Intermediate Microeconomic Theory

Tools and Step-by-Step Examples

Chapter 3: Consumer Choice

Outline

- Budget Constraint
- Utility Maximization Problem (UPM)
- Utility Maximization Problem in Extreme Scenarios
- Revealed Preference
- Kinked Budget Lines
- Appendix A. Lagrange Method to Solve the UPM
- Appendix B. Expenditure Minimization Problem

Budget Constraint

Budget Constraint

- The budget constraint is the set of bundles that the consumer can afford, given the price of each good and her income.
  - *Example*:

    The budget set for good x (food) and y (clothing) is

    $$p_x x + p_y \leq I.$$

    where p_x is the price of each unit of food;

    p_y is the price of each unit of clothing;

    I is the consumer's available income to spend on food and clothing.

Budget Constraint

The budget set says that the total \$ the consumer spends on food, $p_x x$, plus total \$ she spends on clothing, $p_y y$, cannot exceed her available income, I.

If $p_x = \$10$ and $p_y = \$0$, and $I = \$400$, her budget constraint is

$$10x + 20y \leq 400.$$

Budget Constraint

- Bundles (x, y) that satisfy:
  - $p_x x + p_y y < I$
    - the consumer does not use all her income.
  - $p_x x + p_y y = I$
    - the consumer spends all her income.
- We refer to $p_x x + p_y y = I$ as the *budget line*.

Budget Constraint

- Rearranging the budget line and solving for y,

$$p_y y = I - p_x x,$$

$$y = \underbrace{\frac{I}{p_y}}_{\text{Vertical intercept}} - \underbrace{\frac{p_x}{p_y}}_{\text{Slope}} x.$$

- Setting $y = 0$, and solving for x we find the horizontal intercept at

$$p_x x + p_y 0 = I,$$

$$x = \frac{I}{p_x}.$$

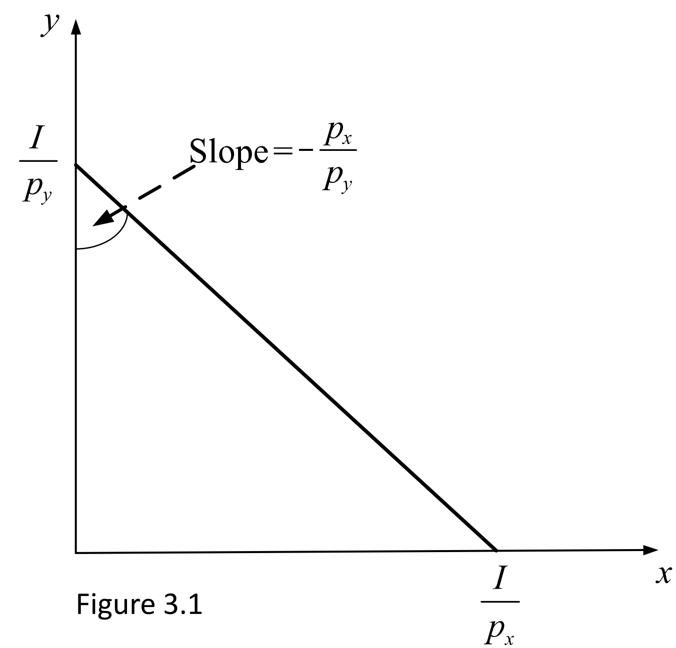

Figure 3.1

Budget Constraint

- The slope of the budget line, tells us how many units of y the consumer must give up to buy 1 more unit of x
- If $p_x = \$10$ and $p_y = \$20$, the slope is

$$-\frac{p_x}{p_y} = -\frac{10}{20} = -\frac{1}{2}.$$

  - The consumer must give up $1/2$ units of good y to acquire 1 more unit of good x, because good y is twice as expensive as good x.
  - Alternatively, she must give up 1 unit of good y to purchase 2 more units of good x.

Budget Constraint

- *Changes in income:*
  - An *increase* in income from I to I', where $I' > I$, shifts the budget line outward in a parallel fashion.
    - As income increase, she can afford a larger set of bundles.

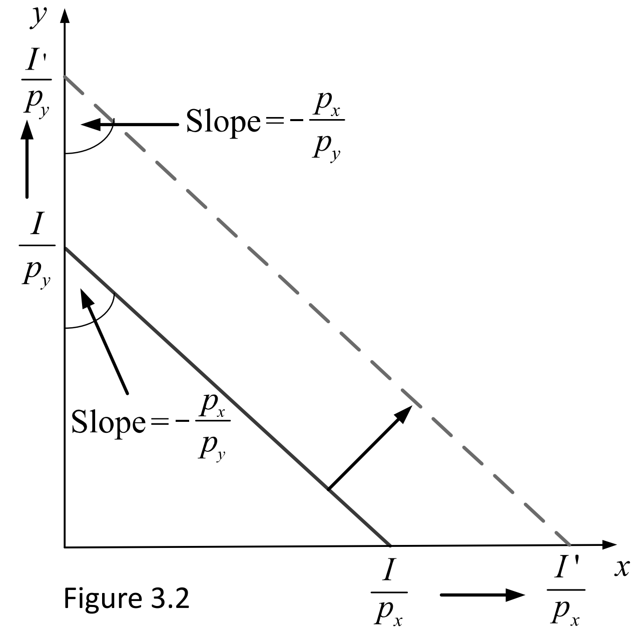

Figure 3.2

Budget Constraint

- *Changes in income* (cont.):
  - A *decrease* in income produces the opposite, a shifting inward in a parallel fashion.

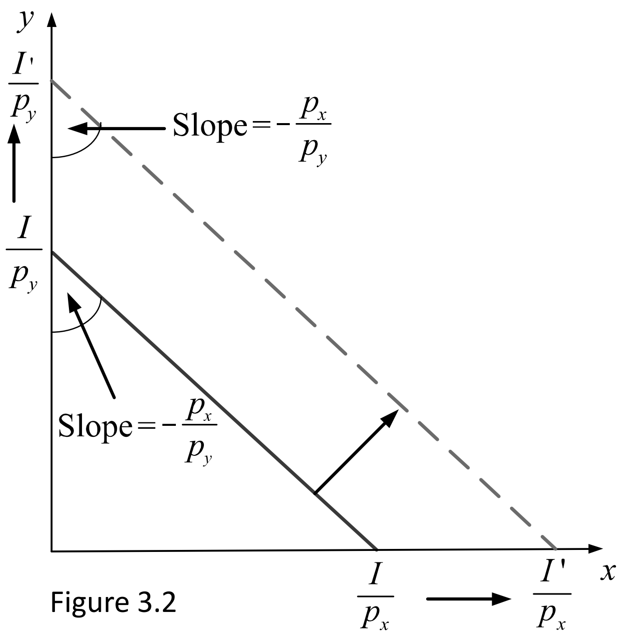

Figure 3.2

Budget Constraint

- *Changes in prices*:
  - An *increase* in the price of one good, such as p_x, pivots the budget line inward.
    - The vertical intercept $\frac{I}{p_y}$ is unaffected.
    - The horizontal intercept $\frac{I}{p_x}$ moves leftward.

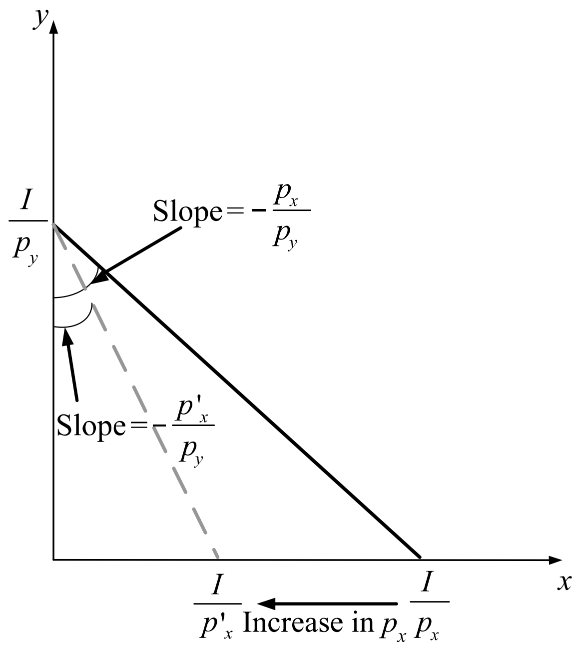

Figure 3.3a

Budget Constraint

- *Changes in prices* (cont.):
  - An *increase* in the price of one good, such as p_x, pivots the budget line inward.
    - The consumer faces a more expensive good, shrinking the set of bundles she can afford.
    - A *decrease* of p_x has the opposite effect, moving the horizontal intercept rightward.

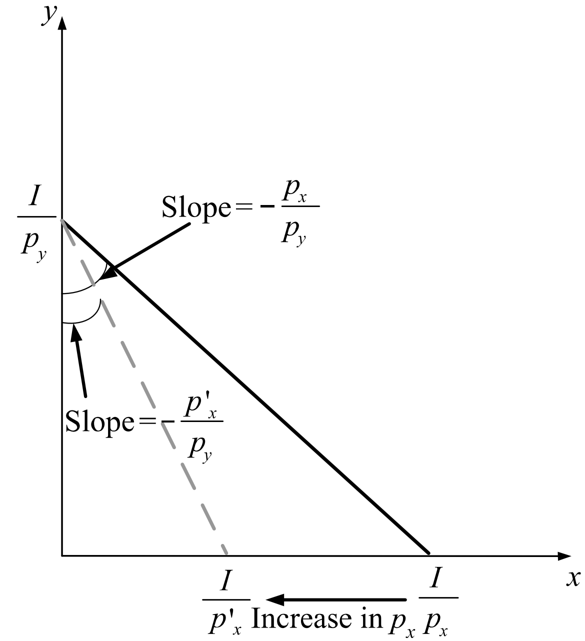

Figure 3.3a

Budget Constraint

- *Changes in prices* (cont.):
  - A similar argument applies if the price of good y, p_y, increases.
    - The horizontal intercept $\frac{I}{p_x}$ is unaffected.
    - The vertical intercept $\frac{I}{p_y}$ moves down.

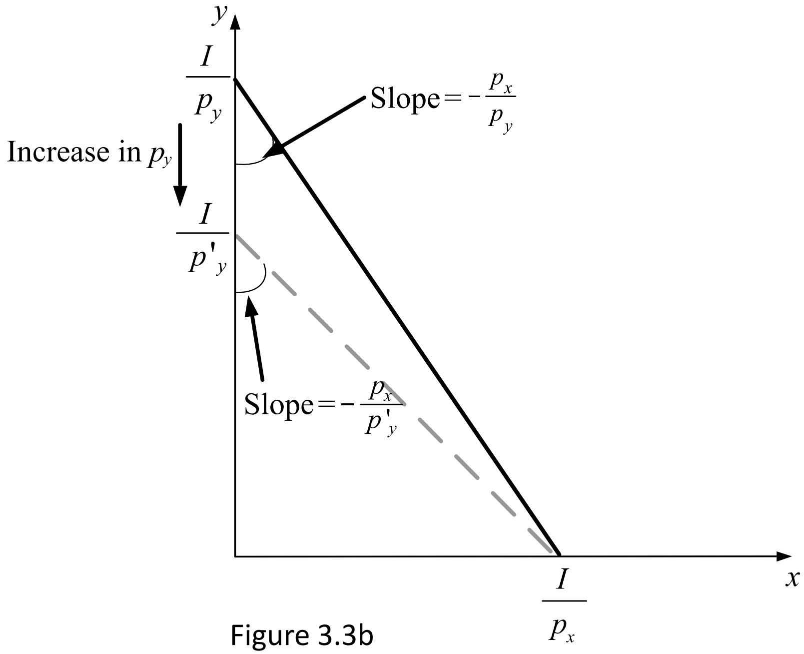

Figure 3.3b

Budget Constraint

- *Changes in prices* (cont.):
  - A similar argument applies if the price of good y, p_y, increases.
    - A decrease in p_y moves the vertical intercept up.

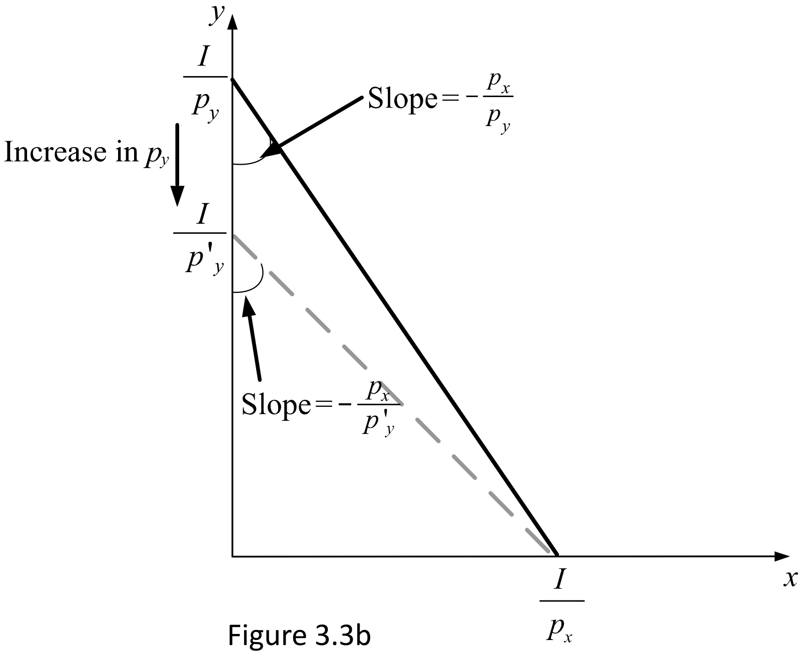

Figure 3.3b

Budget Constraint

- *Query:*

What would happen if both income and the price of all goods were doubled?

- The budget line is unaffected!
  - The vertical intercept of the budget line would become $\frac{2I}{2p_y}$, which simplifies to $\frac{I}{p_y}$ → no change in its position.
  - The horizontal intercept is now $\frac{2I}{2p_x}$, reducing to $\frac{I}{p_x}$.
  - And the slope does not change either, $-\frac{2p_x}{2p_y} = -\frac{p_x}{p_y}$.

This argument applies to any common increase (decrease) in all prices and income.

Utility Maximization Problem

Utility Maximization Problem

- The process by which the consumer chooses utility-maximizing bundles, that are bundles that maximize her utility among all of those she can afford.

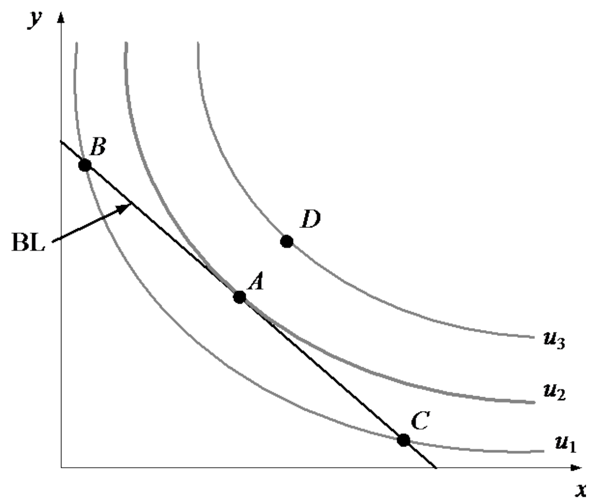

Figure 3.4

- Let's test if points $A - D$ are utility-maximizing for the consumer.

Utility Maximization Problem

- Bundles C and B cannot be optimal. She reaches u_1 spending all her income, $p_x s + p_y = I$. But at bundle A (with same spending) she reaches a higher utility u_2, $u_2 > u_1$.
- Bundle D cannot be optimal. It yields a higher utility than A, but it is unaffordable.
- Only bundle A is optimal, where the budget line and indifference curves are *tangent* each other.

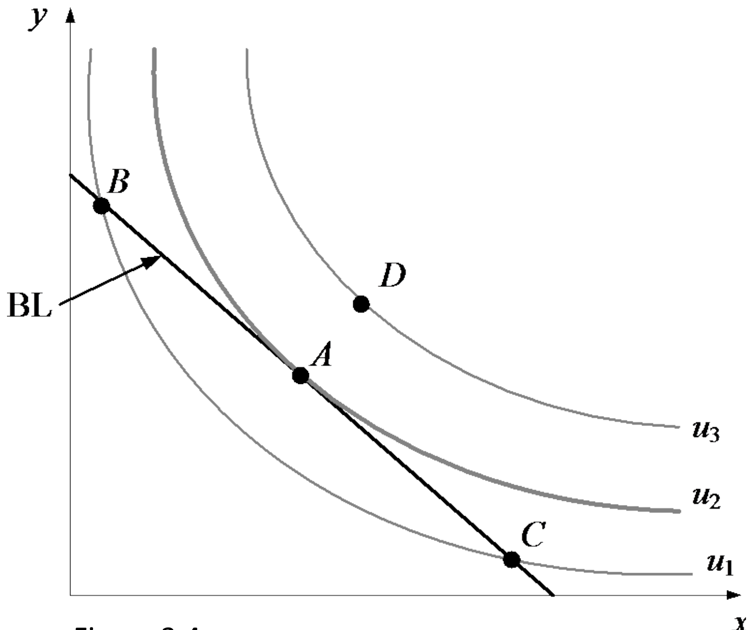

Figure 3.4

Utility Maximization Problem

- This tangency condition requires that the slope of the budget line at bundle A, $\frac{p_x}{p_y}$, is equal to the slope of the indifference curve,

$$MRS = \frac{MU_x}{MU_y}.$$

- Therefore, utility-maximizing bundles must satisfy

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y} \text{ or after rearranging } \frac{MU_x}{p_x} = \frac{MU_y}{p_y}.$$

  - This condition states that marginal utility per dollar spent on the last unit of good x must be equal to that of good y → *bang for the buck* must coincide across all goods.

- If $\frac{MU_x}{p_x} > \frac{MU_y}{p_y}$, the consumer would obtain a larger bang for the buck from x than y, providing incentives to spend more \$ in x.

Utility Maximization Problem

- Tool 3.1. *Procedure to solve the UMP*:
  1. Set the tangency condition as $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$. Cross-multiply and simplify.
  2. If the expression for the tangency condition:
     a. Contains both unknowns (x and y), solve or x, and insert the resulting expression into the budget line $p_x x + p_y y = I$.
     b. Contains only one unknown (x or y), solve for that unknown, and insert the result into the budget line $p_x x + p_y y = I$.

Utility Maximization Problem

- Tool 3.1. *Procedure to solve the UMP* (cont.):
  2. If the expression for the tangency condition:
     c. Contains no good x or y, compare $\frac{MU_x}{p_x}$ against $\frac{MU_y}{p_y}$.
        - If $\frac{MU_x}{p_x} > \frac{MU_y}{p_y}$, set good $y = 0$ in the budget line and solve for good x (corner solution where the consumer purchases only good x).
        - If $\frac{MU_x}{p_x} < \frac{MU_y}{p_y}$, set $x = 0$ in the budget line and solve for y (corner solution where she purchases only good y).

Utility Maximization Problem

- Tool 3.1 *Procedure to solve the UMP* (cont.):
  3. If, in step 2, you find that one of the goods is consumed in negative amounts (e.g., $x = -2$), then set the amount of this good equal to 0 on the budget line (e.g., $p_x 0 + p_y y = I$), and solve for the remaining good.
  4. If you haven't found the values for all the unknowns, use the tangency conditions from step 1 to find the remaining unknown.

Utility Maximization Problem

- *Example 3.1*: *UMP with interior solutions–I.*
  - Consider an individual with Cobb-Douglas utility function
  $$u(x, y) = xy.$$
  facing $p_x = \$20$, $p_y = \$40$, and $I = \$800$.
  - *Step 1*. We use the tangency condition to find optimal bundle
  $$\frac{MU_x}{MU_y} = \frac{p_x}{p_y},$$
  $$\frac{y}{x} = \frac{20}{40} \Longrightarrow \frac{y}{x} = \frac{1}{2},$$
  $$2y = x.$$
  This result contains both x and y, so we move to step 2a.

Utility Maximization Problem

- *Example 3.1* (continued):
  - *Step 2a*. From the budget line, $20x + 40y = 800$.

    Inserting $2y = x$ into the budget line,

$$20\underbrace{(2y)}_{x} + 40y = 800,$$

$$80y = 800,$$

$$y = \frac{800}{80} = 10 \text{ units.}$$

    Because the consumer purchases 10 units of y, we move to step 4 (recall that we only need to stop at step 3 if x or y are negative in step 2).

  - *Step 4*. To find the optimal consumption of x, we use the tangency condition $x = 2y = 2 \times 10 = 20$ units.

Utility Maximization Problem

- *Example 3.1* (continued):
  - *Summary*. The optimal consumption bundle is $(20,10)$. The slope of the indifference curve, $\frac{y}{x} = \frac{10}{20} = \frac{1}{2}$, coincides with that of the budget line, $\frac{p_x}{p_y} = \frac{1}{2}$.

Utility Maximization Problem

- *Example 3.2*: *UMP with interior solutions–II*.
  - Consider an individual with Cobb-Douglas utility function

$$u(x, y) = x^{1/3}y^{2/3}$$

facing $p_x = \$10$, $p_y = \$20$, and $I = \$100$.

Before using the tangency condition $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$, we first find

$$\frac{MU_x}{MU_y} = \frac{\frac{1}{3}x^{\frac{1}{3}-1}y^{\frac{2}{3}}}{\frac{2}{3}x^{\frac{1}{3}}y^{\frac{2}{3}-1}} = \frac{\frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}}{\frac{2}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}} = \frac{y^{\frac{2}{3}+\frac{1}{3}}}{2x^{\frac{1}{3}+\frac{2}{3}}} = \frac{y}{2x}.$$

Utility Maximization Problem

- *Example 3.2* (continued):
  - *Step 1*. We use the tangency condition $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$

$$\frac{y}{2x} = \frac{10}{20},$$

$$y = x.$$

This result contains x and y, so we move to step 2a.

Utility Maximization Problem

- *Example 3.2* (continued):
  - *Step 2a*. Inserting $y = x$ into the budget line,

$$10x + 20y = 100,$$

$$20(y) + 20y = 100,$$

$$30y = 100,$$

$$y = \frac{100}{30} \simeq 3.33 \text{ units.}$$

Utility Maximization Problem

- *Example 3.2* (continued):
  - *Step 4*. The optimal consumption of x can be found by using the tangency condition

    $$y = x \simeq 3.33 \text{ units.}$$

  - *Summary*. The optimal consumption bundle is $(3.33, 3.33)$.

Utility Maximization Problem

- *Example 3.2* (continued):
  - We can find the budget shares of each good, that is the % of income the consumer spends on good x and good y:

$$\frac{p_x x}{I} = \frac{10 \times 3.33}{100} = \frac{1}{3},$$
$$\frac{p_y y}{I} = \frac{20 \times 3.33}{100} = \frac{2}{3}.$$

    which coincides with the exponent of each good in the Cobb-Douglas utility function $u(x, y) = x^{1/3} y^{2/3}$.
  - This result can be generalized to all types of Cobb-Douglas utility functions $u(x, y) = Ax^{\alpha} y^{\beta}$, where $\mathrm{A}, \alpha, \beta > 0$.
    - The budget share of good x is α, and of good y is β.

Utility Maximization Problem

- *Example 3.3*: *UMP with corner solutions*.
  - Consider a consumer with utility function $u(x, y) = xy + 7x$, and facing $p_x = \$1$, $p_y = \$2$, and $I = \$10$.
  - *Step 1*. Using the tangency condition $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$,

$$\frac{y + 7}{x} = \frac{1}{2},$$

$$2y + 14 = x.$$

This result contains x and y, so we move to step 2a.

Utility Maximization Problem

- *Example 3.3* (continued):
  - *Step 2*. Inserting $2y + 14 = x$ into the budget line

$$x + 2y = 10,$$

$$\underbrace{(2y + 14)}_{x} + 2y = 10,$$

$$4y = -4,$$

$$y = -1.$$

Utility Maximization Problem

- *Example 3.3* (continued):
  - *Step 3*. Because the amounts of x and y cannot be negative, the consumer would like to reduce her consumption of good y as much as possible (i.e., $y = 0$). Inserting this result into the budget line

    $$x + (2 \times 0) = 10 \rightarrow x = 10 \text{ units.}$$

  - *Summary*. We have found a corner solution, where the consumer uses all her income to purchase good alone.
  - Graphically, her optimal budget $(x, y) = (10{,}0)$ is located in the horizontal intercept of her budget line.

Utility Maximization Problem

- *Example 3.3* (continued):
  - At the corner solution, the tangency condition does not hold,

$$\frac{MU_x}{p_x} \neq \frac{MU_y}{p_y} \quad \Rightarrow \frac{y+7}{1} \neq \frac{x}{2}.$$

At $(x, y) = (10,0)$, $\frac{0+7}{1} > \frac{10}{2}$.

- $MU_x > MU_y$, inducing the consumer to increase her consumption of x and decrease that of y.
- Once she reaches $y = 0$, she cannot longer decrease her consumption of y.

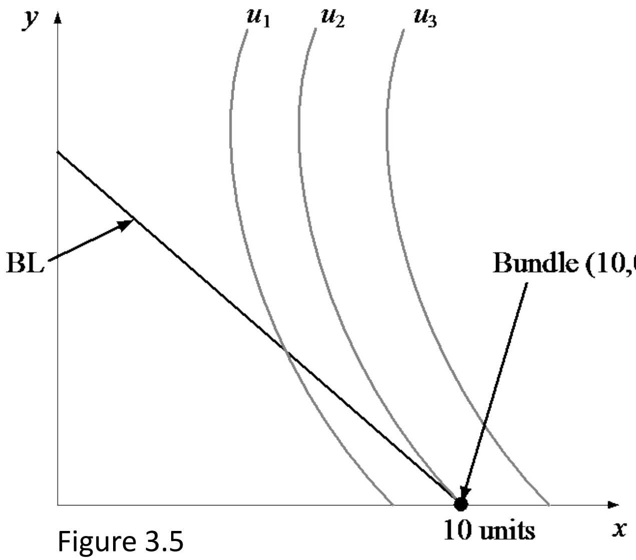

Figure 3.5

UMP in Extreme Scenarios

UMP in Extreme Scenarios

- Goods are regarded as perfect substitutes:
  - Consider two brands of mineral water. This utility function takes the form $u(x, y) = ax + by$, where $a, b > 0$.
  - In this scenario, $\frac{MU_x}{MU_y} = \frac{a}{b}$.
  - Three cases can emerge:
    1. $\frac{a}{b} > \frac{p_x}{p_y}$.
    2. $\frac{a}{b} < \frac{p_x}{p_y}$.
    3. $\frac{a}{b} = \frac{p_x}{p_y}$.

UMP in Extreme Scenarios

- Goods are regarded as perfect substitutes (cont.):

1. If $\frac{a}{b} > \frac{p_x}{p_y}$, the IC is steeper than the budget line, producing a corner solution. The consumer spends all income on x.

   Using the "bang for the buck" approach:

$$\frac{a}{p_x} > \frac{b}{p_y},$$

   the bang for the buck from x is larger than that of y. So she consumer would like to increase her consumption of x while decreasing that of y.

UMP in Extreme Scenarios

- Goods are regarded as perfect substitutes (cont.):

2. If $\frac{a}{b} < \frac{p_x}{p_y}$, a corner solution exists, where the consumer spends all her income on good y.

   The optimal consumption bundle lies on the vertical intercept of the budget line.

UMP in Extreme Scenarios

- Goods are regarded as perfect substitutes (cont.):

3. If $\frac{a}{b} = \frac{p_x}{p_y}$, the slope of the indifference curves and the budget line coincide, yielding a complete overlap.

   Tangency occurs at all points of the budget line → a continuum of solutions exists, any bundle (x, y) satisfying $p_x x + p_y y = I$ is utility maximizing.

UMP in Extreme Scenarios

- Goods are regarded as perfect complements:
  - Consider cars and gasoline. This utility function takes the form $u(x, y) = Amin\{ax, by\}$, where $\mathrm{A}, a, b > 0$.
  - The ICs are L-shaped, and have a kink at a ray from the origin with slope a/b.
  - The MRS of this function is undefined, because the kink could admit any slope.
    - We cannot use the tangency condition as we cannot guarantee that the MRS takes specific numbers for all bundles.
  - Optimal bundles require to identify bundles for which we cannot increase the consumer's utility given her budget constraint.

UMP in Extreme Scenarios

- Goods are regarded as perfect complements (cont.):
  - She consumes the bundle at the kink of her IC where it intersects her budget line.
  - Mathematically, it requires
    - $ax = by \Rightarrow y = \frac{b}{a}x$, for the bundle to be at the kink;
    - $p_x x + p_y y = I$, for the bundle to be on the budget line.
  - We have system of two equations and two unknowns.
  - Inserting the first equation into the second,

$$p_x x + p_y \underbrace{\frac{a}{b}x}_{y} = I \quad \Rightarrow x = \frac{I}{p_x + p_y \frac{a}{b}} = \frac{bI}{bp_x + ap_y}.$$

UMP in Extreme Scenarios

- Goods are regarded as perfect complements (cont.):
  - The optimal amount of y becomes

$$y = \frac{a}{b} + \underbrace{\frac{bI}{bp_x + ap_y}}_{x} = \frac{aI}{bp_x + ap_y}.$$

  - If $a = b = 2$ (when the individual needs to consume the same amount of each good), and $p_x = \$10$, $p_y = \$20$, and $I = \$100$, the optimal consumptions of x and y are

$$x = \frac{bI}{bp_x+ap_y} = \frac{2\times100}{(2\times10)+(2\times20)} = \frac{10}{3} \text{ units,}$$

$$y = \frac{aI}{bp_x+ap_y} = \frac{2\times100}{(2\times10)+(2\times20)} = \frac{10}{3} \text{ units.}$$

Revealed Preferences

Revealed Preference

- Previously, we have analyzed how to find optimal bundles, assuming we observe consumer's preferences represented with her utility function.
- *What if we only know which choices she made when facing different combinations of prices and income?*
- We still can check if the consumer made optimal choices using the Weak Axiom of Revealed Preferences (WARP).

Revealed Preference

- Consider:
  - $A = (x_A, y_A)$ be the optimal bundle when facing initial prices and income (p_x, p_y, I).
  - $B = (x_B, y_B)$ be the optimal bundle when facing final prices and income (p_x', p_y', I').

Revealed Preference

- Weak Axiom of Revealed Preference (WARP). If optimal consumption bundles A and B are both affordable under initial prices and income (p_x, p_y, I), then bundle A cannot be affordable under final prices and income (p_x', p_y', I'):
  - If $p_x x_A + p_y y_A \leq I$ and $p_x x_B + p_y y_B \leq I$,
  - then $p_x' x_A + p_y' y_A > I'$.

Revealed Preference

- Weak Axiom of Revealed Preference (WARP) (cont.).
  - If both bundles are initially affordable, and the consumer selects A, she is "revealing" her preference for A over B.
  - WARP requires A is not affordable under final prices and income, otherwise the consumer should still select the original bundle A.
- Think on WARP as a *consistency* requirement in consumer's choices when facing different prices and incomes.

Revealed Preference

- Tool 3.2. *Checking for WARP*:
  1. *Checking the premise*. Check if bundles A and B are initially affordable → they lie on or below the budget line, BL, (p_x, p_y, I).

     1a. If step 1 holds, move to step 2.

     1b. If step 1 does not hold, stop. We can only claim that the consumer choices *do not violate* WARP.
  2. *Checking the conclusion*. Check that bundle *A* is no longer affordable → it lies strictly above the final budget line BL', (p'_x, p'_y, I').

     2a. If step 2 holds, WARP is *satisfied*.

     2b. If step 2 does not hold, WARP is *violated*.

Revealed Preference

- *Example 3.4: Testing for WARP.*
  - Consider a change in the budget line, from BL to BL', due to a simultaneous decrease in p_x and I.
  - For instance,
    - Initial bundle line BL, $p_x = \$2$, $p_y = 2\$$, and $I = \$100$.
    - Final bundle line BL', $p_x' = \$1$, $p_y = 2\$$, and $I' = \$100$.

Revealed Preference

- *Example 3.4* (continued):
  - The vertical intercept of the budget line decreases from $\frac{I}{p_y} = \frac{10}{2} = 5$ units to $\frac{I'}{p_y} = \frac{7}{2} = 3.5$ units.
  - The vertical intercept of the budget line increases from $\frac{I}{p_x} = \frac{10}{2} = 5$ units to $\frac{I'}{p'_x} = \frac{7}{1} = 7$ units.

Revealed Preference

- *Example 3.4* (continued):
  - Scenario (a). WARP *is satisfied*.

Step 1 holds. Bundles A and B are affordable under BL:

- A lies on BL.
- B lies strictly below BL.

Step 2 holds. Bundle *A* is unaffordable under BL':

- A lies strictly above BL'.

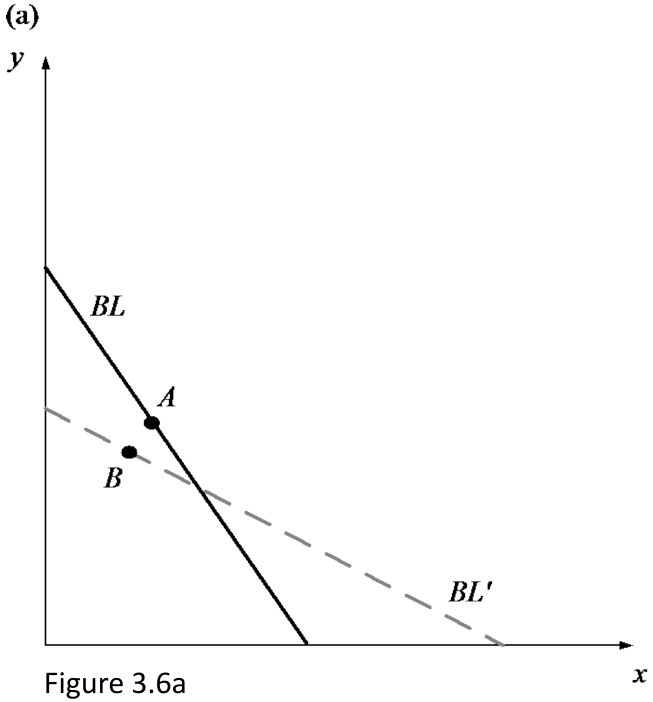

Figure 3.6a

Revealed Preference

- *Example 3.4* (continued):
  - Scenario (b). WARP *is violated*.

<u>Step 1</u> holds. Bundles A and B are affordable under BL:

- A lies on BL.
- B lies strictly below BL.

<u>Step 2</u> does not hold. Bundle *A* is affordable under BL':

- A lies strictly below BL'.

The consumer is not consistent in her choices.

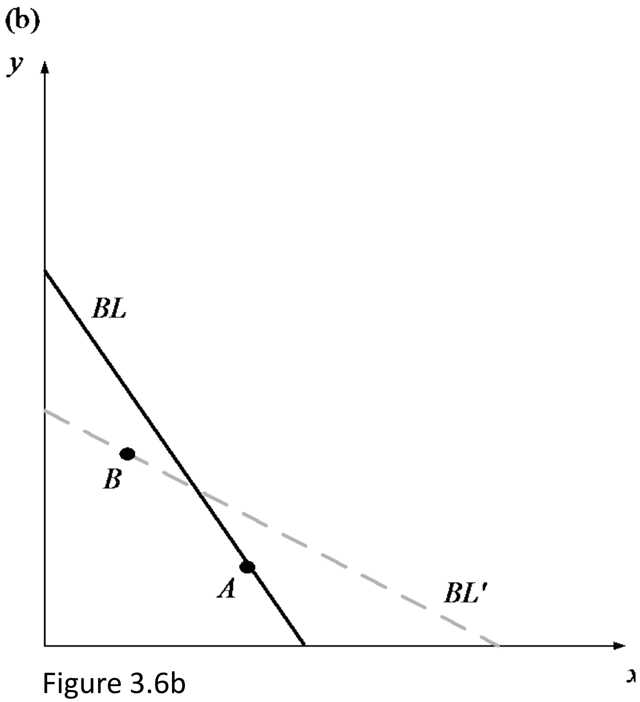

Figure 3.6b

Revealed Preference

- *Example 3.4* (continued):
  - Scenario (c). WARP *is not violated*.

Step 1 does not hold.

Bundle A is affordable under BL but B is unaffordable:

- A lies on BL.
- B lies strictly above BL.

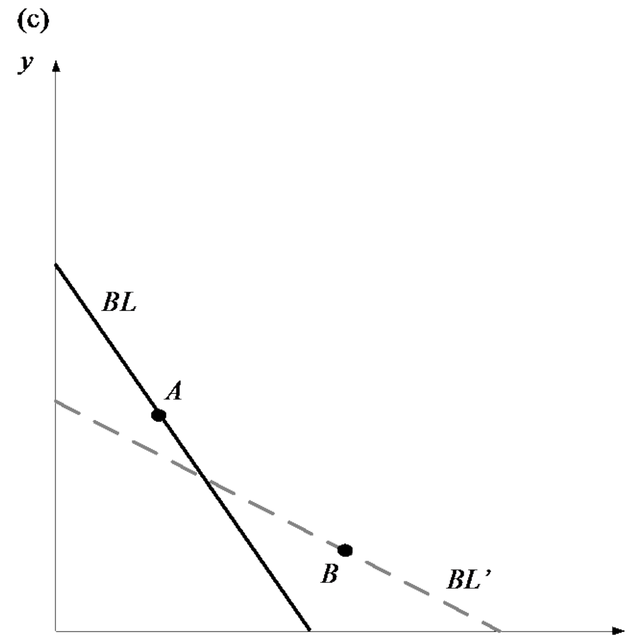

Figure 3.6c

Revealed Preference

- *Example 3.4* (continued):
  - Scenario (d). WARP *is not violated*.

Step 1 does not hold.

Bundle A is affordable under BL but B is unaffordable :

- A lies on BL.
- B lies strictly above BL.

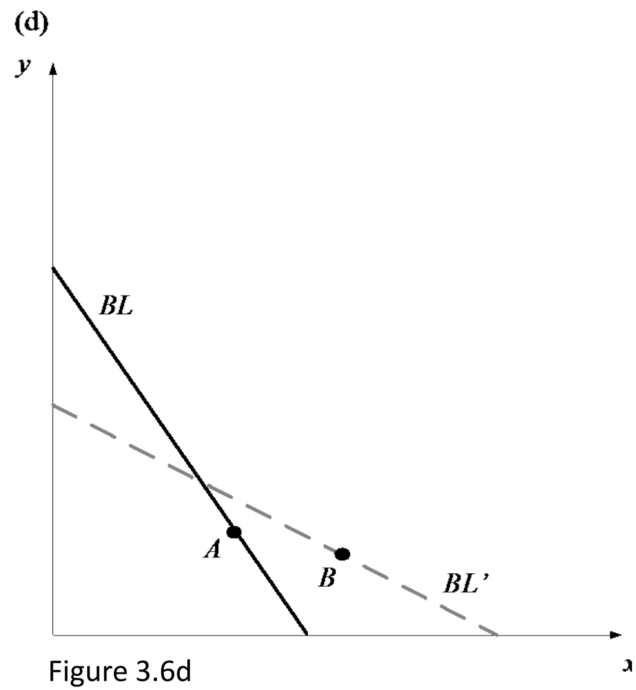

Figure 3.6d

Kinked Budget Lines

Quantity Discounts

- Sellers offer quantity discounts making first units more expensive than each unit afterwards.

  Formally,

  - the consumer faces a price p_x for all units of x between 0 and $\bar{x}$ (i.e., for all $x \leq \bar{x}$);
  - but she faces a lower price p_x', where $p_x' < p_x$, for each subsequent unit (i.e., for all $x > \bar{x}$).

Quantity Discounts

- Graphically,

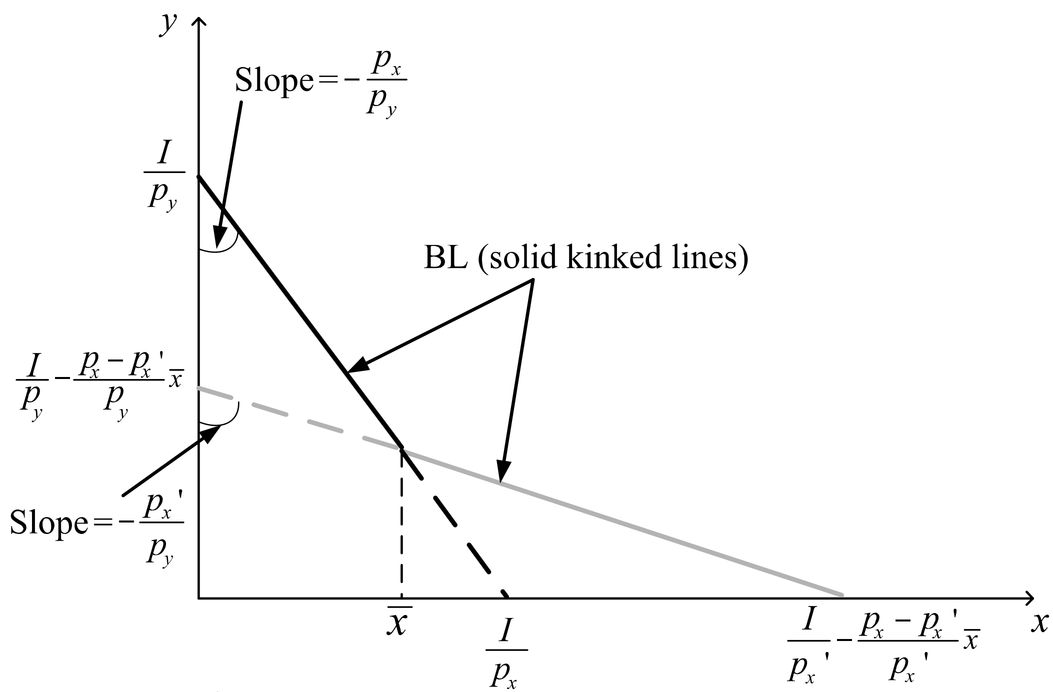

Figure 3.7

Quantity Discounts

- Mathematically, the equation of the budget line is
  - For all $x \le \bar{x}$,

$$y = \underbrace{\frac{I}{p_y}}_{\text{Vertical intercept}} \underbrace{-\frac{p_x}{p_y}}_{\text{Slope}} x.$$

  - For all $x > \bar{x}$,

$$y = \underbrace{\left(\frac{I}{p_y} - \frac{p_x - p_x'}{p_y}\bar{x}\right)}_{\text{Vertical intercept}} \underbrace{-\frac{p_x'}{p_y}}_{\text{Slope}} x.$$

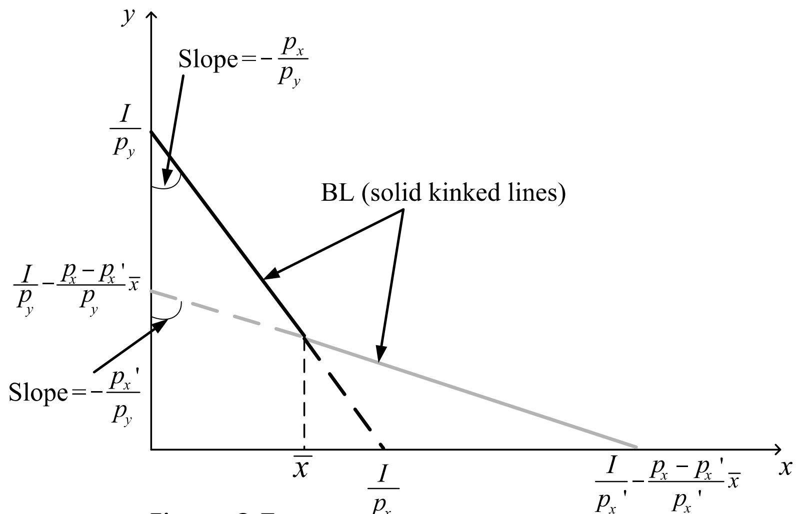

Figure 3.7

Note $\frac{p_x'}{p_y} < \frac{p_x}{p_y}$, and $\frac{I}{p_y} - \frac{p_x - p_x'}{p_y}\bar{x} < \frac{I}{p_y}$.

Quantity Discounts

- Effect of a large or small price discount:
  - A large discount makes the difference $p_x - p_x'$ larger, shifting the vertical intercept downward and flattering the right segment of the budget line.
  - A small discount produces a small difference $p_x - p_x'$, pushing the vertical intercept upward and steepening the right segment of the budget line.

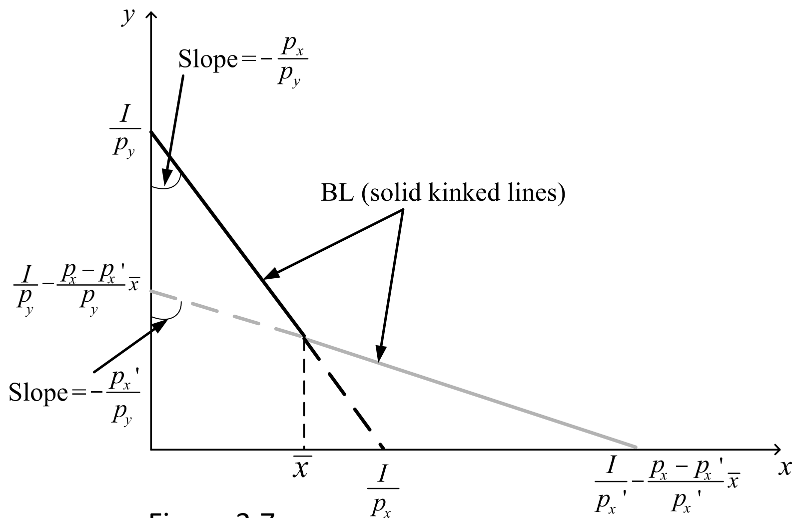

Figure 3.7

Quantity Discounts

- *Example 3.5:* *Quantity discounts*.
  - Eric has $I = \$100$ to purchase video games (good x) and food (good y).
  - The price of food is $p_y = \$5$, regardless of how many units he buys.
  - The price of video games is $p_x = \$4$ for the first 2 units, but $p_x' = \$1$ for unit 3 and beyond.
  - Cutoff is at $\bar{x} = 2$.

Quantity Discounts

- *Example 3.5* (continued):
  - Then, Eric's budget line is:
    - For all $x \leq 2$,

$$y = \frac{100}{5} - \frac{4}{5}x$$
$$= 20 - \frac{4}{5}x.$$

    - For all $x > 2$,

$$y = \left(\frac{100}{5} - \frac{4-1}{5}2\right) - \frac{1}{5}x$$
$$= \left(20 - \frac{3}{5}2\right) - \frac{1}{5}x$$
$$= \frac{94}{5} - \frac{1}{5}x.$$

Quantity Discounts

- *Example 3.5* (continued):
  - Graphically,
    - For $x \leq 2$, the budget line originates at $\frac{I}{p_y} = \frac{100}{5} = 20$ units in the vertical axis and decreases at a rate of $-\frac{p_x}{p_y} = -\frac{4}{5} = -0.8$.
    - For $x > 2$, the budget line originates at $y = \frac{94}{5} \cong 18.8$ units, has a slope of $-\frac{p_x'}{p_y} = -\frac{1}{5}$, becoming flatter, and cross the horizontal axis at $x = \frac{I}{p_x'} - \frac{p_x - p_x'}{p_x'}\bar{x} = \frac{100}{1} - \left(\frac{(4-1)}{1} \times 2\right) = 100 - 6 = 94$ units.

Introducing Coupons

- Consider a market where the government offers coupons, letting consumers purchase the first $\bar{x}$ units of good x for free.

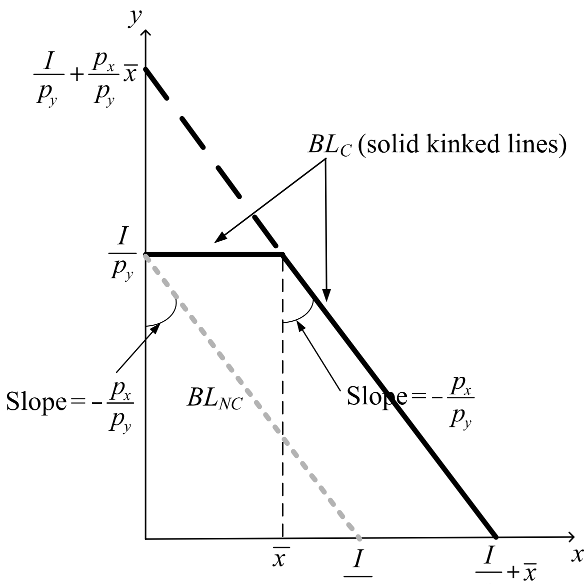

Figure 3.8

BL_C, budget line with coupons.

BL_{NC}, budget line without coupons.

- The coupons expand the set of bundles the consumer can afford.

Introducing Coupons

- Mathematically, this kinked budget line BL_C is

$$BL_C \begin{cases} p_y y = I \text{ for all } x < \bar{x}, \text{ and} \\ p_x(x - \bar{x}) + p_y y = I \text{ for all } x \geq \bar{x}. \end{cases}$$

  - For $x < \bar{x}$, the consumer faces $p_x = \$0$, thanks to the coupons. Then BL_C is $p_y y + 0x = I \quad \Rightarrow p_y y = I$.
  - For $x \geq \bar{x}$, the consumer exhausted all coupons and faces market prices p_x and p_y. Then, BL_C becomes $p_x(x - \bar{x}) + p_y y = I$.

Introducing Coupons

- Solving for y, we can represent BL_C as
  - For $x < \bar{x}$, $y = \frac{I}{p_y}$
  - For $x \geq \bar{x}$, $y = \frac{I}{p_y} + \frac{p_x}{p_y}(x - \bar{x})$

or

$$y = \underbrace{\left(\frac{I}{p_y} + \frac{p_x}{p_y}\bar{x}\right)}_{\text{Vertical intercept}} - \underbrace{\frac{p_x}{p_y}}_{\text{Slope}} x$$

Setting $y = 0$, and solving for x, we find the horizontal intercept at $x = \frac{I}{p_x} + \bar{x}$.

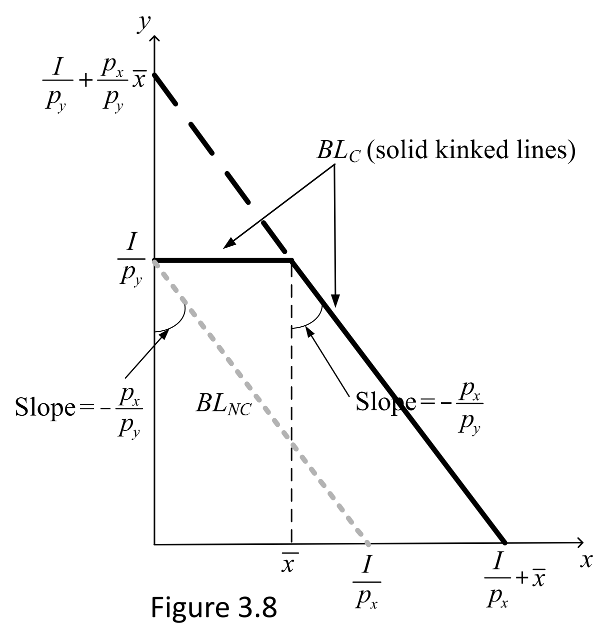

Figure 3.8

Introducing Coupons

- Example 3.6: *Coupons*.
  - John income is $I = \$100$, the price of electricity is $p_x = \$1$, and the price of bikes is $p_y = \$4$.
  - The government agency distributes coupons for the first 200 kWh per month, making them free.
  - Because $\bar{x} = 200$, John's budget line BL_C is
    - For $x < 200$, $y = \frac{I}{p_y} = \frac{100}{4} = 25$ units.
    - For $x \geq 200$, $y = \left(\frac{I}{p_y} + \frac{p_x}{p_y}\bar{x}\right) - \frac{p_x}{p_y}x = \left(\frac{100}{4} + \frac{1}{4}200\right) - \frac{1}{4}x = 75 - \frac{1}{4}x$.

Introducing Coupons

- *Example 3.6* (continued):
  - Graphically, the dashed segment of the BL_C
    - originates at $y = 75$,
    - decreases at a rate of $\frac{1}{4}$,
    - and hits the horizontal axis at

$$x = \frac{I}{p_x} + \bar{x} = \frac{100}{1} + 200 = 300 \text{ units.}$$

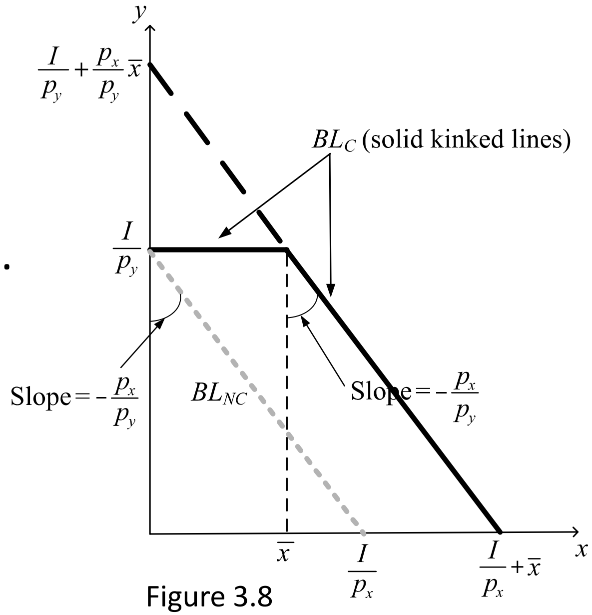

Figure 3.8

Appendix A.
Lagrange Method to Solve UMP

A. Lagrange Method to Solve UMP

- We have used the tangency condition $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$ to find optimal consumption bundles.
- Now, we show that this condition must be satisfied at the optimum of the UMP. The UMP can be expressed as

$$\max_{x,y} u(x, y)$$

$$\text{subject to } p_x x + p_y y = I.$$

  - We use the budget line $p_x x + p_y y = I$, rather than the budget constraint $p_x x + p_y y \leq I$, because the consumer will always spend all her available income.
  - The consumer faces a "constrained maximization problem."

A. Lagrange Method to Solve UMP

- Constrained maximization problems are often solved by setting up a Lagrangian function,

$$\mathcal{L}(x, y; \lambda) = u(x, y) + \lambda[I - p_x x - p_y y],$$

where λ represents the Lagrange multiplier, which multiplies the budget line.

- To solve this problem, we take FOP with respect to x, y, and λ,

$$\frac{\partial \mathcal{L}}{\partial x} = MU_x - \lambda p_x = 0,$$

$$\frac{\partial \mathcal{L}}{\partial y} = MU_y - \lambda p_y = 0, \text{ and}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0.$$

A. Lagrange Method to Solve UMP

- The first and the second conditions can be rearranged to

$$\frac{MU_x}{p_x} = \lambda \text{ and } \frac{MU_y}{p_y} = \lambda.$$

- Because both conditions are equal to λ, we obtain

$$\frac{MU_x}{p_x} = \lambda = \frac{MU_y}{p_y}$$

  This is the "bang for the buck" coinciding across goods.

- Alternatively, this condition can be expressed as

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y},$$

  which coincides with the tangency condition used in the previous analysis.

Appendix B.
Expenditure Minimization Problem

Expenditure Minimization Problem

- The UMP considers a fixed budget and finds which bundle provides the consumer with the highest utility.
- Alternatively, the consumer could minimize her expenditure while reaching a fixed utility level.
- This is the approach that the expenditure minimization problem (EMP) follows.

Expenditure Minimization Problem

- Graphically, the EMP is understood as the consumer seeking to reach an IC with a target utility level $\bar{u}$, but shifting her budget line as close to the origin as possible.
  - Bundles B or C cannot be optimal despite reaching $\bar{u}$. She spends more income than in A.
  - Bundle D cannot be optimal. She can find cheaper bundles and reach $\bar{u}$.

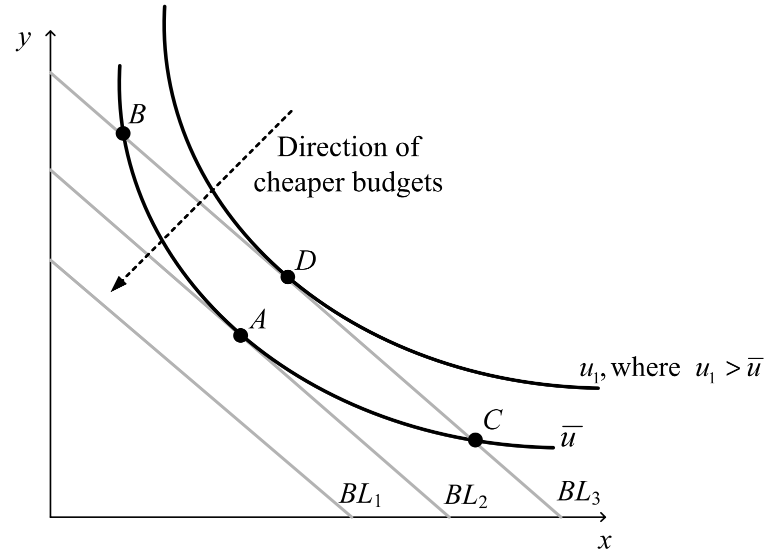

Figure 3.9

Expenditure Minimization Problem

- Bundle A must be optimal. There are no other bundles reaching at a lower expenditure than BL_2.

  At A, the indifference curve and the budget line are tangent,

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}.$$

  Her constraint is $u(x, y) = \bar{u}$, rather than $u(x, y) \geq \bar{u}$. She would never choose bundle satisfying $u(x, y) > \bar{u}$, such as D.

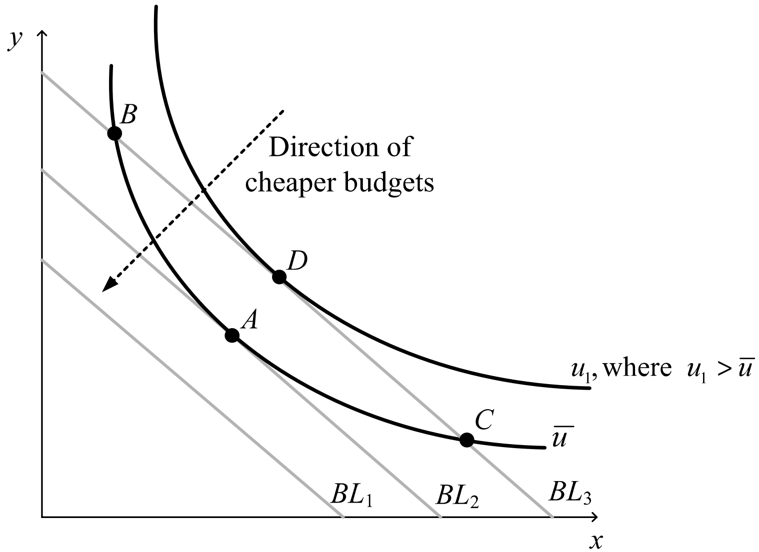

Figure 3.9

Expenditure Minimization Problem

- Bundle D cannot be optimal. She can find cheaper bundles and reach $\bar{u}$. These bundles that still satisfy the constraint and can be purchased at lower cost.

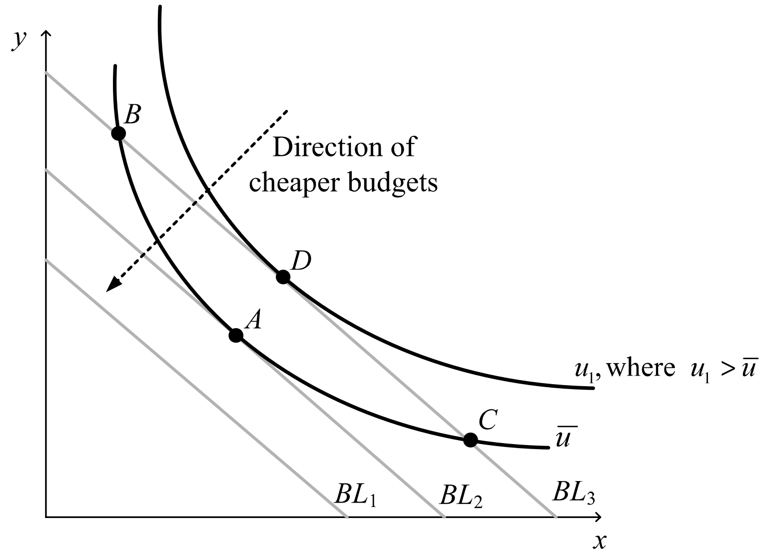

Figure 3.9

Expenditure Minimization Problem

- Tool 3.3. *Procedure to solve the EMP*:
  1. Set the tangency condition as $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$. Cross-multiply and simplify.
  2. If the expression for the tangency condition:
     a. Contains both unknowns (x and y), solve or y, and insert the resulting expression into the utility constraint $u(x, y) = \bar{u}$.
     b. Contains only one unknown (x or y), solve for that unknown, and insert the result into the utility constraint $u(x, y) = \bar{u}$.

Expenditure Minimization Problem

- Tool 3.3. *Procedure to solve the EMP* (cont.):

  2. If the expression for the tangency condition:

     c. Contains no good x or y, compare $\frac{MU_x}{p_x}$ against $\frac{MU_y}{p_y}$.

     - If $\frac{MU_x}{p_x} > \frac{MU_y}{p_y}$, set good $y = 0$ in the utility constraint and solve for good x
     - If $\frac{MU_x}{p_x} < \frac{MU_y}{p_y}$, set $x = 0$ in the utility constraint and solve for y.

Expenditure Minimization Problem

- Tool 3.3. *Procedure to solve the EMP* (cont.):

  3. If, in step 2, you find that one of the goods is consumed in negative amounts (e.g., $x = -2$), then set the amount of this good equal to 0 on the utility constraint (e.g., $u(0, y) = \bar{u}$), and solve for the remaining good.

  4. If you haven't found the values for all the unknowns, use the tangency conditions from step 1 to find the remaining unknown.

Expenditure Minimization Problem

- *Example 3.7*: *EMP with a Cobb-Douglas utility function*.
  - Consider an individual with Cobb-Douglas utility function
    $$u(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}},$$
    facing $p_x = \$10$, $p_y = \$20$, and a utility target $\bar{u}$.
  - We seek to apply the tangency condition, $\frac{MU_x}{Mu_y} = \frac{p_x}{p_y}$. We first need to find $\frac{MU_x}{Mu_y}$,
    $$\frac{MU_x}{Mu_y} = \frac{\frac{1}{3} x^{-\frac{2}{3}} y^{\frac{2}{3}}}{\frac{2}{3} x^{\frac{1}{3}} y^{-\frac{1}{3}}} = \frac{y}{2x}.$$
    Next, we apply the steps in Tool 3.3.

Expenditure Minimization Problem

- *Example 3.7* (continued):
  - *Step 1*. The tangency condition reduces to

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y},$$

$$\frac{y}{2x} = \frac{10}{20} \quad \Longrightarrow y = x.$$

  This result contains both x and y, so we move to step 2a.

  - *Step 2a.* The utility constraint $u(x, y) = \bar{u}$ becomes $x^{\frac{1}{3}}y^{\frac{2}{3}} = \bar{u}$. Inserting $y = x$,

$$x^{\frac{1}{3}}\underbrace{(x)}_{y}^{\frac{2}{3}} = \bar{u} \quad \Longrightarrow x = \bar{u}.$$

  For instance, if $\bar{u} = 5$, the optimal amount of x is $x = 5$.

Expenditure Minimization Problem

- *Example 3.7* (continued):

  Because we found a positive amount of good x, we move to step 4.

  - *Step 4*. Using the tangency condition, $y = x$,

    $$y = \bar{u}.$$

  - *Summary*. The optimal consumption bundle is $x = y = \bar{u}$, consuming the same amount of each.

    For instance, if the consumer seeks to reach a utility target of $\bar{u}$, the optima bundle is $(5,5)$.

Expenditure Minimization Problem

- *Example 3.8*: *EMP with a quasilinear utility*.
  - Consider the quasilinear utility from example 3.3
    $$u(x, y) = xy + 7x,$$
    facing $p_x = \$1, p_y = \$2$, and a utility target $\bar{u} = 70$.
  - *Step 1*. The tangency condition reduces is
    $$\frac{MU_x}{MU_y} = \frac{p_x}{p_y},$$
    $$\frac{y + 7}{x} = \frac{1}{2} \quad \Longrightarrow 2y + 14 = x.$$
    This result contains both x and y, so we move to step 2a.

Expenditure Minimization Problem

- *Example 3.9* (continued):
  - *Step 2a.* Inserting the result from the tangency condition, $2y + 14 = x$, into the utility target $xy + 7x = 70$,

$$\underbrace{(2y+14)}_{x}y + 7\underbrace{(2y+14)}_{x} = 70,$$

$$2(7+y)^2 = 70 \Longrightarrow (7+y)^2 = 35,$$

$$\sqrt{(7+y)^2} = \sqrt{35} \Longrightarrow 7 + y = \sqrt{35},$$

$$y \simeq -1.08 \text{ units.}$$

Because we found negative units of at least one good, we need to apply step 3 next.

Expenditure Minimization Problem

- *Example 3.9* (continued):
  - *Step 3.* The individual consumes 0 amounts of y, and dedicates all her income to buy x. $MU_x > MU_y$, regardless of the amount consumed, which drives her to purchase only good x.

    Because $y = 0$, her utility constraint becomes $u(x, 0) = 70$, or

    $$x0 + 7x = 70,$$

    $$x = 10 \text{ units.}$$

  - *Summary.* The optimal consumption bundle is $x = 10$ and $y = 0$, regardless of the utility target the individual seeks to reach.

Relationship between UMP and EMP

- Similarities and differences of UMP and EMP approaches:

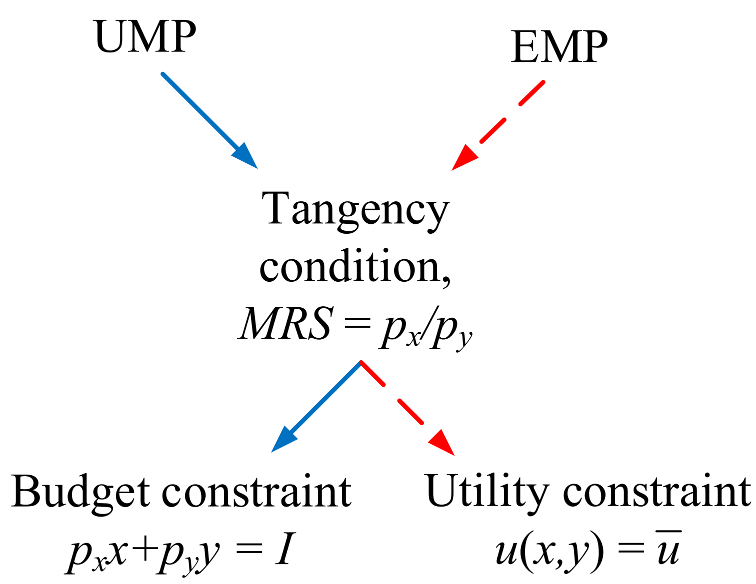

Figure 3.10

Relationship between UMP and EMP

- Both approaches lead as to the same optimal consumption bundle. The EMP is dual representation of UMP.
- Consider a consumer that solves her UMP and finds optimal bundle

$$(x^U, y^U).$$

- In this situation, the utility she can reach when purchasing this bundle is

$$u(x^U, y^U).$$

Relationship between UMP and EMP

- If we ask the consumer to solver her EMP to reach a target utility level of

$$u(x^U, y^U) = \bar{u},$$

  the bundle that solves her EMP coincides with that of UMP.

- We can draw the opposite relationship, starting from EMP.
- Let (x^E, y^E) be the optimal bundle solving EMP.

Relationship between UMP and EMP

- Let I^E be the income the consumer needs to purchase her optimal bundle (i.e., $p_x x^E + p_y y^E = I^E$).
- If we ask her to solve her UMP, giving an income of $I = I^E$, the optimal bundles solving her UMP,

$$(x^U, y^U),$$

coincides with that solving her EMP,

$$(x^E, y^E).$$

Relationship between UMP and EMP

- *Example 3.9:* *Dual problems.*

  From UMP to EMP:

  - Solving the UMP in example 3.2, $(x^U, y^U) = (3.33, 3.33)$, which yields a utility level of $u = 3.33$.
  - If we go to the EMP in example 3.7, and her to a target of a utility level of $\bar{u} = 3.33$.

    Then, her optimal bundle becomes
    $$(x^E, y^E) = (3.33, 3.33),$$
    because in example 3.7 we found $x = y = \bar{u}$.
  - Hence, optimal bundles in UMP and EMP coincide.

Relationship between UMP and EMP

- *Example 3.9* (continued):

  From EMP to UMP:

  - We approach the consumer again, giving her the income that she would need to purchase the optimal bundle found in EMP of example 3.7,

  $$p_x x^E + p_y y^E = \$100.$$

  - Solving her UMP, she obtains

  $$(x^U, y^U) = (3.33, 3.33),$$

  which coincides with the optimal bundle solving the EMP.