Strictly Competitive Games

- Definition. **Strictly Competitive Games.** A two-player game is strictly competitive if, for every two strategy profiles s and s',
  - if $u_1(s) > u_1(s')$ then $u_2(s) < u_2(s')$; and
  - if $u_1(s) = u_1(s')$, then $u_2(s) = u_2(s')$.

- Intuitively:
  - If player 1 prefers strategy profile s to s', then player 2 has the opposite preference order: preferring s' over s; and
  - if player 1 is indifferent between s and s', player 2 must also be indifferent between these two strategy profiles.
- *Example:* The penalty kicks game is an example of a strictly competitive game where we can test the above definition (next slide).

Strictly Competitive Games

		Kicker	
		Aim left (l)	*Aim Right (r)*
Goalie	Dive Left (L)	0,0	-10,16
	Dive Right (R)	-10,16	0,0

- Comparing (L, l) and (L, r), we see that the goalie prefers the former, since $0 > -10$, while the kicker prefers the latter because $0 < 16$.
- Comparing (L, l) and (R, l), we find that the goalie prefers the former, since $0 > -10$, while the kicker prefers the latter because $0 < 16$.
- Comparing (L, l) and (R, r), we see that the goalie is indifferent, and so is the kicker, both players earning a payoff of zero in both strategy profiles.
- Comparing (R, l) and (L, r), we find that the goalie is indifferent between these two strategy profiles, earning -10 in both of them. A similar argument applies to the kicker, who earns a payoff of 16 in both strategy profiles.
- We can confirm the definition of strictly competitive games (i.e., opposite preferences of players 1 and 2) holds for every two strategy profiles, s and s'.

Games that are not strictly competitive

- A two-player game is *not strictly competitive* if, for at least two strategy profiles, s and s', every player $i's$ utility satisfies $u_i(s) > u_i(s')$.

- Example

		Criminal	
		Street A	Street B
Police	Street A	10,0	-1,6
	Street B	0,8	7,-1

Matrix 5.6. Police and Criminal Game

  - Comparing strategy profiles (A, A) and (B, B), along the main diagonal, we can see that the police prefers (A, A) to (B, B), since her payoff satisfies $10 > 7$.
    - Similar argument applies for the criminal, as her payoff satisfies $0 > -1$.
  - Because we found that players' preferences over strategy profiles are aligned, rather than misaligned, we can already claim that the game is *not* strictly competitive without having to compare other pairs of strategy profiles.

Constant-sum Games

- Definition. **Constant-sum games.** A two-player game is a constant-sum game if, for every strategy profile s, player's payoffs satisfy

$$u_1(s) + u_2(s) = K, \quad \text{where } K > 0 \text{ is a constant.}$$

- Then, players' payoffs must add up to the same constant across all cells in the matrix.
- If, instead, players' payoffs add up to a different number in at least one of the cells, then we can claim that the game is *not* constant sum.
  - It can still be strictly competitive, but not constant sum.

Constant-sum Games

Counterexample:

		Player 2	
		l	r
Player 1	U	10,0	9,3
	D	9,3	10,0

Matrix 5.7. A strictly competitive game that is non constant-sum

- The game is strictly competitive (check as practice).
- It is not a constant-sum game since players payoff in strategy profiles like (U, l) and (D, r) add up to 10, while those strategy profiles (U, r) and (D, l) add up to 12.

Constant-sum Games

- Constant-sum games are always strictly competitive:
  - Condition $u_1(s) + u_2(s) = K$ can be rewritten as $u_1(s) = K - u_2(s)$.
  - Then, if player 1's payoff increases when moving from s to s', then player 2's payoff must decrease.

- We now introduce a special class of constant-sum games, those in which K=0, called zero-sum games.

Zero-sum Games

- Definition. **Zero-sum games.** A two-player game is a zero-sum game if, for every strategy profile s, player's payoffs satisfy

$$u_1(s) + u_2(s) = 0.$$

- Alternatively, condition $u_1(s) + u_2(s) = 0$ can be expressed as $u_1(s) = -u_2(s)$.
  - Intuitively, every dollar that player 1 earns comes from the same dollar that player 2 loses and vice versa.

Player 2

		$Heads$	$Tails$
Player 1	$Heads$	1,-1	-1,1
	$Tails$	-1,1	1,-1

Matrix 5.8. Matching Pennies Game

  - Matching pennies game is zero-sum game. Rock-paper-scissors is another example.
  - Specifically, in Matrix 5.8, we have that either $1 + (-1) = 0$ or $-1 + 1 = 0$.

Security Strategies

- Definition. **Security Strategies.** In a two-player game, player $i's$ security strategy, i**,** solves
$$\max_{\sigma_i} \min_{\sigma_j} u_i(\sigma_i, \sigma_j)$$
    - Consider the "worst-case scenario" $w_i(\sigma_i) = \min_{\sigma_j} u_i\left(\sigma_i, \sigma_j\right)$
    - Player i anticipates that player j chooses her strategy σ_j to maximize her own payoff, which entails minimizing i's payoff, $u_i(\sigma_i, \sigma_j)$.
        - This is because players interact in a strictly competitive game.
    - Player i then chooses her strategy σ_i to maximize the payoff across all worst-case scenarios.
    - Intuitively, player i seeks to find the strategy σ_i that provides her with the "best of the worst" payoffs, as represented with the max-min problem.
        - This explains why security strategies are sometimes known as max-min strategies.

Tool 5.2. How to find security strategies in a two-player game

1. Find the expected utility of player 1's randomization, fixing player 2's strategy.
2. Repeat step 1 until you considered all strategies of player 2, fixing one at a time.
3. *"Min" part.* Find the lower envelope of player 1's expected utility. That is, for each strategy σ_1, find the lowest expected utility that player 1 earns.
4. *"Max" part.* Find the highest expected utility of the lower envelope identified in step 3, and the corresponding strategy σ_1. This is player 1's security strategy, σ_1^{sec}.
5. To find the security strategy for player 2, follow a similar process in steps 1-4 above.

Example 5.3. Finding Security Strategies

Example

		Player 2	
		l	r
Player 1	U	10,0	9,3
	D	9,3	10,0

Matrix 5.9. A Strictly Competitive Game that is non constant-sum

To find the security strategy for player 1, we follow the next steps:

1. We find player 1's expected utility of randomizing between U and D, with associated probabilities p and $1-p$, respectively. First, we fix player 2's strategy at column l, which yields:

$$EU_1(p|l) = p \times 10 + (1-p) \times 9 = 9 + p$$

Example 5.3. Finding Security Strategies

2. We now find her expected utility of randomizing, but fixing player 2's strategy at column r, as follows:

   $EU_1(p|r) = p \times 9 + (1-p) \times 10 = 10 - p$

3. To find the lower envelope of the previous two expected utilities, we can depict each line as a function of p, as we do in Figure 5.5. The lower envelope is the segment $9 + p$ for all $p \leq \frac{1}{2}$, but segment $10 - p$ otherwise.

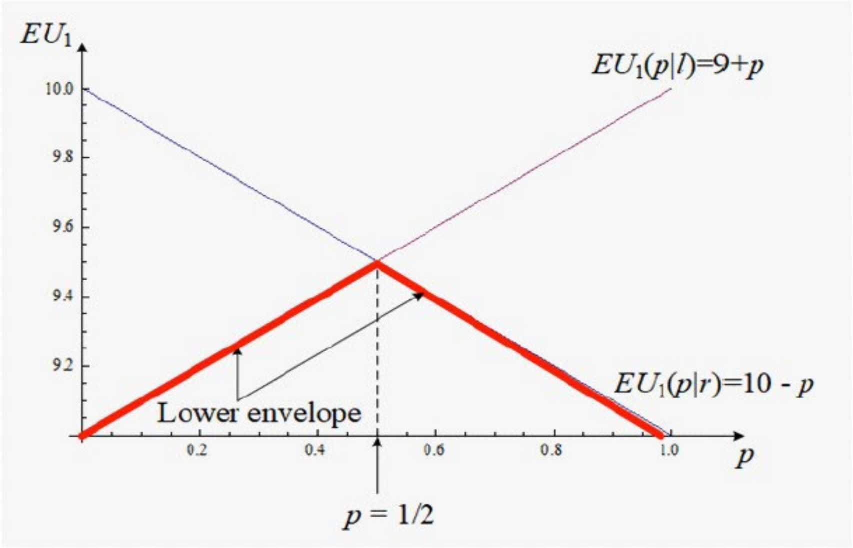

Figure 5.5. Lower envelope and security strategies.

Example 5.3. Finding Security Strategies

4. Among all points in the lower envelope, player 1 enjoys the highest utility at $p = \frac{1}{2}$, which yields an expected payoff $EU_1(p|l) = 9 + \frac{1}{2} = 9.5$, as illustrated in Figure 5.5 by the height of the crossing point between $EU_1(p|l)$ and $EU_1(p|r)$. This is player 1's security strategy, $p^{Sec} = \frac{1}{2}$.
5. Following the same steps for player 2, we find that, since payoffs are symmetric, her security strategy is $q^{Sec} = \frac{1}{2}$.

Security Strategies and NE

- At this point, you may be wondering about the relationship between security strategies and msNE.
- We obtain the same equilibrium result from both solution concepts, but only for two-player strictly competitive games.
- Consider the previous example from Matrix 5.9:

$$(p^{sec}, q^{sec}) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Let us now confirm that the msNE produces the same result.

Example 5.4. Solved by msNE

- Assuming that player 1 randomizes between U and D with probabilities p and $1-p$, respectively, and player 2 mixes l and r with probabilities q and $1-q$, respectively,
  - We find that player 1's expected utility from choosing U is:
$$EU_1(U) = q \times 10 + \ (1-q)\ \times\ 9 = 9 + q$$
  - Similarly, player 1's expected utility from choosing D is:
$$EU_1(D) = q \times 9 + \ (1-q)\ \times\ 10 = 10 - q$$
  - Therefore, player 1 randomizes between U and D when she is indifferent between these two pure strategies $EU_1(U) = EU_1(D)$, which entails
$$9 + q = 10 - q \ \Rightarrow q = \frac{1}{2}$$

Example 5.4. Solved by msNE

- Player 2's expected utilities
  - $EU_2(l) = p \times 0 + (1-p) \times 3 = 3 - 3p$, and
  - $EU_2(r) = p \times 3 + (1-p) \times 0 = 3p$
  - $EU_2(l) = EU_2(r) \Rightarrow 3 - 3p = 3p \Rightarrow p = \frac{1}{2}$

- Summarizing, we can claim that the msNE of this game is $(p, q) = \left(\frac{1}{2}, \frac{1}{2}\right)$, which coincides with the security strategies we found in example 5.3.

Example 5.5. Security strategies and msNE yield different equilibrium outcomes

		Player 2	
		l	r
Player 1	U	3,5	-1,1
	D	2,6	1,2

Matrix 5.10. A Game that is not strictly competitive

- The above game is not strictly competitive. We can find strategy profile where players' interests are aligned; both players prefer, for instance, (U, l) to (D, r).
- Since the game is not strictly competitive, we can expect that security strategies may produce a different equilibrium prediction than msNE.

Example 5.5. Security strategies and msNE yield different equilibrium outcomes

- For player 1:
  - When player 2 chooses l, player 1's expected payoff from randomizing between U and D with probabilities p and $1-p$ respectively,
  $$EU_1(p|l) = p \times 3 + (1-p) \times 2 = 2 + p$$
  - When player 2 chooses r, player 1's expected utility is
  $$EU_1(p|r) = p \times (-1) + (1-p) \times 1 = 1 - 2p$$

Example 5.5. Security strategies and msNE yield different equilibrium outcomes

- $EU_1(p|l)$ lies above $EU_1(p|r)$ for all $p \in [0,1]$.
- This means that the lower envelope coincides with $EU_1(p|r) = 1 - 2p$ for all values of p.
- The highest point of this lower envelope occurs at $p = 0$, so player 1 assigns no probability weight to U or, alternatively, that she plays D in pure strategies.
- This means that D is player 1's security strategy.

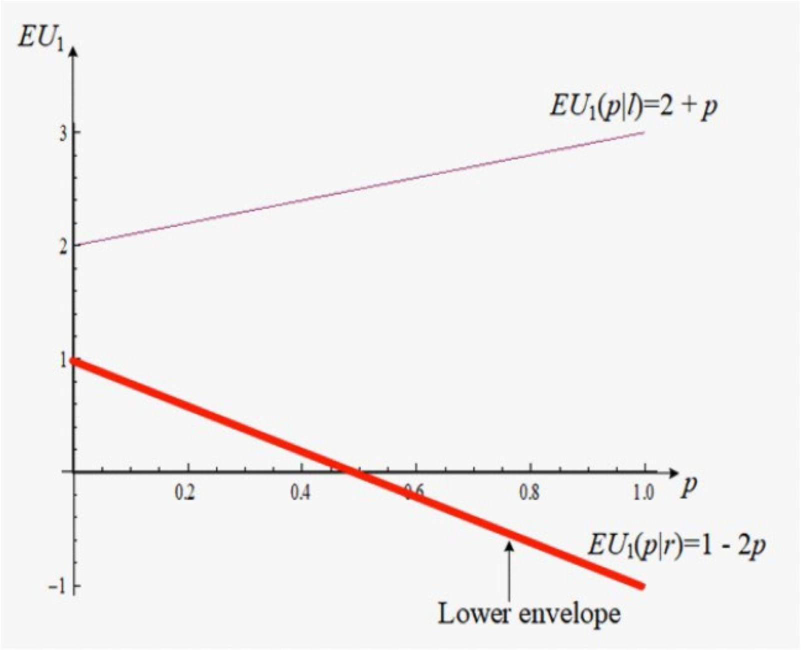

Figure 5.6. Lower envelope and security strategies - Corner solution.

Security strategies and msNE yield different equilibrium outcomes

Similarly, for <u>player 2</u>

- $EU_2(q|U) = q \times 5 + (1-q) \times 1 = 1 + 4q,$ and
- $EU_2(q|D) = q \times 6 + (1-q) \times 2 = 2 + 4q.$

- Since $EU_2(q|U)$ < $EU_2(q|D)$ for all values of q,we can claim that U is the lower envelope.
- We can, then, notice that the highest point of $1 + 4q$ occurs at $q = 1$, meaning that player 1 puts full probability weight on l, which becomes his security strategy.

In summary, the security strategy profile in this game is (D, l).

Example contd. & solving by msNE

- For **msNE**
  - We can facilitate our analysis by noticing that strategy l strictly dominates r since it yields a strictly higher payoff than r regardless of the row that player 1 chooses $(5 > 1 \text{ and } 6 > 2)$.
  - We know players put no probability weight in strictly dominated strategies, so we can delete column r from the matrix and obtain:

Player 2

		l
Player 1	U	3,5
	D	2,6

Matrix 5.10. A Game that is not strictly competitive – After deleting column r

  - Turning now to player 1, we do not need to consider his randomization since, at this point, he has a clear best response to l, U. Therefore, the psNE (no msNE) is (U, l).
  - This equilibrium outcome does not coincide with the security strategy profile (D, l).