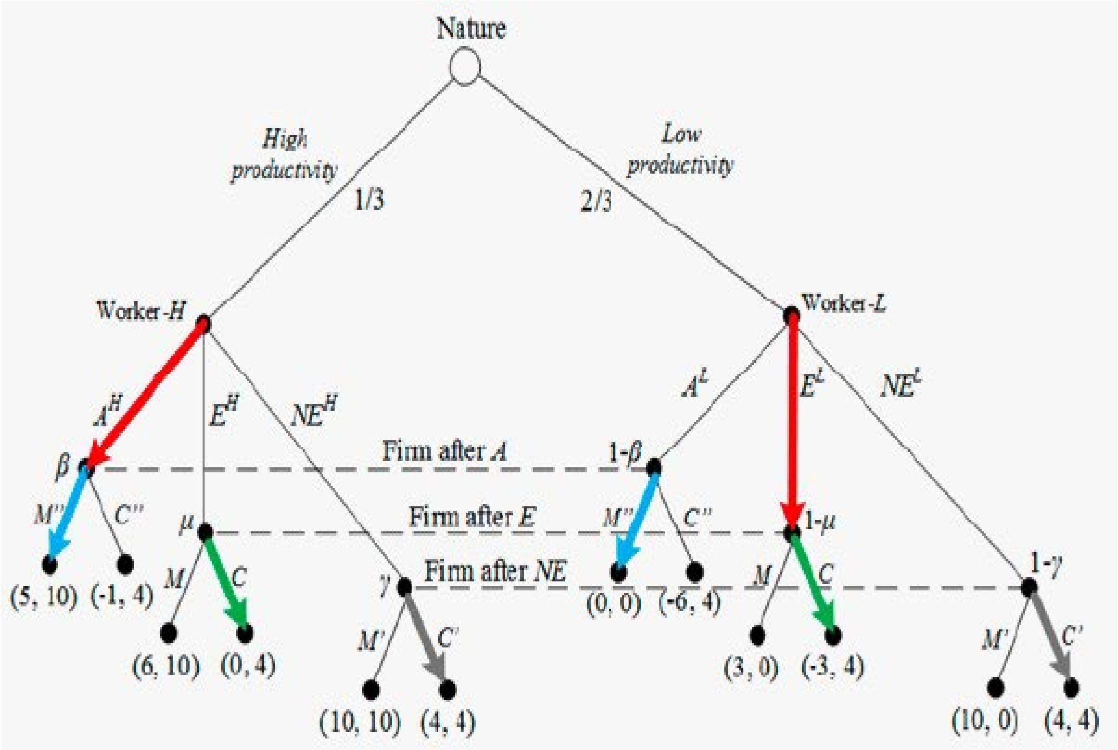


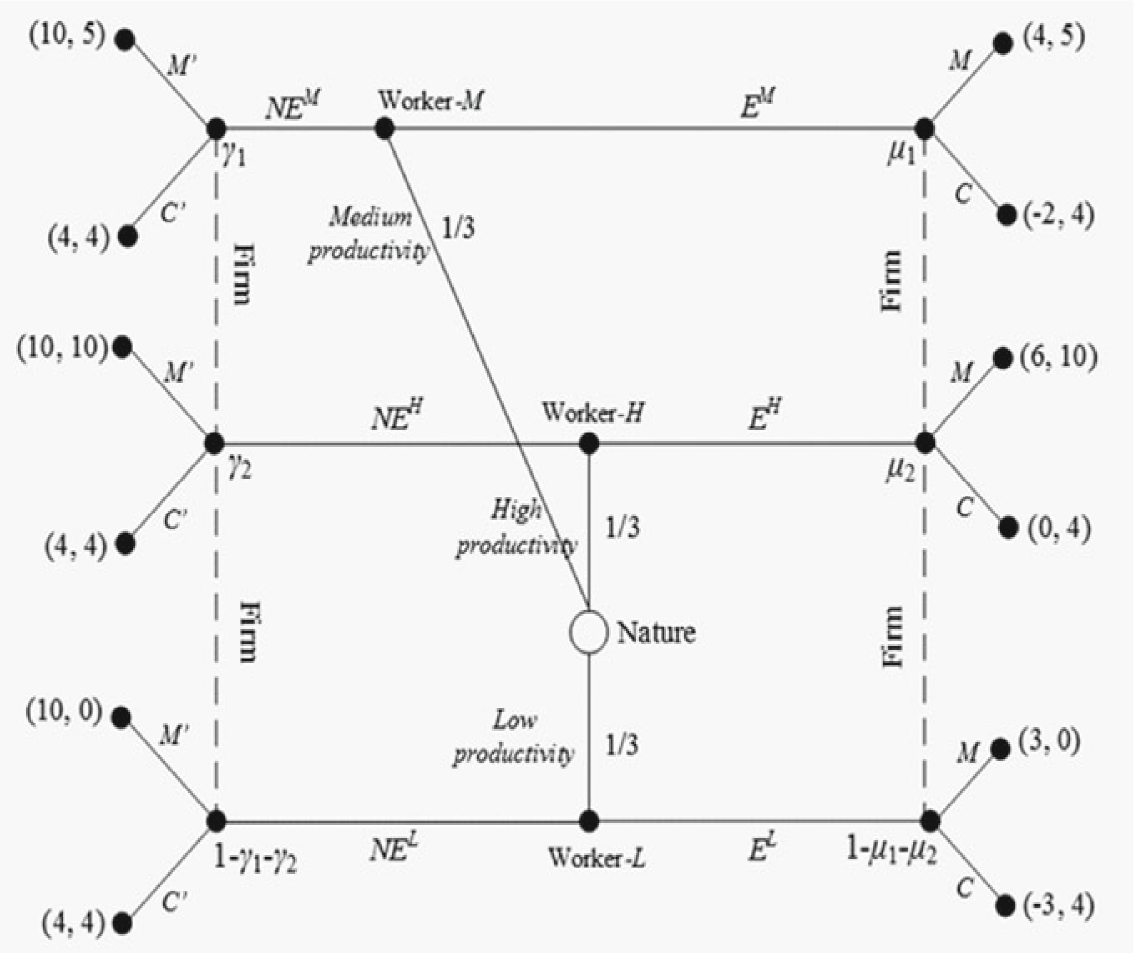
Figure 10.9c. Labor-market signaling game with three messages – Optimal messages

How is Tool 10.1 affected by more available messages?

5. *Summary.*
    5. From Steps 4a-4b, we found that:
        - one sender type (the low-productivity worker)
        - deviates from the strategy profile prescribed in Step 1, (A^H, E^L),
        - implying that this profile cannot be supported as a PBE.

What if the sender has more than two types?

- Finally, Figure 10.10 extends the signaling game of Figure 10.2 to allow for the worker to have more than two types, such as high, medium, or low productivity.
- Relative to the game tree in Figure 10.2:
    - a new branch stems from the initial node where nature draws the worker's type (at the top of the figure), corresponding to medium productivity.



What if the sender has more than two types?

- For clarity, Figure 10.10a depicts the “vertical” version of the game tree in Figure 10.10.
- This representation helps illustrate that:
  - after the initial move of nature,
  - every type of worker privately observes her type, and
  - chooses whether to acquire education, E^k, or not, NE^k, where $k = \{H, M, L\}$.

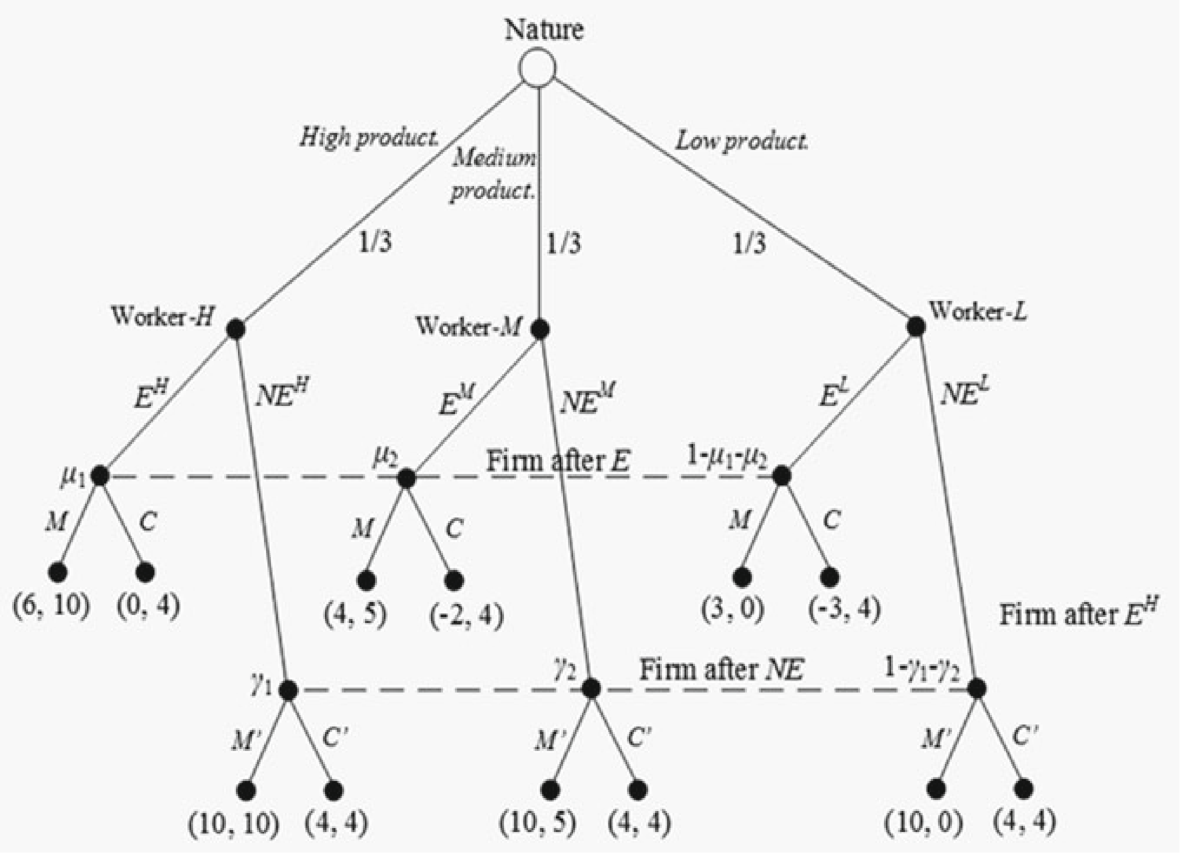


Figure 10.10a. Labor-market signaling game with three worker types – vertical version

What if the sender has more than two types?

- Two information nodes still.
- But each information set connects *three* nodes.
  - One for each sender type.
- This means that, upon observing a worker with education:
  - the firm does not know whether her productivity is high, medium, or low;
  - as depicted by the dashed line with the label "Firm after E".
- A similar argument applies when the firm observes a worker with no education,
  - in the information set with the label "Firm after NE"
  - at the bottom of Figure 10.10a.

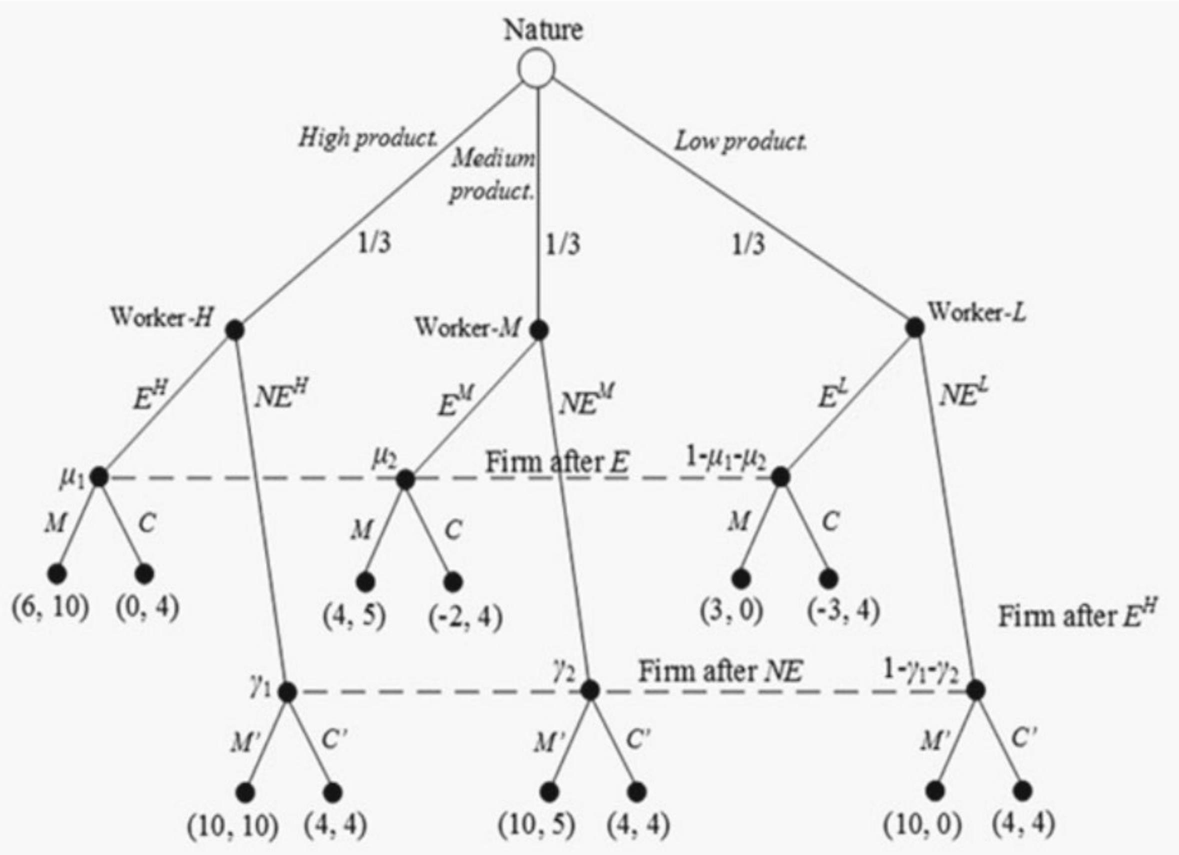


Figure 10.10a. Labor-market signaling game with three worker types – vertical version

What if the sender has more than two types?

- Regarding players' payoffs:
  - First, note that the medium-productivity worker incurs a cost of 6 when acquiring education,
  - which is in-between that of the low-productivity worker (7) and that of the high-productivity worker (4).
- The firm's profits are always 4 when hiring the applicant as a cashier,
  - regardless of her type (high, medium, or low) and regardless of her education.
- When hiring her as a manager, however, the firm's profit is:
  - 10 when the applicant's productivity is high,
  - 5 when it is medium, and
  - 0 otherwise.

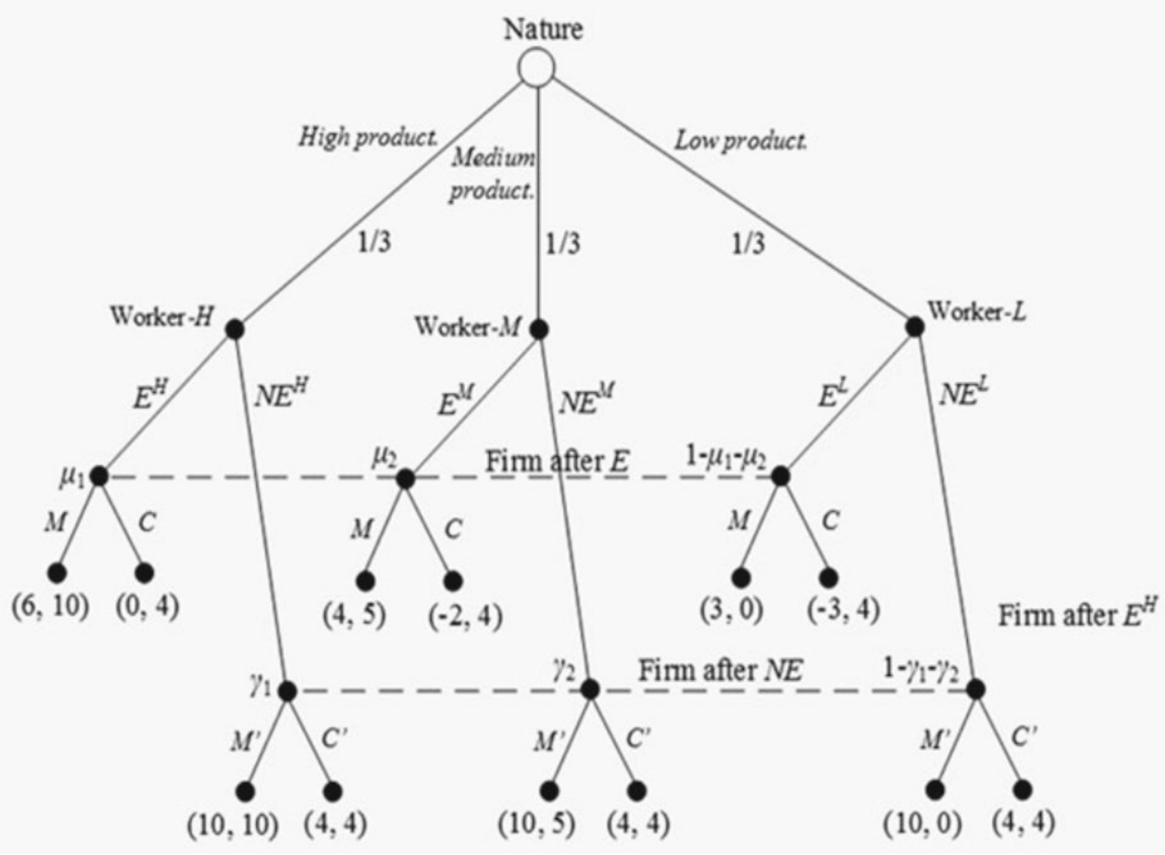


Figure 10.10a. Labor-market signaling game with three worker types – vertical version

How is Tool 10.1 affected by more than two types?

- In this setting, we have different profiles to test:

$$(E^H, E^M, NE^L), (E^H, NE^M, E^L), (E^H, NE^M, NE^L), (NE^H, E^M, E^L),$$
$$(NE^H, E^M, NE^L), (NE^H, NE^M, E^L),$$
$$(E^H, E^M, E^L) \text{ and } (NE^H, NE^M, NE^L)$$

- The last two strategy profiles are clearly pooling,
  - because all worker types choose the same message (all acquire education or none do),
- The first six strategy profiles are neither fully separating:
  - as two worker types pool into sending the same message,
- Not fully pooling either:
  - as one worker type chooses a different message than the other two types.
- We may refer to them as “partially separating” profiles.

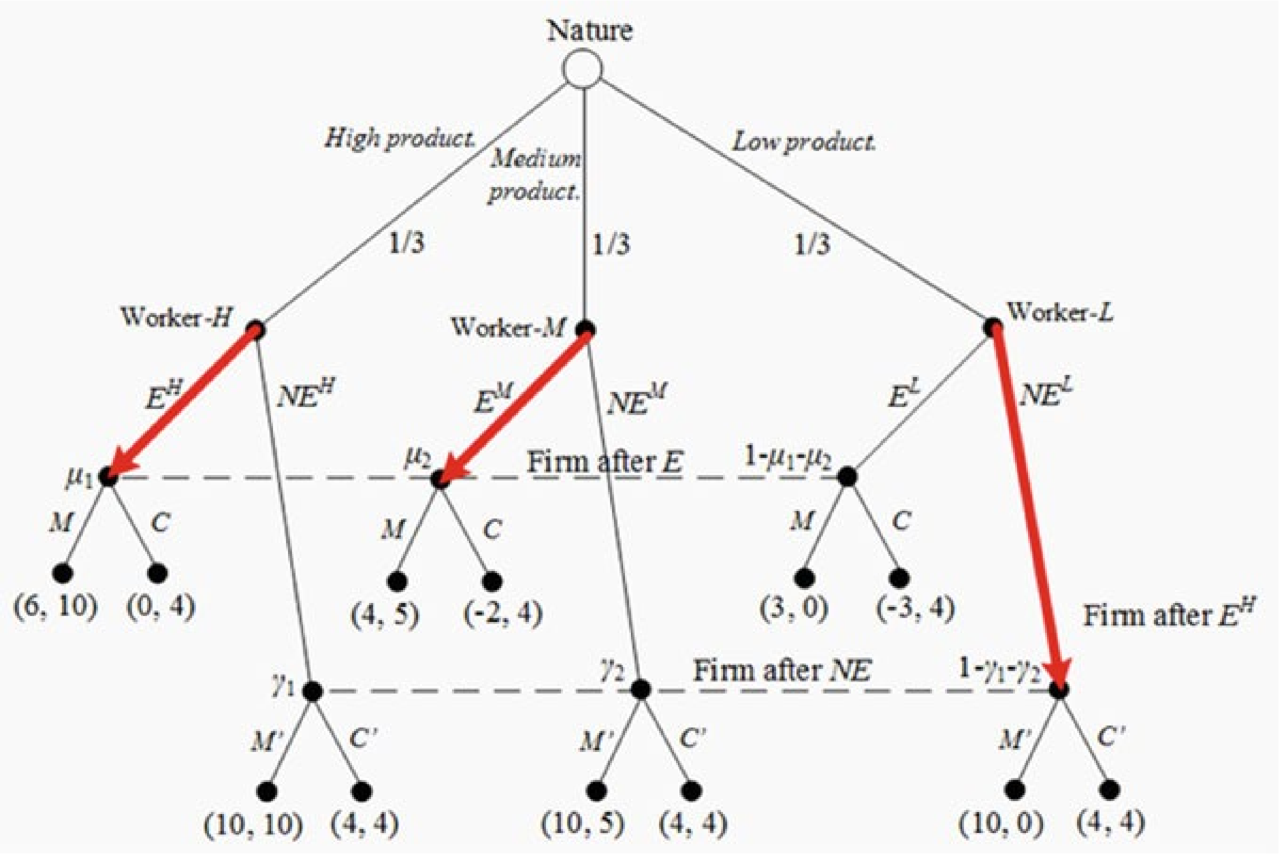
How is Tool 10.1 affected by more than two types?

- Following our discission in section 10.10.1:
  - if there are $x = 3$ sender types and each sender has $y = 3$ available messages,
  - then there are a total of $2^3 = 8$ different strategy profiles, as identified in our list above.
- For illustration purposes, we focus on (E^H, E^M, NE^L).
  - Let’s test if it can be sustained as a PBE.

How is Tool 10.1 affected by more than two types?

1. *Specifying a strategy profile.*

- Consider strategy profile (E^H, E^M, NE^L), as in Figure 10.10b, where:
  - both the high- and medium-productivity worker acquire education
  - while the low-productivity worker does not.



How is Tool 10.1 affected by more than two types?

2. *Bayes' Rule.* In this setting, the firm's updated beliefs are:

$$\mu(H|E) = \frac{\frac{1}{3}\alpha^H}{\frac{1}{3}\alpha^H + \frac{1}{3}\alpha^M + \frac{1}{3}\alpha^L}$$

where Figure 10.10b assumes, for simplicity, that all types are equally likely.

- In strategy profile (E^H, E^M, NE^L), for instance, $\alpha^H = \alpha^M = 1$ but $\alpha^L = 0$, yielding an updated belief

$$\mu(H|E) = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3} + 0} = \frac{1}{2}.$$

- Intuitively, the firm knows that education must originate from either the high or medium types.
  - Since both are equally likely, the probability of facing a high-productivity worker is 50 percent.

How is Tool 10.1 affected by more than two types?

2. *Bayes' Rule.* In this setting, the firm's updated beliefs are:

$$\mu(H|E) = \frac{\frac{1}{3}\alpha^H}{\frac{1}{3}\alpha^H + \frac{1}{3}\alpha^M + \frac{1}{3}\alpha^L}$$

where Figure 10.10b assumes, for simplicity, that all types are equally likely.

- Similarly,

$$\mu(M|E) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}+0} = \frac{1}{2}, \quad \text{but} \quad \mu(L|E) = \frac{0}{\frac{1}{3}+\frac{1}{3}+0} = 0$$

since the low-productivity worker does not acquire education in this strategy profile.

- Upon not observing education, however, we obtain more concentrated beliefs, that is, $\mu(H|NE) = \mu(M|NE) = 0$ whereas $\mu(L|NE) = 1$, as the firm is convinced of facing a low-productivity worker.

How is Tool 10.1 affected by more than two types?

3. *Optimal Responses.* Given our results from Step 2, we now analyze the firm's responses upon observing each of the three possible messages.

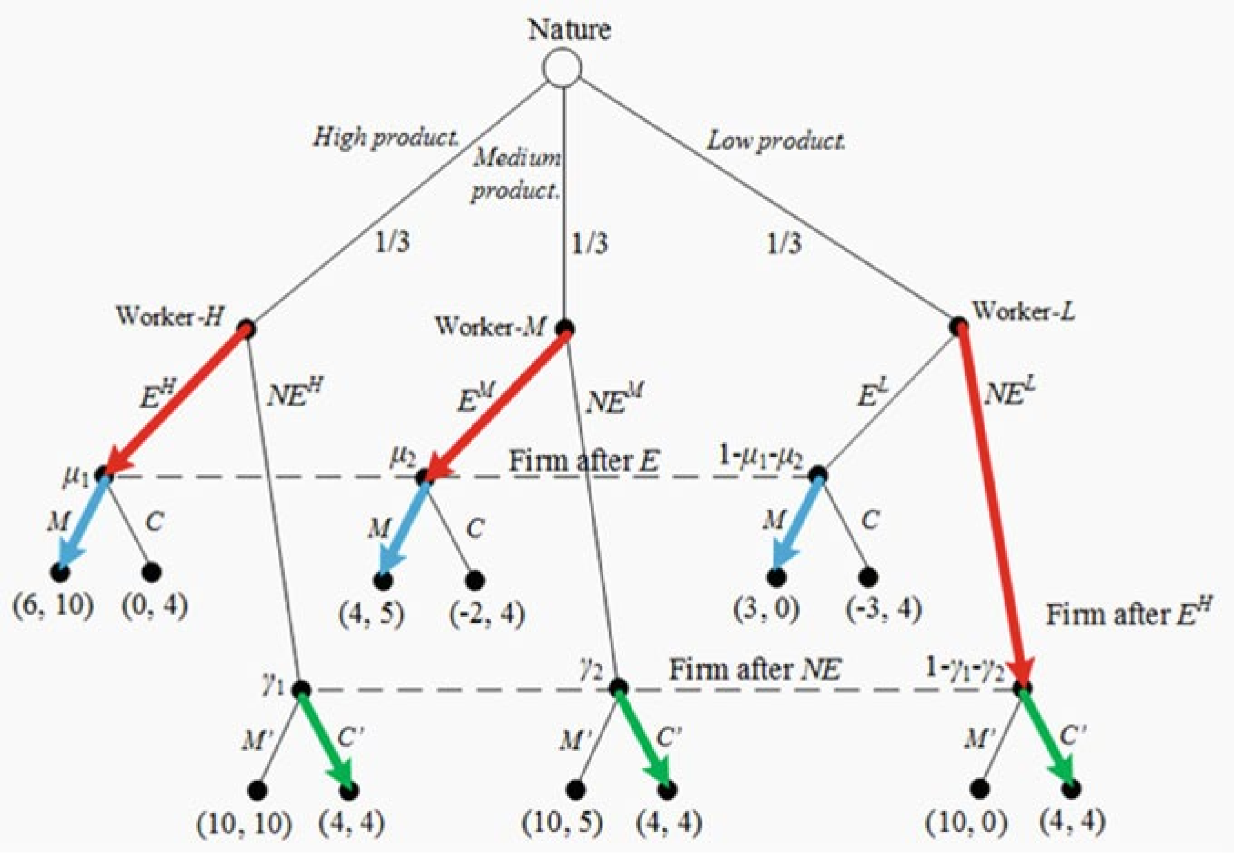
a. Upon observing E, the firm responds hiring the worker as a manager, M, since

$$\underbrace{\frac{1}{2}10 + \frac{1}{2}5}_{=7.5} > 4$$

at the right side of the game tree.

The firm now does not have concentrated beliefs upon observing E, as it can originate from the high or medium type.

Therefore, it must compute its expected profit from each of its decisions.

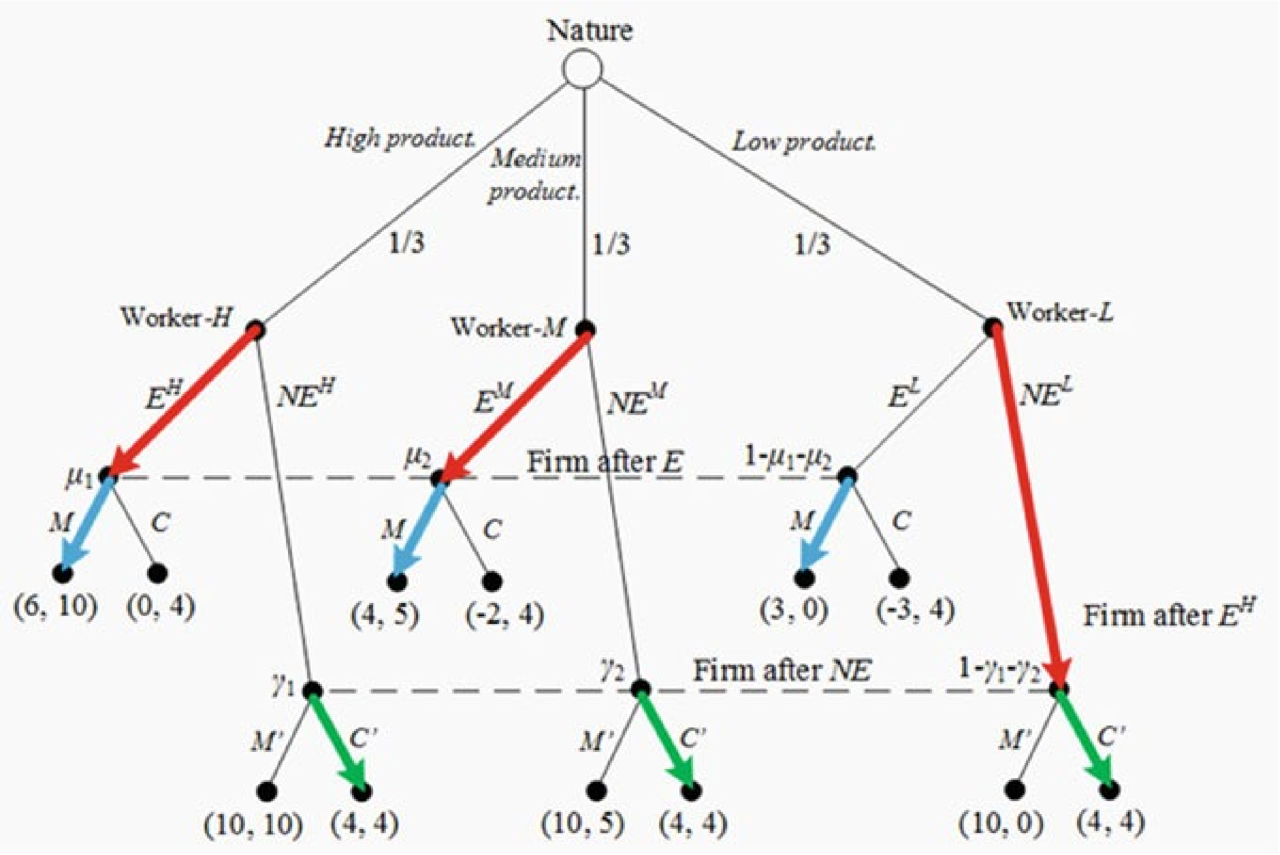


How is Tool 10.1 affected by more than two types?

3. *Optimal Responses.*

b. Finally, upon observing NE, the firm responds hiring the worker as a cashier too, C', since we imposed that off-the-equilibrium beliefs satisfy $\mu(H|NE) = \mu(M|NE) = 0$.

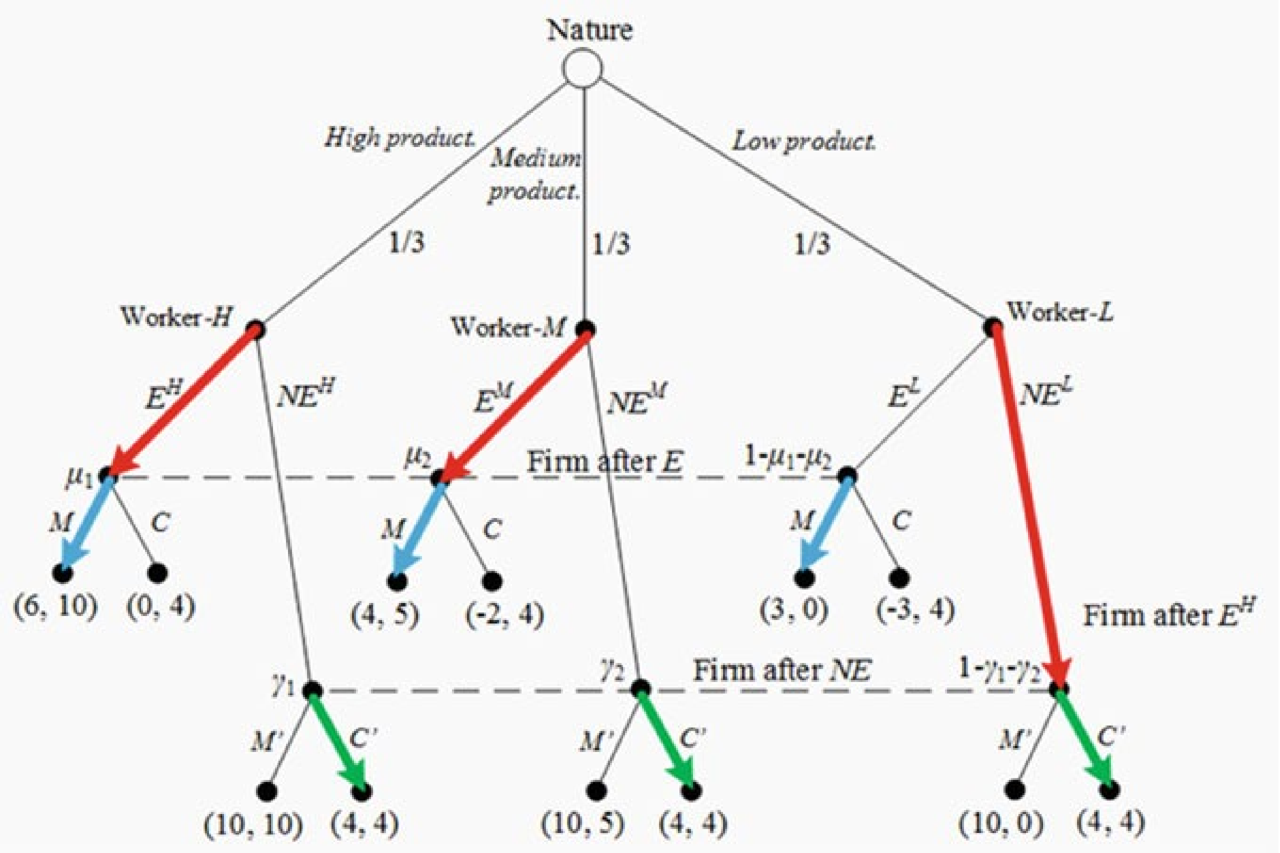
Figure 10.10c shades the branches corresponding to M and C'.



How is Tool 10.1 affected by more than two types?

4. *Optimal Messages.*

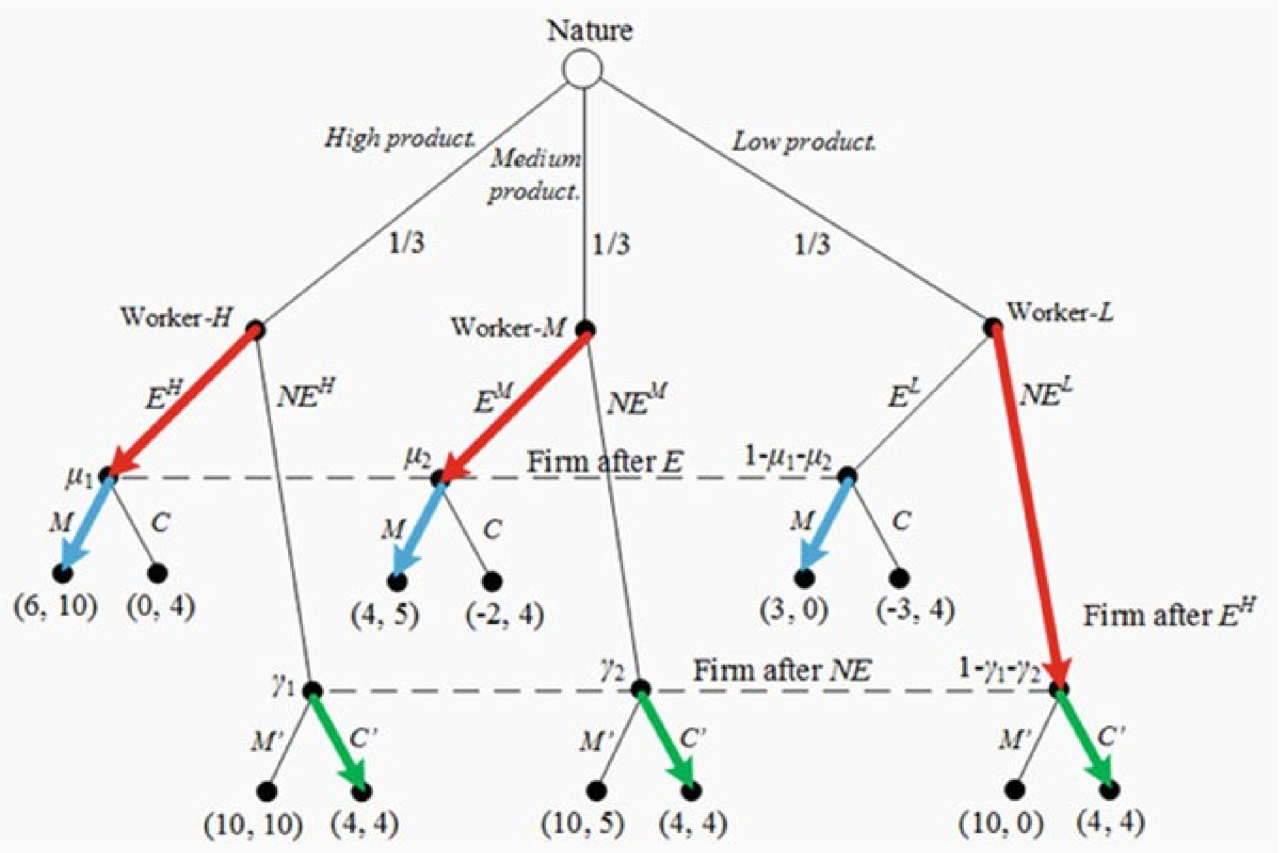
a. *High productivity.* At the left of the game tree, the high-productivity worker chooses education:
   - Her payoff from doing so, 6, exceeds
   - Her payoff from not acquiring education, 4.



How is Tool 10.1 affected by more than two types?

4. *Optimal Messages.*

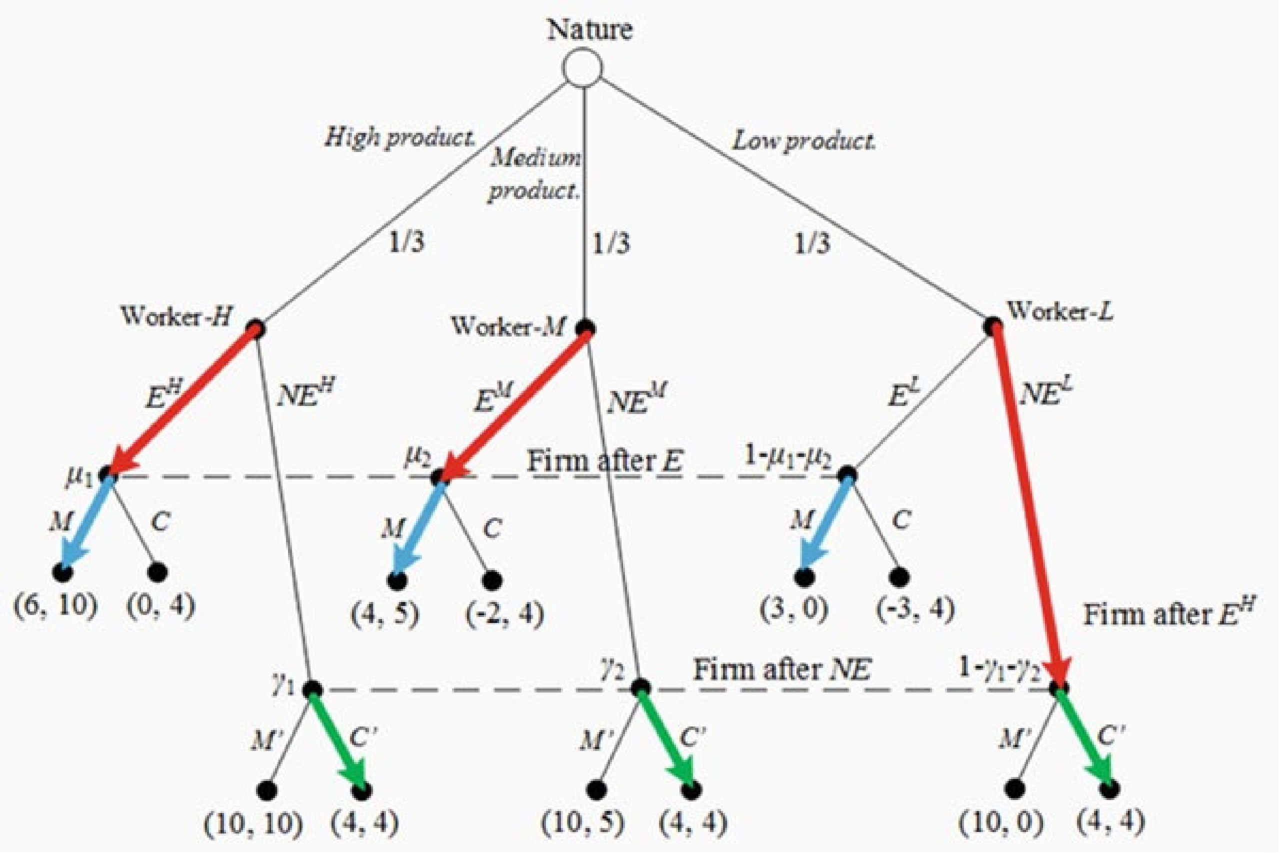
a. *Medium productivity.* At the center of the tree, the medium-productivity worker is indifferent between:
   - choosing education, earning 4, or
   - not acquiring education, also earning 4.



How is Tool 10.1 affected by more than two types?

4. *Optimal Messages.*

c. *Low productivity.* At the right side of the tree, the low-productivity worker does not acquire education:
   - Her payoff from No education, 4, exceeds
   - Her payoff from Education, 3.



How is Tool 10.1 affected by more than two types?

5. *Summary.*

- From Steps 4a-4b, we found that all sender types prefer to behave as prescribed in Step 1, (E^H, E^M, NE^L), implying that:
  - this separating strategy profile can be supported as a PBE,
  - with the firm holding beliefs $\mu(H|E) = \mu(M|E) = \frac{1}{2}$ and $\mu(L|E) = 0$ upon observing education,
  - and $\mu(H|NE) = \mu(M|NE) = 0, \mu(L|NE) = 1$ after no education,
  - responding with (M, C'), i.e., hiring the worker as a manager upon observing education but as a cashier otherwise.

Other Extensions

- **Productivity-enhancing education.**
  - Previous sections assumed, for simplicity, that education does not affect a worker’s job productivity.
  - We said productivity wasn’t productivity enhancing, thus only serving as a signal of the worker’s type.
  - In many real-life settings, however, education makes the worker more productive.
  - We explore this possibility in Exercises 10.2 and 10.3.

Other Extensions

- **More general cost differentials across types.**
  - Our above model assumed that the cost of acquiring education for the high (low) productivity worker was $c_H = 4\ (c_L = 7, \text{respectively})$.
  - Exercise 10.5 allows, for generality, that the low-productivity worker's cost is $c_L > 4$, so the difference $c = c_L$- $c_H > 0$ represents the cost differential that the low-productivity worker suffers relative to the high productivity worker when acquiring education.
  - That is, a higher cost differential may deter the low-productivity worker from mimicking the high-type's decision to acquire education, ultimately facilitating the emergence of separating PBEs.

Other Extensions

- **More general profit differentials across types.**
  - The standard labor-market signaling game considered that hiring a high (low) productivity worker as a manager provides the firm with a profit of $d_H = 10$ ($d_L = 0$, respectively);
    - hiring her as a cashier produces a profit of 4 regardless of the the worker's type.
  - In Exercise 10.6, we allow for a more general profit differential between worker types when hired as a manager, where we still assume that $d_H = 10$ but $d_L = 10 - d$, where $d = d_H - d_L$ and $0 \leq d \leq 10$.
  - Intuitively, as d increases, the firm hiring the low-productivity worker as a manager entails a larger loss, which the firm seeks to avoid;
    - while when $d = 0$, the firm earns the same profit hiring both worker types as a manager.