Appendices for
“Strategic Merger Approvals Under Incomplete Information”

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March 19, 2024

Appendix

Appendix 1 - Semiseparating equilibria

In order to check for a semiseparating equilibrium, we consider that the high-type firm randomizes between submitting and not submitting with probability $\sigma_H$ and $(1 - \sigma_H)$, respectively, while the low-type firm randomizes with probabilities $\sigma_L$ and $(1 - \sigma_L)$. The CA approves the merger with probability $\sigma_{CA}$ and blocks it with probability $1 - \sigma_{CA}$. Importantly, note that if the low-type firm is indifferent between submitting and not submitting, then the high-type firm must strictly prefer to submit, that is,

$$\sigma_{CA} \left[ \left( \frac{1-c + x_L}{2} \right)^2 - R \right] + (1 - \sigma_{CA})^2 \left( \frac{1-c}{3} \right)^2 = 2 \left( \frac{1-c}{3} \right)^2$$

entails that

$$\sigma_{CA} \left[ \left( \frac{1-c + x_H}{2} \right)^2 - R \right] + (1 - \sigma_{CA})^2 \left( \frac{1-c}{3} \right)^2 > 2 \left( \frac{1-c}{3} \right)^2$$

which implies that $\sigma_H = 1$.

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**First step.** For the CA to mix, his beliefs $\mu$ must satisfy

\[
\frac{1 - c + x_H}{2} + (1 - \mu) \frac{1 - c + x_L}{2} = k \frac{1 - c}{3}
\]

where the left side represents the expected output when the CA approves the merger and the right side indicates its certain output when the CA blocks the merger. Rearranging, yields

\[
\frac{\mu x_H + (1 - \mu) x_L}{1 - c} = \frac{1}{3} \equiv \bar{\theta}
\]

which we can also express as

\[
E[\theta] \equiv \frac{\mu x_H}{1 - c} + (1 - \mu) \frac{x_L}{1 - c} = \mu \theta_H + (1 - \mu) \theta_L = \bar{\theta}.
\]

or, after solving for $\mu$, we obtain that the CA’s beliefs must satisfy $\mu = \frac{\bar{\theta} - \theta_L}{\theta_H - \theta_L} \equiv \hat{\mu}$; otherwise, it would not be mixing between approving the merger (if $\mu > \hat{\mu}$) and blocking it (if $\mu < \hat{\mu}$).

**Second step.** Given the CA’s beliefs, we find from Bayes’ rule that

\[
\mu = \frac{\bar{\theta} - \theta_L}{\theta_H - \theta_L} = \frac{p}{p + (1 - p) \sigma_L}
\]

where the numerator captures the probability that the high-type firm submits a merger request (since $\sigma_H = 1$), while the denominator reflects the probability that the CA receives a merger request from any firm type. Solving for probability $\sigma_L$, yields

\[
\sigma^*_L = \frac{p}{1 - p} \frac{\theta_H - \bar{\theta}}{\bar{\theta} - \theta_L} \tag{1}
\]

which is unambiguously positive, and less than 1 if $\frac{p}{1 - p} < \frac{\bar{\theta} - \theta_L}{\theta_H - \bar{\theta}}$.

**Third step.** Given our above results about $\mu$ and $\sigma_L$, we can now find $\sigma_{CA}$. The low-type firm mixes if and only if

\[
\sigma_{CA} \left[ \left( \frac{1 - c + x_L}{2} \right)^2 - R \right] + (1 - \sigma_{CA})2 \left( \frac{1 - c}{3} \right)^2 = 2 \left( \frac{1 - c}{3} \right)^2
\]

where the left (right) side denotes the expected (certain) profit from submitting (not submitting) a merger request, which is approved with probability $\sigma_{CA}$. However, after rearranging, we find that

\[
\sigma_{CA} \left[ \left( \frac{1 - c + x_L}{2} \right)^2 - R - k \left( \frac{1 - c}{3} \right)^2 \right] = 0.
\]

If $\theta_L > \hat{\theta}$, the term in brackets is positive, entailing that the CA’s probability, $\sigma_{CA}$, becomes $\sigma_{CA} = 0$. In other words, the CA blocks all merger requests, implying that no firm type would have incentives to spend $R$ into the submission process, that is, $\sigma_H = \sigma_L = 0$, as in the PE where no firm type submits a merger request (see Proposition 3). Therefore, a semiseparating PBE cannot be sustained when $\theta_L > \hat{\theta}$. 

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If, instead, \( \theta_L < \theta \) holds, the term in brackets is negative, implying that probability \( \sigma_{CA} \) would have to be negative too, which cannot occur, entailing that a semiseparating PBE cannot be supported in this case either.

Finally, if \( \theta_L = \theta \), the term in brackets is exactly zero, implying that probability \( \sigma_{CA} \) is undefined, \( \sigma_{CA} \in [0,1] \). In this context, a semiseparating PBE can be sustained, where the firm randomizes with probability \( \sigma^*_H = 1 \) and \( \sigma^*_L = \frac{p}{1-p} \frac{\theta_H - \theta}{\theta_H - \theta_L} \) and the CA responds approving mergers with any probability \( \sigma_{CA} \in [0,1] \), if and only if \( \theta_H > \theta_L = \theta \) holds.

**Appendix 2 - Allowing for more merging firms**

**No merger.** In a case of no mergers, every firm \( i \) solves,

\[
\max_{q_i \geq 0} (1 - q_i - Q_{-i})q_i - cq_i
\]

where \( Q_{-i} \) denotes the aggregate output of firm \( i \)'s rivals. Differentiating with respect to \( q_i \), and solving for \( q_i \), we find firm \( i \)'s best response function

\[
q(Q_i) = \frac{1 - c}{2} - \frac{1}{2}Q_{-i}
\]

In a symmetric equilibrium, \( q_i = q_j = q \) for every two firms \( i \neq j \), which entails \( Q_{-i} = (n - 1)q \). Therefore, the equilibrium output in this setting is

\[
q_{iNM} = \frac{1 - c}{n + 1}
\]

and equilibrium profits become \( \pi_{iNM} = \left( \frac{1 - c}{n+1} \right)^2 = (q_{iNM})^2 \).

**Merger.** Now consider a merger of \( k \) firms is approved. The merging entity solves

\[
\max_{q^M \geq 0} (1 - q^M - Q_{-i})q^M - (c - x)q^M
\]

where \( q^M \) denotes the merging entity’s output, and \( Q_{-i} \) represents the aggregate output of all outsiders combined. Differentiating with respect to \( q^M \), and solving for \( q^M \), we find the best response function

\[
q^M(Q_{-i}) = \frac{1 - c + x}{2} - \frac{1}{2}Q_{-i}.
\]

Similarly, every outsider firm \( i \) solves

\[
\max_{q^M_i \geq 0} (1 - q^M_i - q^M - Q^M_{-i})q^M_i - cq^M_i
\]

where \( Q^M_{-i} \) denotes the aggregate production level of all other \( (n - k) - 1 \) firms that are outsiders in the merger. Differentiating with respect to \( q^M_i \), and solving for \( q^M_i \), we obtain the best response function \( q \)

\[
q^M_i(q^M, Q_{-i}) = \frac{1 - c}{2} - \frac{1}{2} (q^M + Q^M_{-i})
\]
In a symmetric equilibrium, all outsiders produce the same output, \( q^M_i = q^M_j = q^M_O \) for all \( n-k \) firms, implying that \( Q^M_i = [(n-k) - 1]q^M_O \). Inserting this result in the above best response functions, and rearranging, yields equilibrium output levels
\[
q^M = \frac{1 - c + (n-k+1)x}{n-k+2} \quad \text{and} \quad q^M_O = \frac{1 - c - x}{n-k+2}
\]

Finally, equilibrium profits for the merging entity is
\[
\pi^M = (1 - q^M - (n-k)q^M_O)q^M - (c - x)q^M = \left( \frac{1 - c + (n-k+1)x}{n-k+2} \right)^2 = (q^M)^2
\]
whereas every outsider earns
\[
\pi^M_O = (1 - q^M - (n-k)q^M_O)q^M_O - cq^M_O = \left( \frac{1 - c - x}{n-k+2} \right)^2 = (q^M_O)^2.
\]

**Cutoff \( \theta(k,n) \).** In this setting, an increase in consumer surplus is equivalent to an increase in output. In particular, \( q^M \geq kq^{NM}_i \) holds if and only if
\[
\frac{1 - c + (n-k+1)x}{n-k+2} \geq k\frac{1 - c}{n + 1}.
\]
Rearranging, and solving for yields \( x \geq \left( \frac{1 - c}{n + 1} \right) \left( \frac{1}{k} - \frac{1}{n + 1} \right) \) or, alternatively, \( \theta \equiv \frac{x}{1 - c} \geq k \frac{1}{n + 1} \equiv \theta(k,n) \)

**Cutoff \( \tilde{\theta}(k,n) \).** A merger between \( k \) out of \( n \) firms is profitable if the post-merger profits exceed the pre-merger profits, that is, \( \pi^M_i - R \geq k\pi^{NM}_i \), which holds if
\[
\left( \frac{1 - c + (n-k+1)x}{n-k+2} \right)^2 - R \geq k\left( \frac{1 - c}{n + 1} \right)^2
\]
After simplifying, we obtain
\[
\frac{1 - c + (n-k+1)x}{n-k+2} \geq \sqrt{k\left( \frac{1 - c}{n + 1} \right)^2 + R}
\]
and upon further rearranging, we find
\[
\theta \equiv \frac{x}{1 - c} \geq \frac{n - k + 2}{(1 - c)(n - k + 1)} \sqrt{k\left( \frac{1 - c}{n + 1} \right)^2 + R - \frac{1}{n - k + 1}} \equiv \tilde{\theta}(k,n)
\]
which increases in \( R \). Comparing cutoffs \( \tilde{\theta}(k,n) \) and \( \theta(k,n) \), we obtain that \( \tilde{\theta}(k,n) > \theta(k,n) \) holds if and only if
\[
\frac{n - k + 2}{(1 - c)(n - k + 1)} \sqrt{k\left( \frac{1 - c}{n + 1} \right)^2 + R - \frac{1}{n - k + 1}} > \frac{k - 1}{n + 1}
\]
or, rearranging, and solving for \( R \),
\[
R > \left( \frac{1 - c}{n + 1} \right)^2 \left[ \left( \frac{(n + 1) + (n - k + 1)(k - 1)}{(n - k + 2)} \right)^2 - k \right] \equiv \hat{R}.
\]
Therefore, \( \tilde{\theta}(k,n) > \theta(k,n) \) holds if and only if \( R > \hat{R} \).
Appendix 3 - Uninformed outsiders

No merger. In a case of no mergers, every firm $i$ solves the same maximization problem as in Appendix 2, yielding the same equilibrium output, $q_{i,NM}^M = \frac{1-c}{n+1}$, and profits, $\pi_{i,NM}^M = \left(\frac{1-c}{n+1}\right)^2 = (q_{i,NM}^M)^2$.

Merger. When $k$ out of $n$ firms merge, the merging entity solves
\[
\max_{q^M \geq 0} (1 - q^M - Q_{-i})q^M - (c - x)q^M
\]
where $q^M$ denotes the insiders’ output, and $Q_{-i}$ represents the aggregate output of all outsiders combined. Differentiating with respect to $q^M$, and solving for $q^M$, we find the best response function
\[
q^M_H(Q_{-i}) = \frac{1-c+x_H}{2} - \frac{1}{2} Q_{-i}
\]
when the merging entity’s type is high and, similarly, $q^M_L(Q_{-i}) = \frac{1-c+x_L}{2} - \frac{1}{2} Q_{-i}$ when its type is low.

Every outsider (uninformed about the realization of $x$), chooses its output $q_i$ to solve the following expected profit-maximization problem
\[
\max_{q_i \geq 0} p [(1 - q_H - q_i - Q_{-i})q_i - cq_i] + (1 - p) [(1 - q_L - q_i - Q_{-i})q_i - cq_i]
\]
where $q_H$ ($q_L$) denotes the merging entity’s output when its type is high (low, respectively); and $Q_{-i}$ represents the aggregate output of all other outsiders (except for firm $i$). Differentiating with respect to $q_i$, and solving for $q_i^M$, we obtain the best response function
\[
q_i^M(q^M_H, q^M_L, Q_{-i}) = \frac{1-c}{2} - \frac{pq^M_H + (1-p)q^M_L + Q_{-i}}{2}
\]
where $pq^M_H + (1-p)q^M_L$ denotes the merging entity’s expected output. In a symmetric equilibrium, all outsiders produce the same output, $q_i^M = q_i^{M*}$ for all $n-k$ firms, implying that $Q_{-i} = [(n-k)-1]q_i^M$. Inserting this result in the above best response function, and rearranging, yields
\[
q_i^M(q^M_H, q^M_L, Q_{-i}) = \frac{1-c}{n-k+1} - \frac{pq^M_H + (1-p)q^M_L}{n-k+1}.
\]
Simultaneously solving for $q^M_H$, $q^M_L$, and $q_i^M$ in the above three best response functions, yields Bayesian Nash equilibrium outputs
\[
q^M_H = \frac{2(1-c) + (n-k+2)x_H + (n-k)E[x]}{2(n-k+2)},
\]
\[
q^M_L = \frac{2(1-c) + (n-k+2)x_L + (n-k)E[x]}{2(n-k+2)}, \text{ and}
\]
\[
q_i^M = \frac{1-c + E[x]}{n-k+2},
\]
where $E[x] \equiv px_H + (1-p)x_L$ represents the expected cost-reduction effect.
The outsiders’ equilibrium profit does not affect our results regarding regions 1-3, but the merging entity’s does, becoming
\[
\pi^{M,L}(\beta) = (1 - q_L^M - [(n - k) - 1]q_i^M)q_L^M - (c - x_L)q_L^M
\]
\[
= \left( \frac{2(1 - c) + (n - k + 2)x_L + (n - k)E[x]}{2(n - k + 2)} \right)^2
\]
which collapses to that in the previous section when the merging entity’s type is low with certainty, \( p = 0 \), \( \pi^{M,L} = \left( \frac{1-c+(n-k+1)x_L}{n-k+2} \right)^2 \). Therefore, the profit gain for the low-type entity is
\[
\pi^{M,L}(\beta) - k\pi_i^{NM} = \left( \frac{2(1 - c) + (n - k + 2)x_L + (n - k)E[x]}{2(n - k + 2)} \right)^2 - k \left( \frac{1-c}{n+1} \right)^2
\]
which in the case that \( \beta = 0 \) simplifies to \( \pi^{M,L}(0) - k\pi_i^{NM} = \left( \frac{1-c+(n-k+1)x_L}{n-k+2} \right)^2 - k \left( \frac{1-c}{n+1} \right)^2 \); as in section 6.1.
Setting \( \pi^{M,L}(\beta) - k\pi_i^{NM} \geq R \) and solving for \( \theta_L \), we obtain \( \theta_L \geq \hat{\theta}(p) \). When the merging entity’s type is low with certainty, \( p = 0 \), cutoff \( \hat{\theta}(p) \) simplifies to that in section 6.1, that is, \( \hat{\theta}(0) = \hat{\theta}(k,n) \), where
\[
\hat{\theta}(k,n) \equiv \frac{n - k + 2}{(1-c)(n-k+1)} \sqrt{k \left( \frac{1-c}{n+1} \right)^2 + R - \frac{1}{n-k+1}}.
\]

**Appendix 4 - Allowing for continuous responses by the CA**

*Updated beliefs.* In this pooling strategy profile, the CA cannot update its beliefs according to Bayes’ rule. Therefore, upon observing \( R \), where \( R \geq f \), its beliefs are \( \mu(\theta_H|R) = p \) and \( \mu(\theta_L|R) = 1 - p \), whereas upon receiving any off-the-equilibrium message \( R' \neq R \), where \( R' \geq f \), its off-the-equilibrium beliefs are \( \mu(\theta_H|R') = 0 \).

*Receiver’s response.* Given the above beliefs, upon observing \( R \), in equilibrium, the CA responds exerting a challenging effort \( \alpha \), upon observing \( R \), that solves
\[
\max_{\alpha \in [0,1]} \alpha \left( \frac{2}{3} \frac{1-c}{3} + (1 - \alpha) \left( \frac{p-1-c+x_H}{2} + (1-p) \frac{1-c+x_L}{2} \right) \right) - \frac{1}{2} \lambda \alpha^2.
\]
which, differentiating with respect to \( \alpha \), yields
\[
2 \frac{1-c}{3} - \left( \frac{p-1-c+x_H}{2} + (1-p) \frac{1-c+x_L}{2} \right) - \lambda \alpha = 0
\]
and, solving for \( \alpha \), we obtain the CA’s optimal response after observing the pooling submission cost \( R \),
\[
\alpha^* = \frac{1-c - 3E[\theta]}{6\lambda}
\]
where \( E[\theta] \equiv p\theta_H + (1-p)\theta_L \). In addition, \( \alpha^* \) satisfies \( \alpha^* > 0 \) if \( 1-c > 3E[\theta] \) or, after rearranging, \( \frac{1-c}{3} > E[\theta] \), which is incompatible with the initial condition \( E[\theta] > \frac{1}{3} \equiv \tilde{\theta} \) since \( \frac{1}{3} > \frac{1-c}{3} \). Therefore, it is never optimal for the CA to challenge a merger in a pooling equilibrium.
In contrast, upon observing the off-the-equilibrium message $R'$, the CA responds blocking the merger since $\mu(\theta_H|R') = 0$ and $\theta_L < \bar{\theta}$ by assumption.

*Sender’s messages.* Anticipating these responses, the $\theta_H$-type entity invests $R$, as prescribed in this pooling strategy profile, if

$$
\alpha(2\pi_i^{NM}) + (1 - \alpha)\left(\pi_i^{M,H} - R\right) \geq 2\pi_i^{NM},
$$

where the right side assumes that the high-type deviates to zero investment (no merger request) because any deviation to $R' \neq R$ guarantees a merger decline and $R' = 0$ minimizes the firm’s submission cost. Inserting the CA’s optimal response, $\alpha^* = 0$ identified above, into this inequality, we obtain

$$
\pi_i^{M,H} - R \geq 2\pi_i^{NM}.
$$

or, after solving for $R$, we find $R \leq \pi_i^{M,H} - 2\pi_i^{NM}$. Solving for $\theta_H$, we know from Proposition 2 that this inequality yields $\theta_H > \bar{\theta}(R)$.

Similarly, the $\theta_L$-type entity chooses $R$, instead of deviating to any other $R' \neq R$, which guarantees a merger decline, if and only if

$$
\alpha(2\pi_i^{NM}) + (1 - \alpha)\left(\pi_i^{M,L} - R\right) \geq 2\pi_i^{NM},
$$

(The right side of this inequality follows a similar argument as for the high-type firm.). Inserting $\alpha^* = 0$ into this inequality, yields

$$
\pi_i^{M,L} - R \geq 2\pi_i^{NM}
$$

which simplifies to $R \leq \pi_i^{M,L} - 2\pi_i^{NM}$. Solving for $\theta_L$, we know from Proposition 2 that this inequality yields $\theta_L > \bar{\theta}(R)$. Combining the inequalities we found from the high- and low-type firms, we obtain that a PE can be sustained if $R \leq \pi_i^{M,L} - 2\pi_i^{NM}$. Since $\theta_H > \theta_L$ by definition, a sufficient condition for inequalities $\theta_H > \bar{\theta}(R)$ and $\theta_L > \bar{\theta}(R)$ to hold is $\theta_L > \bar{\theta}(R)$, which is equivalent to $\pi_i^{M,L} - 2\pi_i^{NM} \geq R$.

Therefore, equilibrium behavior coincides with that in the pooling PBEs where the CA faces a binary strategy space (approve or block merger requests), surviving both the Intuitive and Divinity Criteria.