Equilibrium Refinements in Strategic-form games (Technical)

- Mixed strategies can help us discard NEs which seem fragile to small strategic mistakes, as if a player's hands could "tremble" when choosing her strategy.

Player 2

Player 1		l	r
	U	$\underline{1},\underline{1}$	$\underline{0}$,0
	D	0,$\underline{0}$	$\underline{0},\underline{0}$

- The above game has two psNE: (U,l) and (D,r).
- The second one seems more fragile to trembles:
  - if player 1 deviates from D to U, even if U only occurs with a small probability, player 2's BR would change from r to l.
- A similar argument applies if player 2 deviates from r to l, by a small prob.
- The issue, of course, is that in (D,r) players use weakly dominated strategies.
- We next seek to rule out psNEs that aren't robust to trembles.

Equilibrium Refinements in Strategic-form games (Technical)

- Definition. **Totally mixed strategy.**
- Player $i's$ mixed strategy, σ_i, is "totally mixed" if it assigns a strictly positive probability weight on every pure strategy, that is $\sigma_i(s_i) > 0$ for all s_i.

- Therefore, all pure strategies happen, even with small probability.
- This allows for trembles, where D could occur with 0.001 probability or less.

Equilibrium Refinements in Strategic-form games (Technical)

- Definition. **Trembling-Hand Perfect equilibrium.**
- A mixed strategy profile $\sigma_i = (\sigma_i, \sigma_{-i})$ is a Trembling-Hand Perfect Equilibrium (THPE) if:

1. There exists a sequence of totally mixed strategies for each player i, $\{\sigma_i^k\}_{k=1}^{\infty}$, that converges to σ_i, and
2. for which $\sigma_i \in BR_i(\sigma_{-i}^k)$ for every k.

- Informally, these two requirements say that:
  1. Every player *i*'s totally mixed strategy (which allows for trembles) must converge to σ_i; and
  2. Strategy σ_i is player *i*'s BR to her rivals' strategy profile σ_{-i}^k at every point of the sequence (i.e., for all *k*).
- Second requirement is a bit trickier to show. (Example in a moment.)

Properties of THPE

1. Every THPE must be a NE.
2. Every strategic-form game with finite strategies for each player has a THPE.
3. Every THPE assigns zero probability weight on weakly dominated strategies.

Intuitively, points (1) and (2) show that THPEs are a subset of the set of all NEs in a strategic-form game.

$$\sigma \text{ is a THPE} \Rightarrow \sigma \text{ is a NE}$$
$$\not\Leftarrow$$

And point (3) helps us rule out strategies D for player 1 and r for player 2 in the 2x2 game we used as a motivation. Therefore, (D,r) is a NE but cannot be supported as a THPE.

Example 5.9. Trembling-hand Perfect Equilibrium

Player 2

Player 1		l	r
	U	$\underline{1},\underline{1}$	$\underline{0},0$
	D	$0,\underline{0}$	$\underline{0},\underline{0}$

Matrix 5.14. A Game with two psNEs, but only (U, l) is THPE

- Consider the following sequence of totally mixed strategies

$$\sigma_i^k = \left(1 - \frac{\varepsilon_k}{2}, \frac{\varepsilon_k}{2}\right) \text{ for every player } i\text{, where } \varepsilon_k = \frac{1}{2^k}.$$

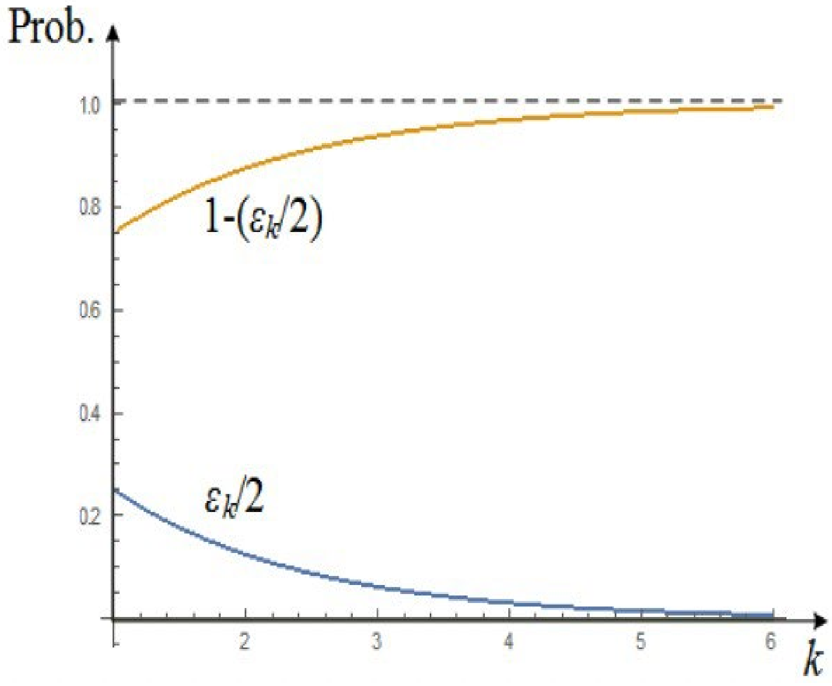
- *Example*:
  - When $k = 1, \varepsilon_1 = \frac{1}{2}$,and σ_i^k becomes $\sigma_i^1 = \left(\frac{3}{4}, \frac{1}{4}\right)$, indicating that every player i makes mistakes with $\frac{1}{4}$ probability.
  - When $k = 2, \varepsilon_2 = \frac{1}{4}$,and σ_i^k becomes $\sigma_i^2 = \left(\frac{7}{8}, \frac{1}{8}\right)$, representing that mistakes are now less likely.
- In the limit, we find that (see figure in next slide)

$$\lim_{k\to+\infty} \sigma_i^k = (1,0) \text{ since } \lim_{k\to+\infty} \varepsilon_k = \lim_{k\to+\infty} \frac{1}{2^k} = 0$$

- which implies that player 1 (2) chooses U (l, respectively) in pure strategies, yielding strategy profile (U, l).

Properties

- Generally, as k increase, mistakes become less likely, and the above totally mixed strategy converges to the psNE (U, l).
- This leads to the following figure:



Example 5.9. Trembling-hand Perfect Equilibrium

- Therefore, the NE (U, l) can be supported as a THPE because:

1. The totally mixed strategy $\sigma_1^k\left(\sigma_2^k\right)$ converges to $U\ (l)$; and
2. $U\ (l)$ is the best response of player 1 (2) to her rival's totally mixed strategy, $\sigma_2^k\left(\sigma_1^k, respectively\right)$ for all *k*.

- To see point (2), note that:
  - When k=1, σ_2^k becomes $\sigma_2^1 = \left(\frac{3}{4}, \frac{1}{4}\right)$, where U is player 1's best response because $EU_1(U|\sigma_2^1) = \frac{3}{4}1 + \frac{1}{4}0 = \frac{3}{4}$ and $EU_1(D|\sigma_2^1) = \frac{3}{4}0 + \frac{1}{4}0 = 0$.
  - When k=2, σ_2^k becomes $\sigma_2^2 = \left(\frac{7}{8}, \frac{1}{8}\right)$, and U is still player 1's best response because $EU_1(U|\sigma_2^2) = \frac{7}{8}1 + \frac{1}{8}0 = \frac{7}{8}$ and $EU_1(D|\sigma_2^1) = \frac{7}{8}0 + \frac{1}{8}0 = 0$.
  - Same argument applies to every *k* since $EU_1\left(U|\sigma_2^k\right) = \left(1 - \frac{\varepsilon_k}{2}\right)1 + \frac{\varepsilon_k}{2}0 = 1 - \frac{\varepsilon_k}{2} > 0$ and $EU_1\left(D|\sigma_2^k\right) = \left(1 - \frac{\varepsilon_k}{2}\right)0 + \frac{\varepsilon_k}{2}0 = 0$.
  - Same argument applies to player 2's best response to σ_1^k being *l* for every *k*. (Check as a practice.)

Example 5.9. Trembling-hand Perfect Equilibrium

- In contrast, (D,r) cannot be sustained as THPE.
  - While we can find converging sequences of totally mixed strategies (first requirement)…
  - Choosing D (r) is *not* player 1's (2's) best response to her rival's totally mixed strategy for every *k* (second requirement).
- To see this point, consider this totally mixed strategy:

$$\sigma_i^k = \left(\frac{\varepsilon_k}{2}, 1 - \frac{\varepsilon_k}{2}\right) \text{ for every player } i, \text{ where } \varepsilon_k = \frac{1}{2^k}.$$

- which assigns the opposite probability weights than that converging to (U,l).
- It converges to psNE (D,r). Check!

Example 5.9. Trembling-hand Perfect Equilibrium

- However, U is player 1's BR to σ_2^k for every k.
- To see this point, consider that:
  - When k=1, σ_2^k becomes $\sigma_2^1 = \left(\frac{1}{4}, \frac{3}{4}\right)$, and U is player 1's best response.
  - When k=2, σ_2^k becomes $\sigma_2^2 = \left(\frac{1}{8}, \frac{7}{8}\right)$, and U is still player 1's best response.
  - Same argument applies for every k.
  - Recall that finding that U is player 1's BR, instead of D, for *at least one value of k* and for *at least one player* would have been enough to show that (D,r) cannot be sustained as THPE.

ε —Proper Equilibrium

- THPE helps us rule out NEs that aren’t robust to trembles.
- But, which trembles do we allow?
- Myerson (1978) suggested that a rational player, while making mistakes, should put:
  - Higher probability weight on strategies yielding higher payoffs.
  - Lower probability weight on strategies yielding lower payoffs.
- Alternatively, players are less likely to make costly mistakes.

ε —Proper Equilibrium

- Definition. **$\boldsymbol{\varepsilon}$ − proper equilibrium**. For any $\varepsilon > 0$, a totally mixed strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is the $\varepsilon -$ proper equilibrium if, for every player i, and for every two pure strategies s_i,and $s_i' \neq s_i$ such that

$$u_i(s_i, \sigma_{-i}) > u_i(s_i', \sigma_{-i}),$$

  - we must have that probabilities of playing s_i and s_i', $\sigma_i(s_i)$ and $\sigma_i(s_i')$ satisfy

$$\varepsilon \times \sigma_i(s_i) \geq \sigma_i(s_i')$$

- Intuitively, if player $i's$ expected payoff from choosing s_i is higher than that from s_i', then…
  - The probability of playing s_i must be at least "ε times higher" than the probability of playing s_i'.

Example 5.10. ε —Proper Equilibrium

		Player 2	
		l	r
Player 1	U	<u>1</u>,<u>1</u>	<u>0</u>,0
	D	0,<u>0</u>	<u>0</u>,<u>0</u>

Matrix 5.14. A Game with two psNEs, but only (U, l) is THPE

- Consider $\sigma_i = \left(1 - \frac{\varepsilon}{a}, \frac{\varepsilon}{a}\right)$ for every player i, where $a \geq 2$ and $0 < \varepsilon < 1$.
- This mixed strategy is an $\varepsilon -$ proper equilibrium because: (1) it is a totally mixed strategy, assigning a positive probability weight to all players' strategies; and (2) for pure strategies U and D, their expected utilities satisfy

$$u_1(U, \sigma_2) = \underbrace{1\left(1 - \frac{\varepsilon}{a}\right)}_{\substack{Player\ 2 \\ chooses\ l}} + \underbrace{0\left(\frac{\varepsilon}{a}\right)}_{\substack{Player\ 2 \\ chooses\ r}} = 1 - \frac{\varepsilon}{a} > 0 = u_1(D, \sigma_2)$$

Example 5.10. ε —Proper Equilibrium Example

And the probabilities of applying U and D are

$$\varepsilon \times \sigma_1(U) = \varepsilon\left(1 - \frac{\varepsilon}{a}\right) = \frac{\varepsilon(a-\varepsilon)}{a} \text{ and}$$

$$\sigma_1(D) = \frac{\varepsilon}{a}$$

which satisfy

$$\varepsilon \times \sigma_1(U) = \frac{\varepsilon(a-\varepsilon)}{a} \geq \frac{\varepsilon}{a} = \sigma_1(D)$$

since, after rearranging, this inequality simplifies to $a \geq \varepsilon$, which holds given that $a \geq 2$ and $0 < \varepsilon < 1$ by assumption.

(Since the game is symmetric, a similar argument applies to player 2's utility from choosing l and r, and its associated probabilities.

Proper Equilibrium

- Definition. **Proper Equilibrium.** A mixed strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is a proper equilibrium if there exists:
  1. A sequence $\{\sigma_i^k\}_{k=1}^{\infty}$ that converges to σ_i for every player i
  2. A sequence $\{\varepsilon_i^k\}_{k=1}^{\infty}$ where $\varepsilon^k > 0$ for all k, that converges to zero
  3. $\{\sigma_i^k\}_{k=1}^{\infty}$ is an ε_k $-$proper equilibrium for every k

- Proper equilibrium are also THPE, but the converse in not necessarily true.
- In other words:
  - If σ is a proper equilibrium, it must be robust to a sequence of decreasing trembles where costly mistakes are less likely to occur;
  - while σ being THPE only requires that it is robust to any sequence of decreasing trembles.

Example 5.11. Proper Equilibrium

		Player 2	
		l	r
Player 1	U	<u>1</u>,<u>1</u>	<u>0</u>,0
	D	0,<u>0</u>	<u>0</u>,<u>0</u>

Matrix 5.14. A Game with two psNEs, but only (U, l) is THPE

- The sequence of totally mixed strategies from example 5.9

$$\sigma_i^k = \left(1 - \frac{\varepsilon_k}{2}, \frac{\varepsilon_k}{2}\right) \text{ for every player } i, \text{ where } \varepsilon_k = \frac{1}{2^k},$$

is a proper equilibrium if it satisfies the three requirements in the above definition:

1. A sequence σ_i^k converges to (U, l)
2. ε^kconverges to zero
3. σ_i^k is an ε_k-proper equilibrium for every k (as shown in Example 5.10).