Chapter 4: Nash equilibria in games with continuous action spaces

Game Theory:

An Introduction with Step-by-Step Examples

by Ana Espinola-Arredondo and Felix Muñoz-Garcia

Introduction

- So far, we studied discrete choice games.
  - Players face a discrete set of actions like “left” or “right”, “high” or “low” etc.
- Players often have continuous action spaces.
- Some examples are
  - Firms choosing prices/quantities.
  - Donors choosing how much to contribute.
  - Participants in an auction choosing how much to bid.
- In this chapter, we will look at some such games.

Topics covered

- Cournot model of quantity competition.
- Bertrand model of price competition.
- Competition among firms with differentiated/heterogeneous goods.
  - Examples include clothing brands, drinks brands etc.
  - Here, products are substitutes of one another, but only to a degree, i.e., if a firms sets a slightly higher price than its rival, it does not drive all of its customers away.
- Donations to a public good.
- Electoral competition

Quantity competition

- Two firms simultaneously and independently choose their output levels.
- Commonly known as the Cournot model of quantity competition, after Antoine Augustin Cournot (1838).
- From example 3.1, we know the best response function.

$$q_i(q_j) = \begin{cases} \frac{1-c}{2} - \frac{1}{2}q_j \text{ if } q_j < 1-c \\ 0 \text{ otherwise.} \end{cases}$$

- Here, firms regard output as “strategic substitutes” as in example 3.1.
- Note that $0 < c < 1$ by assumption throughout this chapter, unless stated otherwise.

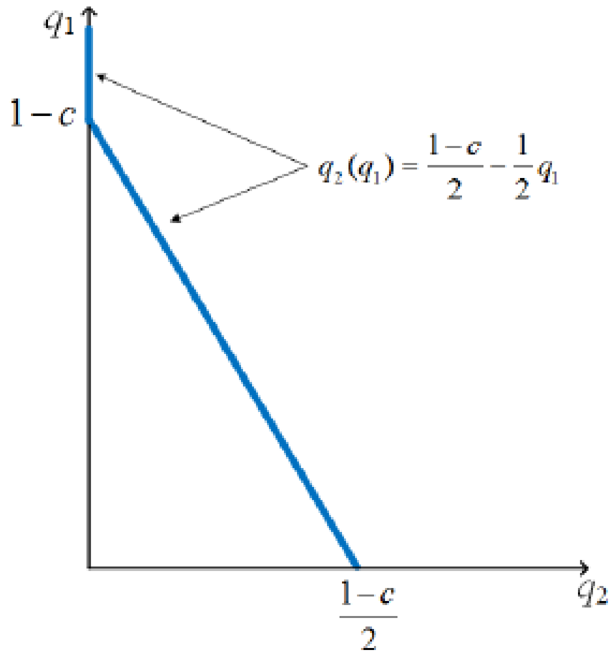
Finding Nash equilibrium: First approach

- We have a system of two equations and two unknowns.
- Solving simultaneously will give a solutions $\left(q_i^*, q_j^*\right)$ that are mutual best responses.
- **First approach:** We can *insert one firm's BRF into another* as follows

$$q_i = \frac{1-c}{2} - \frac{1}{2}\overbrace{\left(\frac{1-c}{2} - \frac{1}{2}q_i\right)}^{q_j(q_i)}$$

- which is a function of q_i alone, and yields $q_i^* = \frac{1-c}{3}$.

Mutual best response (Figures)



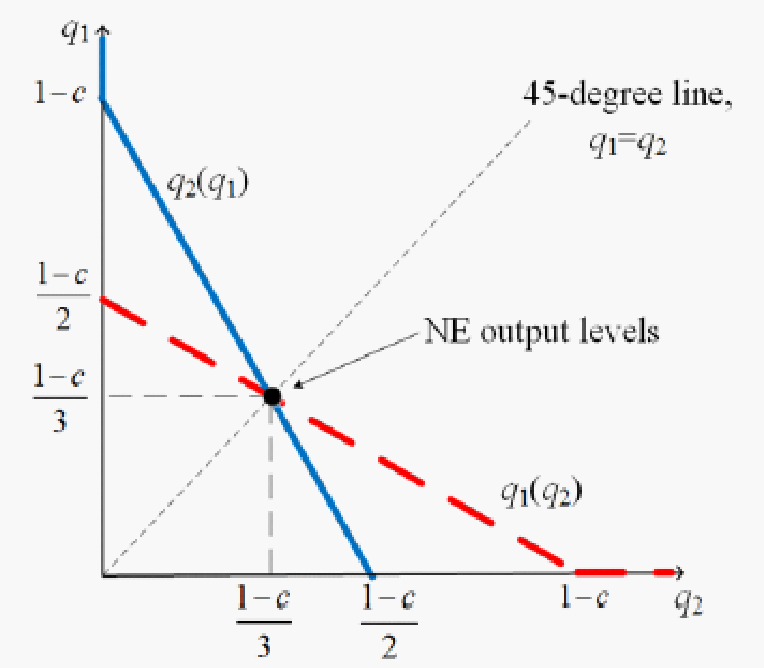


Figure 4.1a. Firm 2's best response function, $q_2(q_1)$.

Figure 4.1b. NE output in the Cournot game.

Finding Nash equilibrium: Second approach

- **Second approach:** An alternative and faster approach is to *recognize the symmetry* in the game (identical demand and cost functions).
- This entails $q_i^* = q_j^* = q^*$.
- Inserting this property into either firm's best response function, yields

$$q^* = \frac{1-c}{2} - \frac{1}{2}q^* \Leftrightarrow q^* = \frac{1-c}{3}$$

- Second approach is useful only if there is symmetry, while the first works regardless of symmetry.
- Second approach:
  - Quick but must be used cautiously (only after being sure the game is symmetric).
  - And don't assume symmetry in the objective function, you can only assume it after FOCs.

Finding payoffs (Equilibrium profits)

$$\pi^* = (1 - q^* - q^*)q^* - cq^*$$

$$= \left(1 - \frac{1-c}{3} - \frac{1-c}{3}\right)\frac{1-c}{3} - c\,\frac{1-c}{3} = \frac{(1-c)^2}{9}$$

- which coincides with the square of the equilibrium output i.e.,

$$\pi^* = (q^*)^2$$

Extending quantity competition to $N \geq 2$ firms

- The same model can be extended to a setting in which there are $N \geq 2$ firms.
- Objective function:

$$\max_{q_i \geq 0} \ (1 - q_i - Q_{-i})q_i - cq_i$$

- where $Q_{-i} = \sum_{j \neq \mathrm{i}} q_j$ denotes aggregate output by firm i's rivals, and q_i is firm i's choice variable.
- Differentiating with respect to q_i yields

$$1 - 2q_i - Q_{-i} = c$$

Solving for q_i, we obtain best response function:

$$q_i(Q_{-i}) = \begin{cases} \frac{1-c}{2} - \frac{1}{2} Q_{-i} & \text{if } Q_{-i} < 1 - c \\ 0 \text{ otherwise.} \end{cases}$$

Same vertical intercept and slope as in a duopoly.

Finding Nash equilibrium output

- We can use the symmetric equilibrium method (second approach) here:
- So Q^*_{-i} must be $Q^*_{-i} = (N-1)q^*$
- Inserting this property into the BRF, yields

$$q^* = \frac{1-c}{2} - \frac{1}{2}(N-1)q^* \Leftrightarrow q^*(N) = \frac{1-c}{N+1}.$$

which coincides with that in a duopoly since $q^*(2) = \frac{1-c}{3}$.

Finding Nash equilibrium output

- Therefore, aggregate output in equilibrium is

$$Q^* = N \times q^*(N) = \frac{N(1-c)}{N+1}$$

which is increasing in N since $\frac{\partial Q^*}{\partial N} = \frac{1-c}{(N+1)^2} > 0$ since $c < 1$ by definition.

- Finding price is now easy by using the inverse demand function:

$$p^* = 1 - Q^* = 1 - \frac{N(1-c)}{N+1} = \frac{1+Nc}{N+1}.$$

which is decreasing in N since $\frac{\partial p^*}{\partial N} = -\frac{1-c}{(N+1)^2} < 0.$

Finding Nash equilibrium output

- Inserting these results in the objective function (profits), we obtain that

$$\pi^* = p^* q^* - c q^* = \frac{1+Nc}{N+1}\frac{1-c}{N+1} - c\,\frac{1-c}{N+1} = \frac{(1-c)^2}{(N+1)^2},$$

or, more compactly, $\pi^* = (q^*)^2$.

Special cases

- When $N = 1$, we get standard results in a monopoly.
  - $q^* = \frac{1-c}{2}$
  - $p^* = \frac{1+c}{2}$
  - $\pi^* = \frac{(1-c)^2}{4}$

- As $N \to +\infty$, we get the results in perfect competition.
  - q^* approaches 0
  - $p^* = c$ (marginal cost pricing).

Quantity competition with heterogeneous goods

- Consider a duopoly, but a modified inverse demand function.

$$p_i(q_i, q_j) = 1 - q_i - dq_j; 1 \geq d \geq 0$$

- The last property means that own-price effects dominate cross-price effects.
- Objective function:

$$\max_{q_i \geq 0} \quad (1 - q_i - dq_j)q_i - cq_i$$

- We follow the same steps to find the best response functions, and get

$$q_i(q_j) = \begin{cases} \frac{1-c}{2} - \frac{d}{2}q_j \quad \text{if } q_j < \frac{1-c}{d} \\ 0 \text{ otherwise.} \end{cases}$$

Intuition for d

- As d goes closer to 1, products are more substitutable (closer to homogenous). The opposite happens as d goes closer to 0.

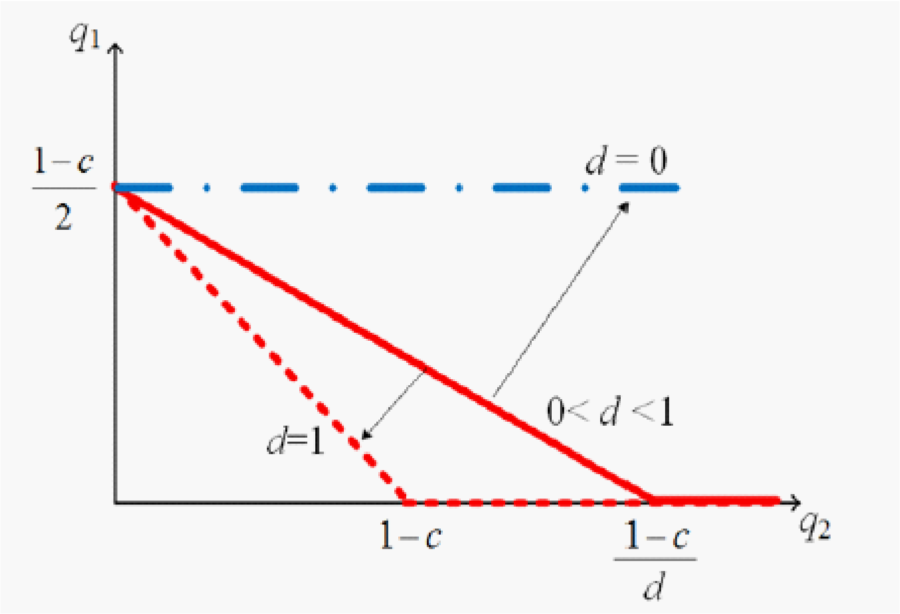
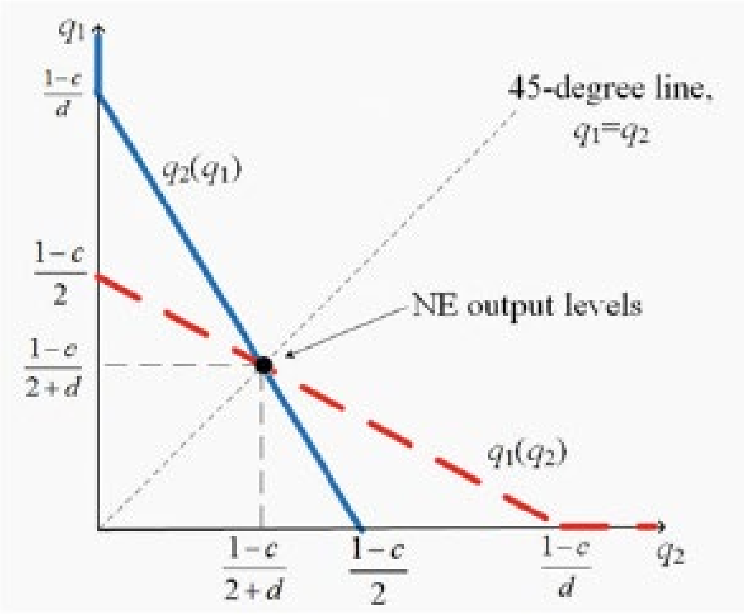


Figure 4.3. Firm i's best response function.

Finding Nash equilibrium output

- We continue to rely on symmetry, because though the goods are not identical, they face the same demand function and costs, and even the d faced by both is identical.
- Therefore, the BRF becomes $q^* = \frac{1-c}{2} - \frac{d}{2}q^*$. Solving for q^*, yields $q^* = \frac{1-c}{2+d}$.



Finding Nash equilibrium output

- Equilibrium output $q^* = \frac{1-c}{2+d}$ is decreasing in d.
  - d going to 1 yields a standard Cournot output with no differentiation, $q^* = \frac{1-c}{3}$.
  - d going to 0 yields a monopoly output as the two goods then become completely unrelated, and serve different markets, $q^* = \frac{1-c}{2}$.

Finding Nash equilibrium output

- Equilibrium price is

$$p^* = 1 - \frac{1-c}{2+d} - d\,\frac{1-c}{2+d} = \frac{1+c(1+d)}{2+d},$$

which is decreasing in parameter d since $\frac{\partial p^*}{\partial d} = -\frac{1-c}{(2+d)^2} < 0$ because $0 < c < 1$ by assumption.

- Equilibrium profits are

$$\pi^* = p^* q^* - c q^* = \frac{1+c(1+d)}{2+d}\,\frac{1-c}{2+d} - c\,\frac{1-c}{2+d} = \frac{(1-c)^2}{(2+d)^2}$$

which, for compactness, can also be expressed as $\pi^* = (q^*)^2$, thus being decreasing as products become more homogeneous (higher d).

Price competition

- Consider 2 firms competing in prices.
- This form of competition is known as Bertrand competition, after Joseph Louis François Bertrand (1883).
- Demand/sales for each firm:
  - Since products are undifferentiated, the cheaper good attracts all the customers while rivals make 0 sales.
  - If firms set $p_i = p_j$, they evenly share market demand.
- We can summarize this demand function as follows:

$$q_i(p_i, p_j) = \begin{cases} 1 - p_i \text{ if } p_i < p_j \\ \dfrac{1 - p_i}{2} \text{ if } p_i = p_j \\ 0 \text{ if } p_i > p_j \end{cases}$$

Best response function

$$p_i(p_j) = \begin{cases} p^m \text{ if } p_j > p^m \\ p_j - \varepsilon \text{ if } c < p_j \leq p^m \\ c \text{ if } p_j \leq c \end{cases}$$

- Here $p^m = \frac{1+c}{2}$ denotes monopoly price.
- If firm j sets a price above p^m, i can set the price as p^m and get maximum profits.
- If j sets a price between c and p^m, i should undercut j marginally, setting $p_i = p_j - \varepsilon$, where $\varepsilon \to 0$.
- If j sets a price below c, any price greater than p_j is a best response. Let us say he sets $p_j = c$.

Best response: Graphically

- The graph below represents a case of “strategic complements”, where the best response of a player increases (decreases) as their rival’s choice increases (decreases). This can be seen from the positive slope.

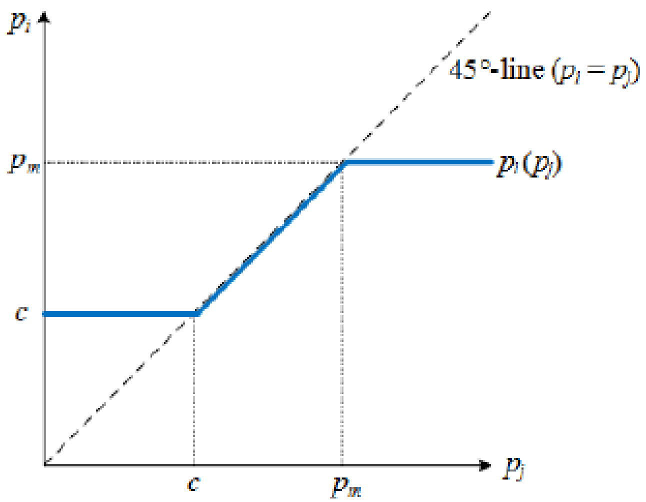


Figure 4.5. Firm i's best response function when competing in prices.

Best response (continued)

- Firm j's best response is symmetric to firm i.
- $p_j(p_i) = \begin{cases} p^m \text{ if } p_i > p^m \\ p_i - \varepsilon \text{ if } c < p_i \leq p^m \\ c \text{ if } p_i \leq c \end{cases}$

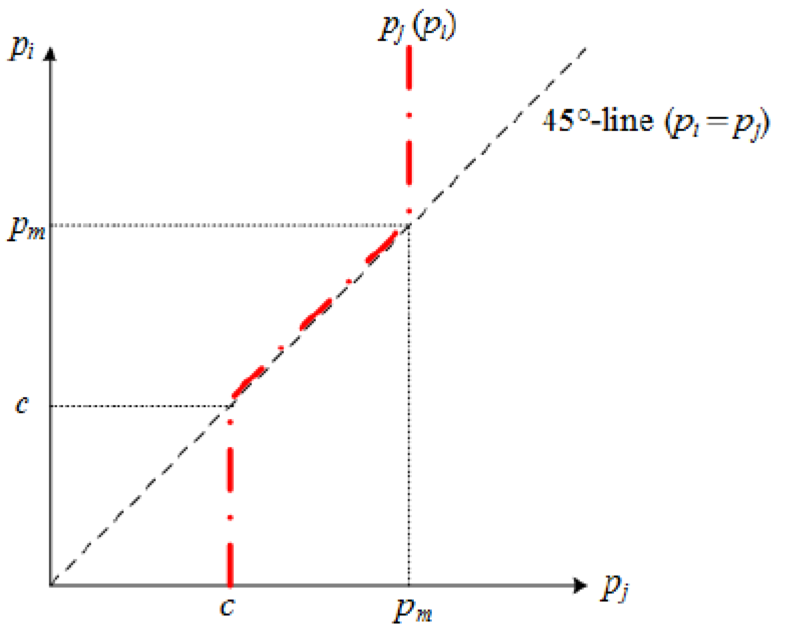


Figure 4.6. Firm j's best response function when competing in prices.

Finding Nash equilibrium prices

- Following a similar method (where the graphs intersect), we find the NE here too.
- $p_i^* = p_j^* = c$, or we can say the NE pair is $(p_i^*, p_j^*) = (c, c)$, and equilibrium profits are 0.
- Super competitive outcome with just two firms.
    - It is actually unaffected by the number of firms (n-firm Bertrand oligopoly), but…
    - It is affected if firms sell differentiated goods, as we examine next.

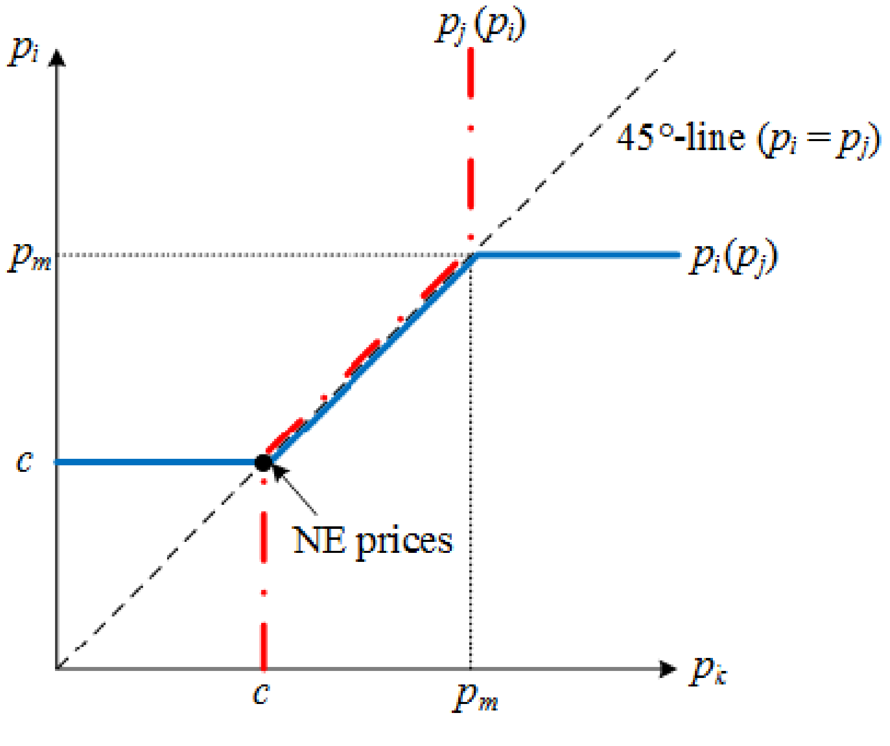


Figure 4.7. Price equilibrium.

Price competition with heterogeneous goods

- Consider two firms firms compete in prices but sell a horizontally differentiated product. Demand is given by

$$q_i(p_i, p_j) = 1 - p_i + \gamma p_j;\ \gamma \in (0,2)$$

- Intuition for parameter γ:
  - When $\gamma \to 0$, j's price does not affect i's sales.
  - When $\gamma = 1$, a change j's price affect i's sales at a similar rate as a change in its own price.
  - When $\gamma > 1$, a change in j's price affects i's sales for than a change in i's own price.

Finding best response functions

- The objective function for every firm i is

$$\max_{p_i \geq 0} \quad \pi_i(p_i) = (p_i - c)\left(1 - p_j + \gamma p_j\right)$$

- Differentiating with respect to p_i, yields

$$1 - 2p_i + c + \gamma p_j = 0$$

- and solving for p_i gives us the best response function:

$$p_i\left(p_j\right) = \frac{1 + c}{2} + \frac{\gamma}{2} p_j$$

Best response function (graph)

$$p_i(p_j) = \frac{1+c}{2} + \frac{\gamma}{2} p_j$$

- When firm j increases its price by \$1, firm i responds increasing its own by $\frac{\gamma}{2}$, which is less than \$1 since $\gamma < 2$ by assumption.
  - (Less-than-proportional response.)
- When γ decreases, sales are less affected by its rival's price, flattening the BRF.

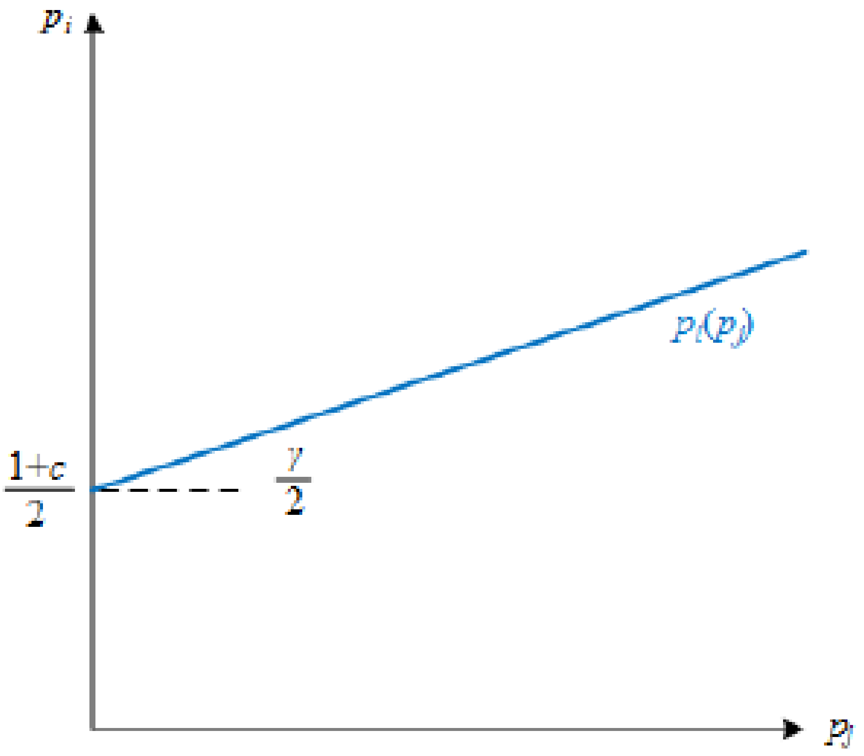


Figure 4.8. Firm i's best response function, $p_i(p_j)$, with heterogeneous goods.

Finding Nash equilibrium prices

- Invoking symmetry, $(p^* = p_1 = p_2)$, the BRF becomes

$$p^* = \frac{1+c}{2} + \frac{\gamma}{2} p^*$$

Solving for p^*, yields the equilibrium price

$$p^* = \frac{1+c}{2-\gamma}$$

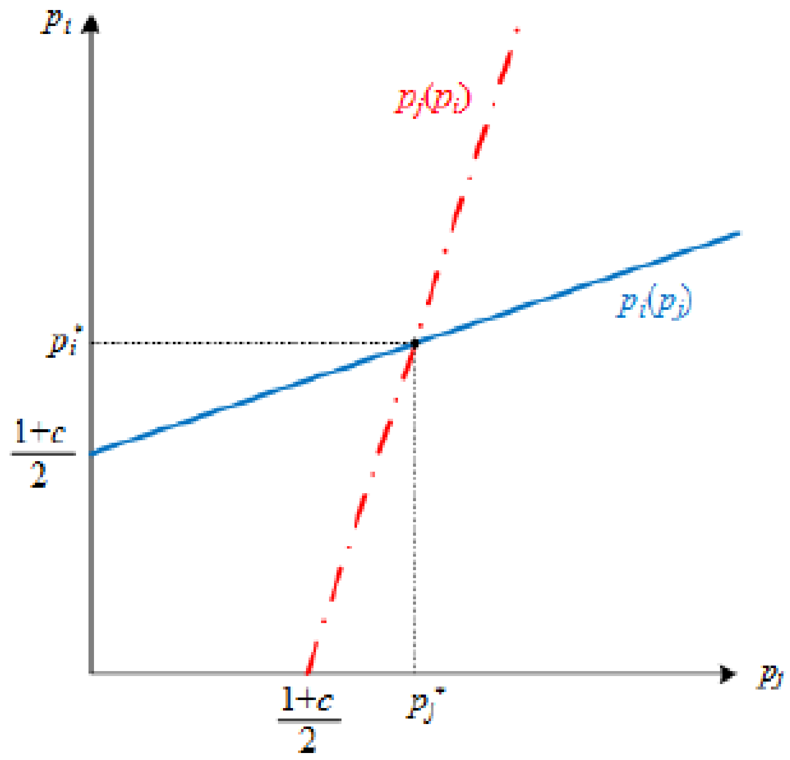


Figure 4.9. Equilibrium price pair with heterogeneous goods.

Finding Nash equilibrium prices

- $p^* = \frac{1+c}{2-\gamma}$ is always positive since γ is less than 2.
- p^* is increasing in γ.
  - (Take the derivative and check! It is greater than 0).
  - Crossing point in the figure of BRFs still happens along the 45-degree line, but moves northeast as γ increases.
- Intuitively, as firm i's sales are more positively affected by its rival's price, p_j, firm i can charge a higher price for its product.

Finding Nash equilibrium prices (graph)

- Additionally, equilibrium output can be found by inserting $p^* = \frac{1+c}{2-\gamma}$ into the demand function, as follows

$$q^* = 1 - p^* + \gamma p^* = 1 - \frac{1+c}{2-\gamma} + \gamma \frac{1+c}{2-\gamma} = \frac{1-(1-\gamma)c}{2-\gamma}.$$

- which is increasing in γ (Check).
  - Therefore, both equilibrium price and output are increasing in product differentiation, γ.
- Equilibrium profits are

$$\pi^* = (p^* - c)q^* = \left(\frac{1+c}{2-\gamma} - c\right)\frac{1-(1-\gamma)c}{2-\gamma} = \left(\frac{1-(1-\gamma)c}{2-\gamma}\right)^2,$$

which, for compactness, can be also expressed as $\pi^* = (q^*)^2$.

- This profit is unambiguously positive, and increasing in γ,
  - exceeding the zero profits under Bertrand competition with homogeneous products.

Public good game, PGG

- Consider a public project (like a highway or a public library),
  - Two individuals simultaneously and independently choose some contribution x_i and x_j,
  - where $X = x_i + x_j$ denotes aggregate contributions/donations.
- Public goods are “non rivalrous”, meaning one person enjoying a good does not exclude the other from doing so (roads, parks etc.)

$$u_i(x_i, x_j) = (w_i - x_i)\sqrt{mX}$$

- $m \geq 0$ represents returns from aggregate contributions X. Note that the level of wealth for each player shown by w_i is common knowledge.

Finding best response functions

$$\max_{x_i \geq 0} \quad u_i(x_i, x_j) = (w_i - x_i)\sqrt{m(x_i + x_j)}$$

- where $X = x_i + x_j$.
- Differentiating with respect to x_i, we find that

$$-\sqrt{m(x_i + x_j)} + \frac{m(w_i - x_i)}{2\sqrt{m(x_i + x_j)}} = 0$$

- Which, after rearranging, yields

$$\frac{m(w_i - 2x_j - 3x_i)}{2\sqrt{m(x_i + x_j)}} = 0$$

which holds if $3x_i = w_i - 2x_j$.

Finding best response functions

- Thus, the BRF is
- $x_i(x_j) = \begin{cases} \frac{w_i}{3} - \frac{2}{3}x_j \text{ if } x_j < \frac{w_i}{2}, & \text{and} \\ 0 \text{ otherwise} \end{cases}$

which:

- originates at $\frac{w_i}{3}$ and
- decreases in player *j*'s contributions, at a rate $\frac{2}{3}$.

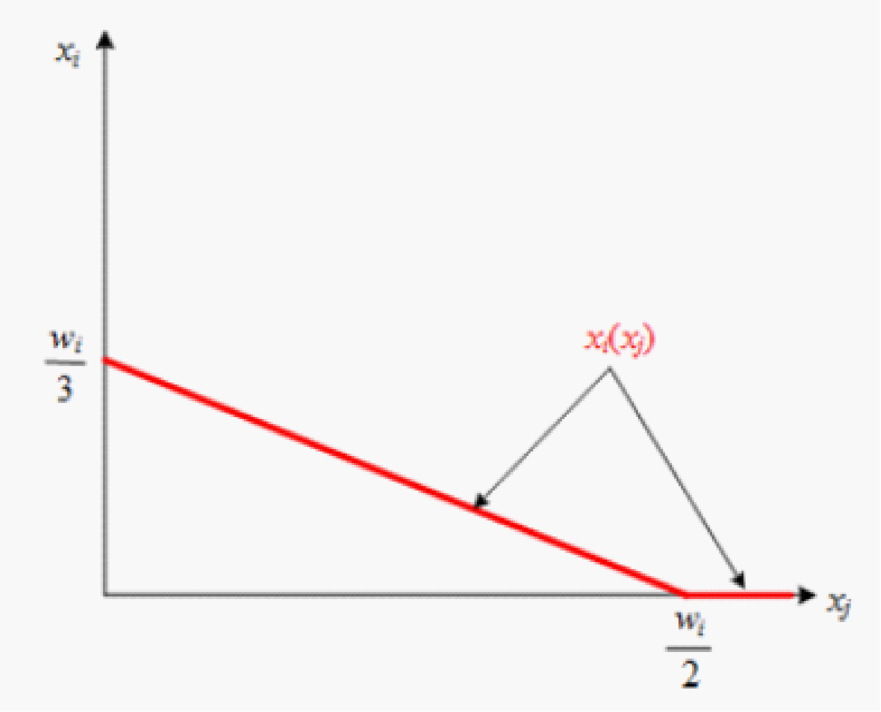


Figure 4.9. Player i's best response function, $x_i(x_j)$.

Finding Nash equilibrium (Symmetric wealth)

- In the symmetric case, $w_1 = w_2 = w$. We can invoke $x_i - x_j = x$. Inserting this into the BRF gives us

$$x = \frac{w}{3} - \frac{2}{3}x$$

- Which yields $x^* = \frac{w}{5}$. This means each player donates a fifth of their wealth. The contribution is increasing in w.

Finding Nash equilibrium (Asymmetric wealth)

- We now consider the more general case where $w_i \neq w_j$. We have to solve the BRFs by substitution.

$$x_i = \frac{w_i}{3} - \frac{2}{3}\underbrace{\left(\frac{w_j}{3} - \frac{2}{3}x_i\right)}_{x_j}$$

- Solving yields $x_i^* = \frac{3w_i - 2w_j}{5}$, which is positive if $w_i > \frac{2}{3}w_j$.
- Similarly, $x_j^* = \frac{3w_j - 2w_i}{5}$, which is positive if $w_i < \frac{3}{2}w_j$.
- *Typical trick*:
  - We solve for the same parameter, w_i, in both equations, so we can obtain two different cutoffs, which we can depict in the (w_i, w_j) quadrant (next slide).

Finding Nash equilibrium (Asymmetric wealth)

- When i's wealth is lower than $\frac{2}{3}w_j$, only j contributes, and vice versa.
- When $\frac{2}{3}w_j \leq w_i < \frac{3}{2}w_i$, both players contribute.
  - A special case of this is symmetric wealth, as seen in the 45° line of the graph.
- When i's wealth is higher than $\frac{3}{2}w_j$, she is the only one making positive contributions.

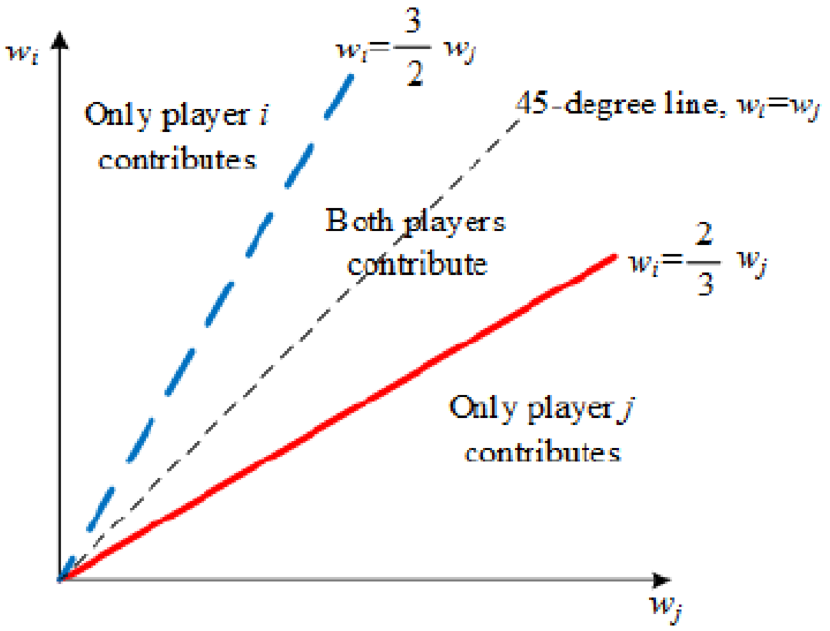


Figure 4.10. Contribution profiles in equilibrium.

Finding Nash equilibrium (Asymmetric wealth)

- Aggregate donations,

$$X^* = \frac{3w_i - 2w_j}{5} + \frac{3w_j - 2w_i}{5} = \frac{w_i + w_j}{5} = \frac{W}{5}.$$

- where $W = w_i + w_j$ denotes aggregate wealth.
- Thus, a fifth of total wealth is contributed to the public good.

- *Example*: if $w_i = \$15 \text{ and } w_j = \$12$,

$$x_i^* = \$4.2,$$
$$x_j^* = \$1.2,$$
$$X^* = \$5.4$$

Inefficient equilibrium

- We can show that the equilibrium donations are “Pareto Inefficient”.
- For simplicity, we can show this result in the case of symmetric wealth. To show this, we first need to maximize joint utility, to find which contribution levels are socially optimal:

$$\max_{x_i\,,\,x_j \geq 0} \quad u_i(x_i, x_j) = (w - x_i)\sqrt{mX} + (w - x_j)\sqrt{mX}$$

- This expression simplifies to

$$\max_{X \geq 0} \quad (W - X)\sqrt{mX}$$

- where $W = w_i + w_j$. We also changed the choice variable to X.

Inefficient equilibrium

- Differentiating with respect to X yields

$$-\sqrt{mX} + \frac{m(W - X)}{2\sqrt{mX}} = 0$$

which simplifies to $\frac{m(W-3X)}{2\sqrt{mX}} = 0$. Solving for X, we obtain the aggregate socially optimal donation:

$$X^{SO} = \frac{W}{3}$$

- Therefore, every player's socially optimal donation is $x_i^{SO} = \frac{W}{6} = \frac{2w}{6} = \frac{w}{3}$, which exceeds that in equilibrium because

$$x_i^{SO} = \frac{w}{3} > \frac{w}{5} = x^*$$

- Hence, the privately optimal outcome is not Pareto efficient/socially optimal.
    - *Alternative*: equilibrium donations are "socially insufficient."
- As expected, aggregate donations satisfy $X^{SO} = \frac{W}{3} > \frac{W}{5} = X^*$

Electoral competition

- Consider two politicians running for president who must simultaneously and independently choose their position in a policy $x_i \in [0,1]$.
- Intuitively, x_i is the proportion of the budget allocated to the policy (0 means none and 1 means the entire budget).
- We assume that the candidates
  - Do not per say have any preferences over the policies, and only want to win
  - Will deliver on the promise (i.e., they are not lying).
- Each voter's ideal policy point is located on this $[0,1]$ interval. Additionally, we assume that voters are
  - Uniformly distributed on $[0,1]$.
  - "Non-strategic", and simply vote for the candidate whose policy is closest to the voters ideal.

Electoral competition

- If $x_i < x_j$
  - All voters to the left of x_i will vote for i.
  - All voters to the right of x_j will vote for j.
  - For those between x_i and x_j, those between x_i and $\frac{x_i+x_j}{2}$ for i while those between $\frac{x_i+x_j}{2}$ and x_j vote for j.

- A candidate wins when
  - If $x_i < x_j$, i wins if $\frac{x_i+x_j}{2} > 1 - \frac{x_i+x_j}{2}$ or $x_i > 1 - x_j$. Otherwise j wins.
  - If $x_i > x_j$, i wins if $x_i < 1 - x_j$. Otherwise j wins.

Finding best response *correspondences*

- For player i
  - If $x_i < x_j$, i sets x_i such that $x_i > 1 - x_j$ and wins as depicted in Region I.
  - If $x_i > x_j$, i sets x_i such that $x_i < 1 - x_j$ and wins as depicted in Region II.
- For player j
  - If $x_j < x_i$, i sets x_j such that $x_j > 1 - x_i$ and wins as depicted in Region III.
  - If $x_j > x_i$, i sets x_j such that $x_j < 1 - x_i$ and wins as depicted in Region IV.

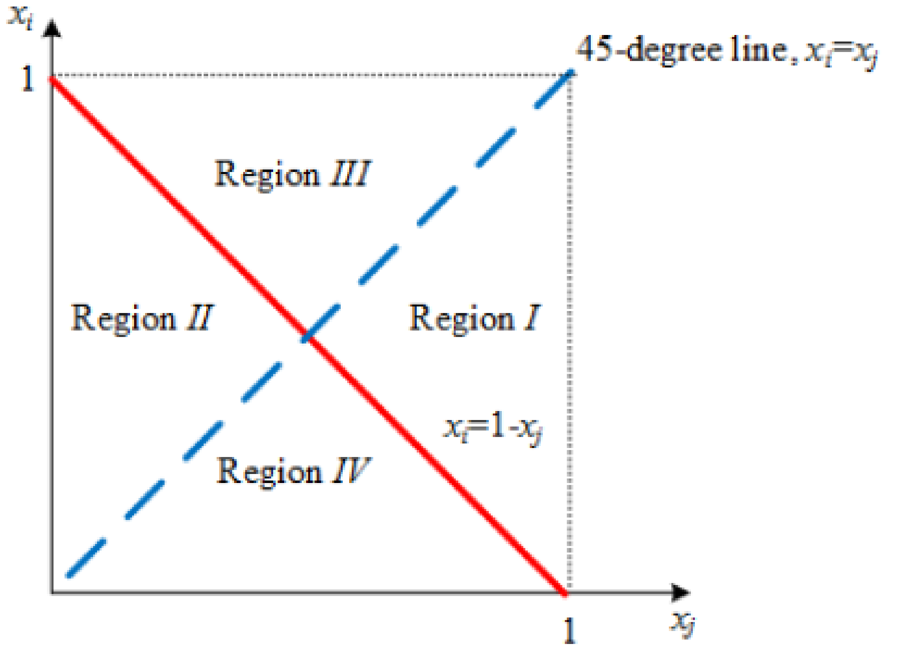


Figure 4.11. Candidate i's best response correspondence.

Finding the Nash equilibrium of the game

- From the above figure 4.11, it is clear that mutually best responses only occur at $x_i^* = x_j^* = \frac{1}{2}$.
- This implies that both candidates would choose the midpoint, converging on their political announcements.
- Since voters' ideal policies are uniformly distributed, this announcement coincides with the ideal policy of the median voter.
  - This result is known as the "median voter theorem".
- The theorem was originally described by Hotelling (1929) and formally shown by Downs (1957).
- For a literature review, see Congleton (2003).

Alternative proof to the electoral competition game

- Consider an asymmetric policy announcement where $x_i \neq x_j$.
- Without loss of generality, assume $x_i < x_j$.
- If $x_i > 1 - x_j$, i wins the election. j has incentive to deviate (move closer to i).
- Similarly, if $x_i > 1 - x_j$, i loses, and has incentive to deviate (move closer to j).

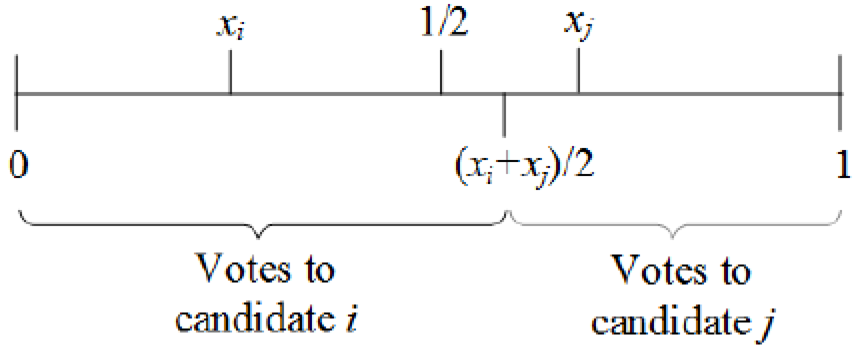


Figure 4.12. Votes to candidate i and j when $x_i < x_j$ and $x_i > 1 - x_j$.

Alternative proof to the electoral competition game

- Now consider $x_i = x_j = x$.
- If $x < \frac{1}{2}$, one candidate would receive more votes, and hence, the other has incentive to deviate.
- Similar argument holds if $x > \frac{1}{2}$.
- However, if $x = \frac{1}{2}$, both players receive the same number of votes and the winner is randomly selected.
- Thus, this is the Nash equilibrium (no incentive to deviate).