Intermediate Microeconomic Theory

Tools and Step-by-Step Examples

Chapter 15:

Games of Incomplete Information and Auctions

Outline

- Incomplete Information
- Extending NE to Games of Incomplete Information
- Auctions
- Second-Price Auctions
- First-Price Auctions
- Efficiency in Auctions
- Common Value Auctions
- A Look at Behavioral Economics–Experiments with Auctions
- Appendix. First-Price Auctions in More General Settings

Incomplete Information

Incomplete Information

- So far, we have learned how to predict equilibrium behavior with 2 tools:
  - Nash equilibrium (NE) solution concept, with the help of best responses.
  - Subgame perfect equilibrium (SPE) concept, by applying backward induction.
- We have explored games of complete information: every player could perfectly predict her opponent's payoff in every contingency.
- However, many strategic settings in real life involve elements of incomplete information.

Incomplete Information

- *Examples*:
  - Firms can observe their own production costs, but do not perfectly observe their rivals' costs.
  - An incumbent firm may have reliable information about market demand, while a new entrant has limited information.
  - Bidders in auctions know how much they are willing to pay for the object being sold, but usually cannot observe other bidders' valuation.
- In these scenarios, players need to compare payoffs in *expectation*.

Extending NE to Games of Incomplete Information

Games of Incomplete Information

- About notation:
  1. A player's "type" is used to represent her private information.
     - With 2 firms privately observing their costs, every firm i's type is its production cost, high c_H or low c_L, where $c_H > c_L \geq 0$.
     - In auctions, a bidder's type denotes her valuation for the object being sold, $v > 0$.
  2. The strategies of player i are expressed as a function of her type.
     - With 2 firms privately informed, a production strategy specifies how many units firm i produces as a function of its costs.
     - In auctions, a bidding strategy specifies how much player i bids as a function of her valuation of the object, $b_i(v)$.

Games of Incomplete Information

- Best response. Player i regards strategy s_i as a "best response" to her rival's strategy s_j if s_i yields a weakly higher *expected* payoff than any other available strategy s_i' against s_j.
  - We are considering expected payoffs.
  - Consider the example of 2 firms:
    - Firm i observes its own production cost, c_H, but does not observe that of its rival.
    - A production strategy $q_i(c_H)$ is its best response to its rival j's output level if $q_i(c_H)$ yields a higher expected profit than any other different production.
    - Firm i must have an optimal production strategy for each of its possible types (e.g., costs).

Games of Incomplete Information

- Bayesian Nash Equilibrium (BNE). A strategy profile (s_i^*, s_j^*) is Bayesian Nash equilibrium if every player chooses a best response (evaluated in expectation) given her rivals' strategies.
  - Players select mutual best responses to each other's strategies, where best responses are "lists" specifying which strategy a player chooses for each of her possible types.

Games of Incomplete Information

- *Example 15.1: Cournot competition, with asymmetric information about costs.*
  - Consider a duopoly game where 2 firms compete on quantities and face inverse demand $p = 1 - q_1 - q_2$.
  - Firm 1 is an incumbent with $MC_1 = 0$, which every firm can accurately estimate.
  - Firm 2 privately observes its marginal costs, which can be low, $MC_2 = 0$, or high, $MC_2 = 1/4$.
  - Because firm 2 is newcomer, firm 1 cannot accurately observes firm 2’s costs, but it assigns equal probability to firm 2 having low and high costs.

Games of Incomplete Information

- *Example 15.1* (continued):
  - *Firm 2's best response.*
    - When firm 2 has low costs ($MC_2 = 0$), its PMP is
$$\max_{q_2^L \geq 0} \pi_2^L = (1 - q_1 - q_2^L)q_2^L \,.$$
    - Differentiating with respect to q_2^L, and solving for q_2^L,
$$1 - q_1 - 2q_2^L = 0 \quad \Longrightarrow q_2^L(q_1) = \frac{1}{2} - \frac{1}{2}q_1. \qquad (BRF_2^L(q_1))$$
    - When firm 2 has high costs ($MC_2 = 1/4$), its PMP is
$$\max_{q_2^H \geq 0} \pi_2^H = (1 - q_1 - q_2^H)q_2^H - \frac{1}{4}q_2^H \,.$$
    - Differentiating and solving for q_2^H,
$$1 - q_1 - 2q_2^H - \frac{1}{4} = 0 \quad \Longrightarrow q_2^H(q_1) = \frac{3}{8} - \frac{1}{2}q_1. \qquad (BRF_2^H(q_1))$$

Games of Incomplete Information

- *Example 15.1* (continued):
  - Comparing the best response function under low and high costs, for a given output level of firm 1,
$$q_2^L(q_1) > q_2^H(q_1).$$
Graphically, $q_2^L(q_1)$ and $q_2^H(q_1)$ are parallel to each other, but $q_2^L(q_1)$ originates at $\frac{1}{2}$, while $q_2^H(q_1)$ originates at $\frac{3}{8} \cong 0.375$.
- *Firm 1.* Firm 1 (uninformed player) seeks to maximize its expected profits because it does not observe firm 2's costs.
  - Firm 1's PMP is

$$\max_{q_1 \geq 0} \pi_1 = \underbrace{\frac{1}{2}(1 - q_1 - q_2^L)q_1}_{\text{if firm 2 has low costs}} + \underbrace{\frac{1}{2}(1 - q_1 - q_2^H)q_1}_{\text{if firm 2 has high costs}} = \left(1 - q_1 - \frac{q_2^L}{2} - \frac{q_2^H}{2}\right)q_1.$$

Games of Incomplete Information

- *Example 15.1* (continued):
    - Differentiating with respect to q_1, and solving for q_1,

$$1 - 2q_1 - \frac{q_2^L}{2} - \frac{q_2^H}{2} = 0,$$

$$q_1(q_2^L, q_2^H) = \frac{1}{2} - \frac{1}{4}q_2^L - \frac{1}{4}q_2^H. \qquad (BRF_1^H(q_2^L, q_2^H))$$

- We found 3 best response functions, which can be solved to obtain the 3 unknown output levels, q_1, q_2^L, and q_2^H.
    - Inserting $q_2^L(q_1)$ and $q_2^H(q_1)$ into $q_1(q_2^L, q_2^H)$, and solving for q_1,

$$q_1 = \frac{1}{2} - \underbrace{\frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}q_1\right)}_{q_2^L(q_1)} - \underbrace{\frac{1}{4}\left(\frac{3}{8} - \frac{1}{2}1_1\right)}_{q_2^H(q_1)},$$

$$q_1 = \frac{9 + 8q_1}{32} \Longrightarrow q_1 = \frac{3}{8}.$$

Games of Incomplete Information

- *Example 15.1* (continued):
  - Inserting this result into firm 2's best response function, first when having low cost,

$$q_2^L\left(\frac{3}{8}\right) = \frac{1}{2} - \frac{1}{2}\frac{3}{8} = \frac{5}{16},$$

  and then when having high costs,

$$q_2^H\left(\frac{3}{8}\right) = \frac{3}{2} - \frac{1}{2}\frac{3}{8} = \frac{3}{16}.$$

- Therefore, the BNE of this duopoly game with incomplete information prescribes production levels

$$(q_1, q_2^L, q_2^H) = \left(\frac{3}{8}, \frac{5}{16}, \frac{3}{16}\right).$$

Auctions

Auctions

- Auctions are a larger part of the economic landscape:
  - Since Babylon in 500 b.c. and during the Roman Empire, in 193 a.c.
  - 1595 the *Oxford English Dictionary* first included the term auction.
  - Auction houses Sotheby's and Christie's founded in 1744 and 1766.
  - Websites such as eBay, with $\$9$ billion in total revenue in 2017 and thousands of employees worldwide, and QuiBids.
  - Also used by governments to sell:
    - Treasure bonds.
    - Airwaves (3G and 4G technology): British 3G telecom licenses generated $\$34$ billion the so-called "the biggest auction ever".

Auctions

- Consider N bidders, each bidder i has a valuation v_i for an object.
- There is one seller.
- We can design many different rules for the auction:

  1. *Firs-price auction (FPA).* The winner is the bidder submitting the highest bid, and she must pay the highest bid (which is hers).
  2. *Second-price auction (SPA).* The winner is the bidder submitting the highest bid, and she must pay the second-highest bid.
  3. *Third-price auction.* The winner is the bidder submitting the highest bid, but she must pay the third-highest bid.
  4. *All-pay auction.* The winner is the bidder submitting the highest bid, but every single bidder must pay the price she submitted.

Auctions

- All auctions can be interpreted as allocation mechanisms with 2 main ingredients:
  1. An *allocation rule* ("who gets the object"):
     - The allocation rule for most auctions determines that the object is allocated to the bidder submitting the highest bid.
     - The object could be assigned through a lottery, where $prob(win) = \frac{b_1}{b_1+b_2+\cdots+b_N}$, as in Chinese auctions.
  2. A *payment rule* ("how much each bidder pays"):
     - In FPA, the individual submitting the highest bid pays here own bid, while everybody else pays zero.
     - In SPA, the individual submitting the highest bid pays the second-highest bid, and everybody else pays zero.
     - In all-pay auction, every individual must pay the bid she submitted.

Second-Price Auctions

Second-Price Auctions

- Bidding your own valuation, $b_i(v_i)$, is a weakly dominant strategy for all players.
  - Submitting a bid equal to your valuation, $b_i(v_i) = v_i$, yields an expected profit equal or higher than that of submitting any other bid, $b_i(v_i) \neq v_i$.
- To show this bidding strategy is an equilibrium outcome,
  1. Examine bidder i's expected payoff $b_i(v_i) = v_i$ ("First case").
  2. Compare with what she would obtain from $b_i(v_i) < v_i$ ("Second case").
  3. Compare with what she would obtain from $b_i(v_i) > v_i$ ("Third case").

Second-Price Auctions

1. *First case*: Bidding your valuation, $b_i(v_i) = v_i$.

   1a) If the highest competing bid lies below her bid, $h_i < b_i$, where $h_i = \max_{j \neq 1}\{b_i\}$,
   - bidder i wins, and obtains a net payoff of $v_i - h_i$.

   2a) If the highest competing bid lies above her bid, $h_i > b_i$,
   - bidder i loses the auction, earning zero payoff.

We do not consider the case when her bid coincides with the highest bid, $b_i = h_i$, and a tie occurs;

- Ties are solved by randomly assigning the object to the bidders who submitted the highest bids.
- Bidder i's expected payoff becomes $\frac{1}{2}(v_i - h_i)$, but earns zero expected payoff because $b_i = h_i$.

Second-Price Auctions

2. *Second case*: Bidding below your valuation, $b_i(v_i) < v_i$.

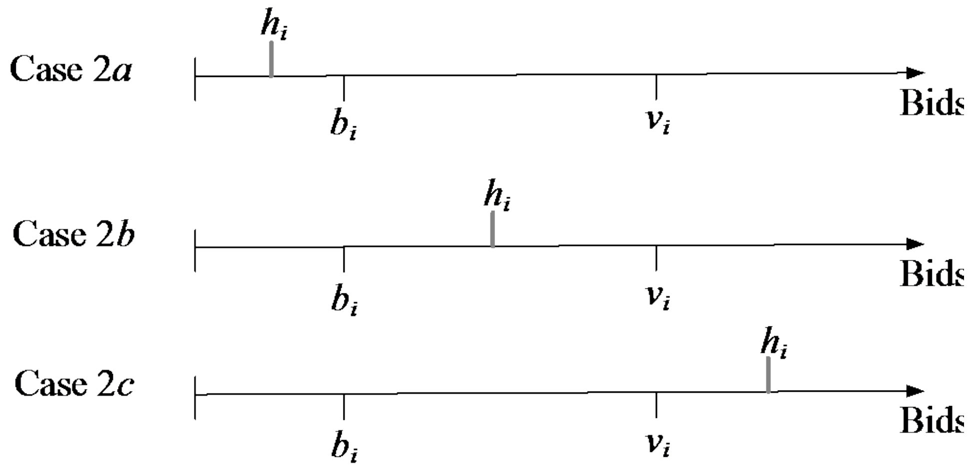

Figure 15.1

2a) If the highest competing bid lies below her bid, $h_i < b_i$,

- bidder i still wins the auction, and obtains the same net payoff as when she does not shade her bid, $v_i - h_i$.

Second-Price Auctions

2. *Second case*: Bidding below your valuation, $b_i(v_i) < v_i$.

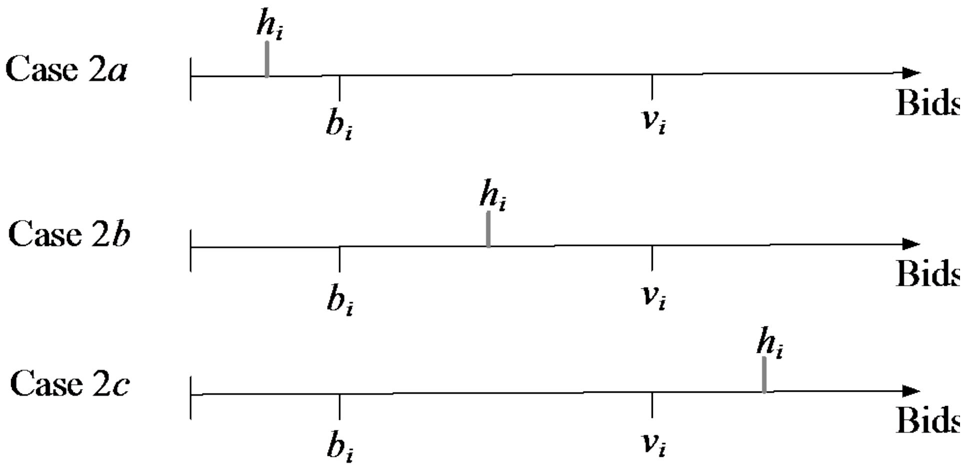

Figure 15.1

2b) If the highest competing bid is between her bid and bidder i's valuation, $b_i < h_i < v_i$,

- bidder i loses, making zero payoff.

Second-Price Auctions

2. *Second case*: Bidding below your valuation, $b_i(v_i) < v_i$.

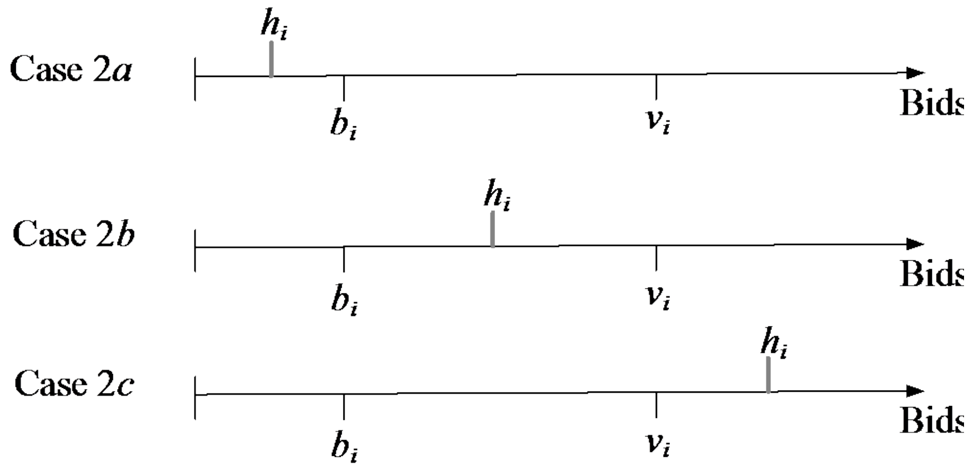

Figure 15.1

2c) If the highest competing bid is higher than her valuation, $h_i > v_i$

- bidder i loses, yielding the same outcome as when $b_i = v_i$.

Second-Price Auctions

3. *Third case*: Bidding above your valuation, $b_i(v_i) > v_i$.

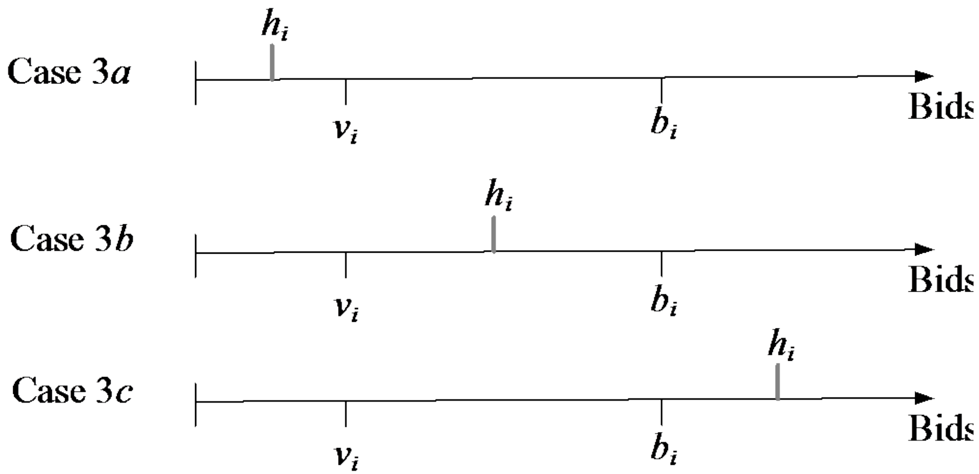

Figure 15.2

3a) If the highest competing bid lies below bidder i's valuation, $h_i < v_i$,

- she still wins, earning a payoff of $v_i - h_i$, which coincides with that when $b_i = v_i$.

Second-Price Auctions

3. *Third case*: Bidding above your valuation, $b_i(v_i) > v_i$.

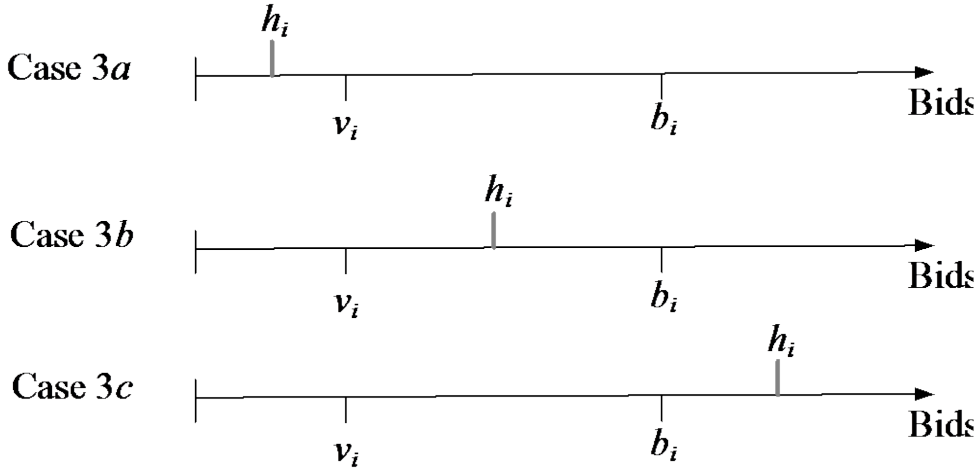

Figure 15.2

3b) If the highest competing bid lies between her valuation and her bid $v_i < h_i < b_i$,

- bidder i still wins the object but earns a negative payoff because $v_i - h_i < 0$.

Second-Price Auctions

3. *Third case*: Bidding above your valuation, $b_i(v_i) > v_i$.

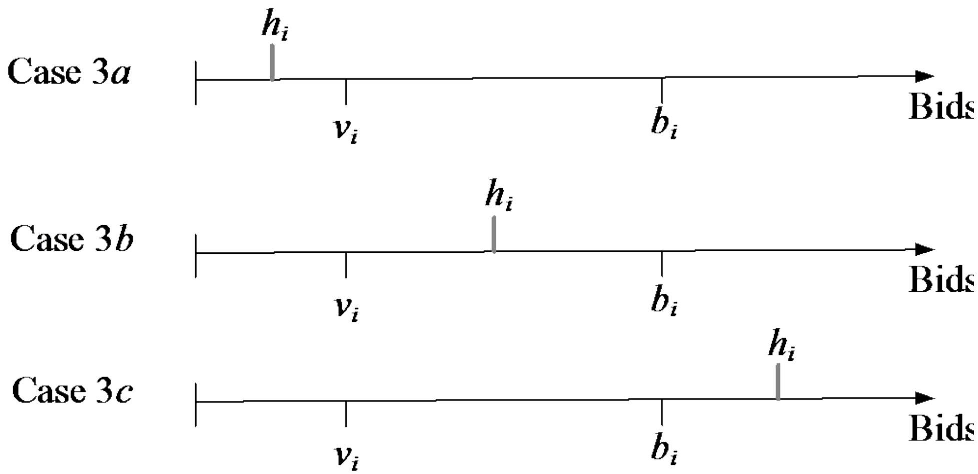

Figure 15.2

3c) If the highest competing bid lies above her bid, $h_i > b_i$,
- bidder i loses, earning a zero payoff.

Second-Price Auctions

- *Summary*:
  - When bidder i shades her bid, $b_i < v_i$, she obtains the same or lower payoff than when she submits a bid that coincides with her valuation, $b_i = v_i$.
    - She does not have incentives to shade her bid.
  - When bidder i submits a bid above her valuation, $b_i > v_i$, her payoff either coincides with her valuation, or becomes strictly lower.
    - She does not have incentives to deviate from her equilibrium bid.
  - Hence, there is no bidding strategy that provides a strictly higher payoff than $b_i(v_i) = v_i$ in the SPA.

Second-Price Auctions

- **Remark:**
  - The equilibrium bidding strategy in the SPA is unaffected by:
    - The number of bidders in the auction, N.
      - An increase in N does not emphasize or ameliorate the incentives that very bidder has to submit $b_i(v_i) = v_i$.
    - Their risk aversion preferences.
      - Results remain when bidders evaluate their net payoff, $v_i - h_i$, according to a concave utility function, such as $u(x) = x^\alpha$. For a given value of h_i, her expected payoff from $b_i(v_i) = v_i$, would be weakly larger than deviating.
    - How valuations for an object are distributed (e.g., uniform, normal or exponential distribution).

First-Price Auctions

Privately Observed Valuations

- Auctions are strategic scenarios where players choose their strategies in an incomplete information context:
  - Every bidder knows her own valuation, v_i, but does not observe other bidders' valuation, v_j.
  - Bidder i knows the probability distribution behind v_j.
    - *Example:*

$$v_i = \begin{cases} \$10 \text{ with probabilty } 0.4 \\ \$5 \text{ with probability } 0.6 \end{cases}$$

    - More generally,

$$F(v) = prob(v_j < v)$$

    - We will assume that every bidder's valuation for the object is drawn from a uniform distribution function between 0 and 1.

Privately Observed Valuations

- Union distribution function, $v_j \sim U[0,1]$.

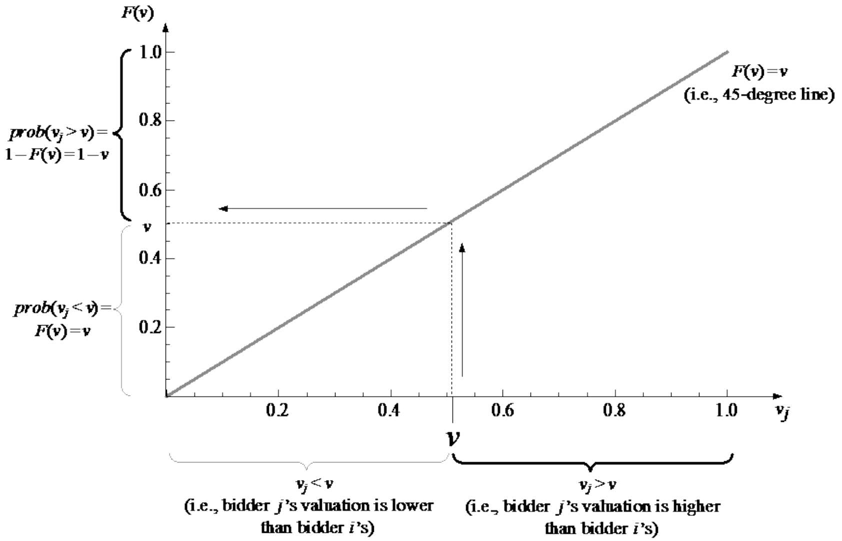

Figure 15.3

- If bidder i's valuation is v, valuations to the left in the horizonal axis represent points where $v_j < v$. The mapping to the vertical axis gives $prob\left(v_j < v\right) = F(v) = v$.

Privately Observed Valuations

- Union distribution function, $v_j \sim U[0,1]$.

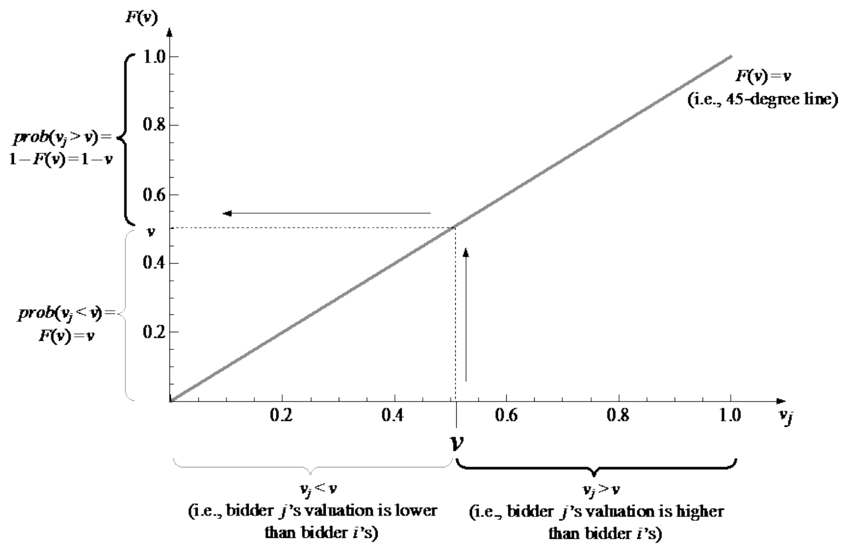

Figure 15.3

  - Valuations to the right side of v describe points where $v_j > v$. Mapping these points into the vertical axis gives $prob(v_j > v) = 1 - F(v) = 1 - v$.

Equilibrium Bidding in First-Price Auctions

- Submitting $b_i > v_i$, is a dominated strategy.
  - Her expected utility becomes,

$$EU_i(b_i|v_i) = prob(win) \times \underbrace{(v_i - b_i)}_{-} + prob(lose) \times 0,$$

  which becomes negative regardless of the probability of wining since $v_i - b_i < 0$.

- Submitting $b_i = v_i$, is also dominated strategy.
  - Her expected utility would be zero,

$$EU_i(b_i|v_i) = prob(win) \times \underbrace{(v_i - b_i)}_{0}.$$

- Equilibrium bidding in FPA imply $b_i > v_i$, known as “bid shading”.

Equilibrium Bidding in First-Price Auctions

- "Bid shading": If bidder i's valuation is v_i, her bid must be a share of her true valuation, $b_i(v_i) = a \cdot v_i$, where $a \in (0,1)$

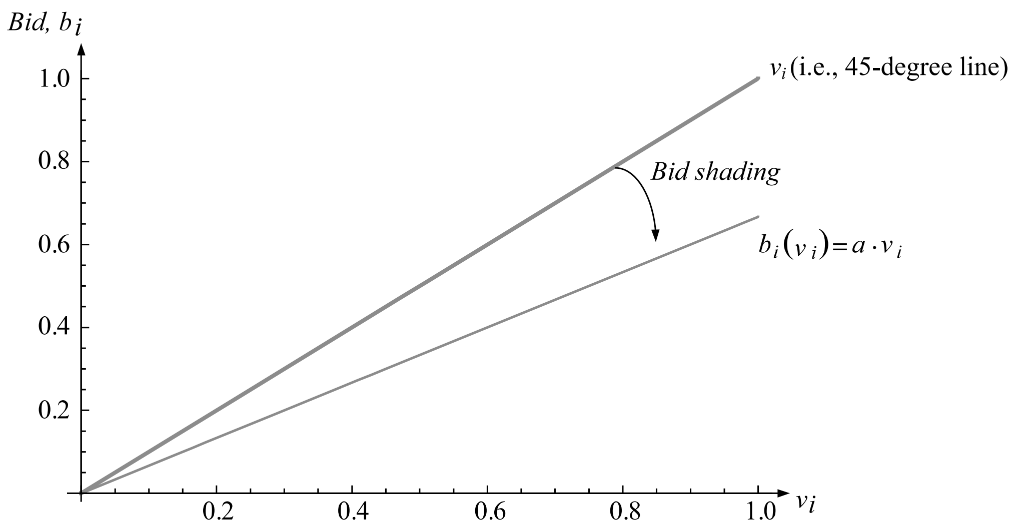

Figure 15.4

Equilibrium Bidding in First-Price Auctions

- What is the precise value of the bid shading parameter a?
- To answer this question, we must describe bidder i's expected utility from submitting a bid x, when her valuation of the object is v_i,

$$EU_i(x|v_i) = prob(win) \times (v_i - x) + prob(lose) \times 0.$$

- We need to characterize $prob(win)$:
  - Upon submitting $b_i = x$, bidder j can anticipate that bidder i's valuation is $\frac{x}{a}$, by inverting the bidding function $b_i(v_i) = x = a \times v_i$.
    - For a bid x, bidder j can use the symmetric bidding function $a \times v_i$ to "recover" bidder i's valuation, $\frac{x}{a}$, that generated a bid of $\$x$.

Equilibrium Bidding in First-Price Auctions

- The probability of winning is

$$prob(b_i > b_j) = prob(x > b_j).$$

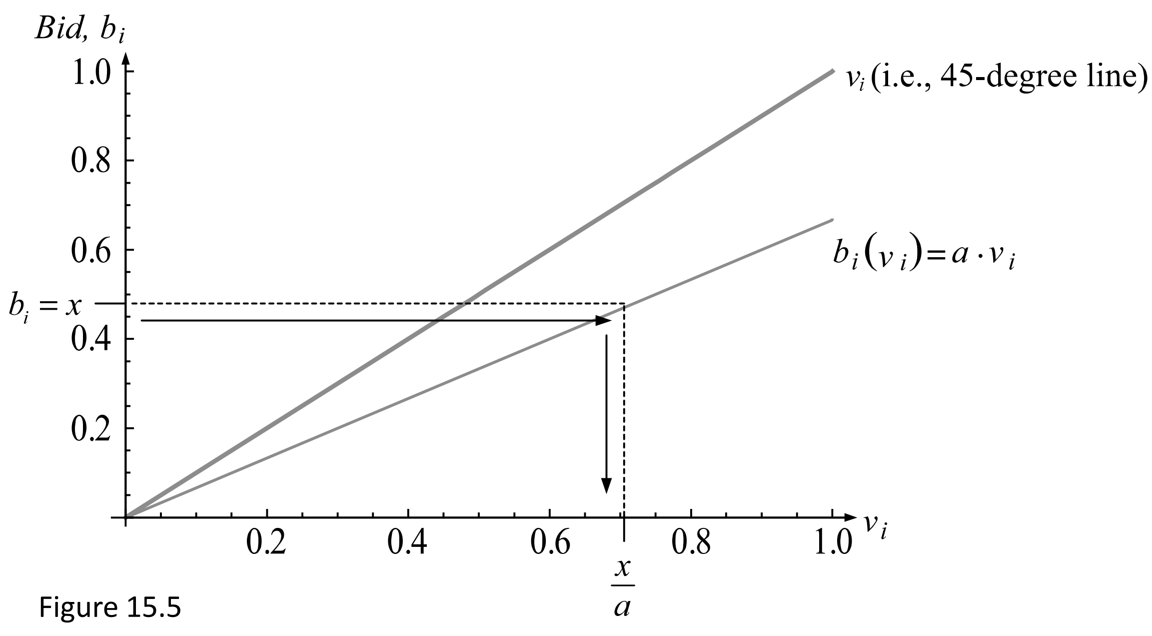

Figure 15.5

Equilibrium Bidding in First-Price Auctions

- Or from the point of view of valuations,

$$prob(b_i > b_j) = prob\left(\frac{x}{a} > v_j\right) = \frac{x}{a} \text{ (since } v_j \sim U[0,1]\text{).}$$

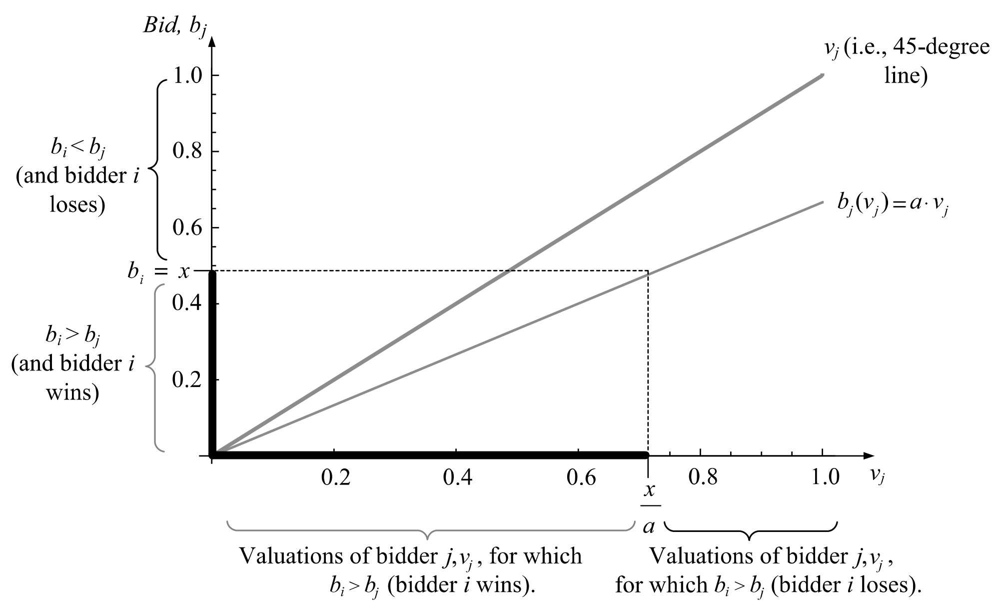

Figure 15.6

Equilibrium Bidding in First-Price Auctions

- Plugging the probability of winning into bidder i's expecting utility from submitting a bid of x in the FPA,

$$EU_i(x|v_i) = \frac{x}{a}(v_i - x) = \frac{v_i x - x^2}{a}.$$

- Taking firs-order conditions with respect to x,

$$\frac{v_i - 2x}{a} = 0,$$

  and solving for x yields bidder i's optimal bidding function:

$$x(v_i) = \frac{1}{2} v_i$$

  - It informs bidder i how much to bid as a function of her privately observed valuation of the object, v_i.

Equilibrium Bidding in First-Price Auctions

- Bidder i's optimal function, $x(v_i) = \frac{1}{2}v_i$.
  - When $N = 2$, bid i shades her bid in half.
    - For instance, when $v_i = \$0.75$, her optimal bid becomes $\frac{1}{2}0.75 = \$0.375$.

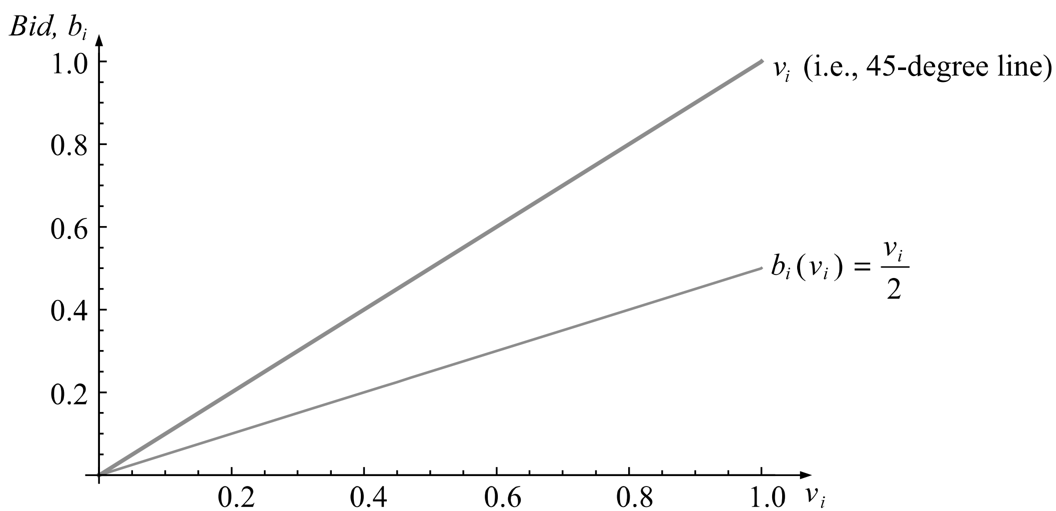

Figure 15.7

First-Price Auctions with N Bidders

- With N bidders, the probability of bidder i winning the auction when submitting a bid of $\$x$ is

$$prob(win) = prob\left(\frac{x}{a} > v_1\right) \cdot \dots \cdot prob\left(\frac{x}{a} > v_{i-1}\right) \cdot prob\left(\frac{x}{a} > v_{i+1}\right) \cdot \dots \cdot prob\left(\frac{x}{a} > v_N\right)$$

$$= \frac{x}{a} \cdot \dots \frac{x}{a} \cdot \frac{x}{a} \cdot \dots \frac{x}{a} = \left(\frac{x}{a}\right)^{N-1},$$

where we evaluate the probability that the valuation of all other $N-1$ bidders lies below the valuation $v_i = \frac{x}{a}$, which generates a bid of $\$x$.

- Hence, bidder i's expected utility from submitting x is

$$EU_i(x|v_i) = \underbrace{\left(\frac{x}{a}\right)^{N-1}}_{prob(win)} (v_i - x).$$

First-Price Auctions with N Bidders

- The bidder expected utility can be rewritten as

$$EU_i(x|v_i) = \frac{1}{a^{N-1}}(x^{N-1}v_i - x^{N-1}x) = \frac{1}{a^{N-1}}(x^{N-1}v_i - x^N)$$

- Taking first-order conditions with respect to x,

$$\frac{1}{a^{N-1}}[(N-1)x^{N-2}v_i - Nx^{N-1}] = 0,$$

- Rearranging and solving for x,

$$\frac{x^{N-1}}{x^{N-2}} = \frac{N-1}{N}v_i,$$

$$\underbrace{x^{(N-1)-(N-2)}}_{x^{(N-1)-(N-2)}=x} = \frac{N-1}{N}v_i,$$

$$x(v_i) = \frac{N-1}{N}v_i.$$

First-Price Auctions with N Bidders

- Optimal bidding function $x(v_i) = \frac{N-1}{N} v_i$.

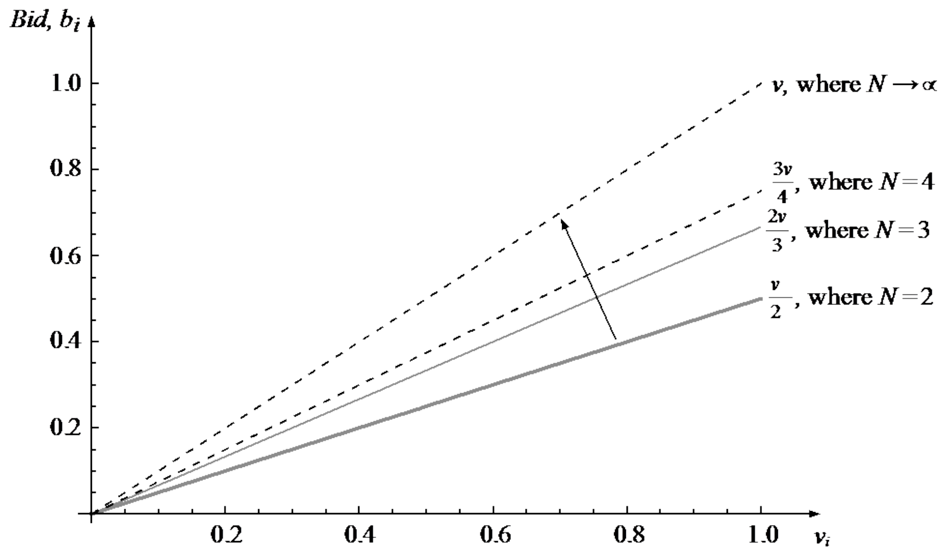

Figure 15.8

  - Bid shading is ameliorated as N increases.
  - When N is extremely large, bidder i's bid almost coincides with her valuation. The bidding function approaches the 45-degree line.

FPA with Risk-Averse Bidders

- Utility function is concave in income, e.g., $u(x) = x^a$.
  - $0 < \alpha \leq 1$ denotes bidder i's risk aversion parameter.
  - When $\alpha = 1$, she is risk neutral.
- $\boldsymbol{N = 2}$**:**
  - The probability of winning is unaffected because a symmetric bidding function $b_i(v_i) = a \cdot v_i$ for every bidder i, where $a \in (0,1)$.
  - The probability that bidder i wins the auction against bidder j is

$$prob(b_i > b_j) = prob(x > b_j) = prob\left(\frac{x}{a} > v_j\right) = \frac{x}{a}.$$

FPA with Risk-Averse Bidders

- $\boldsymbol{N = 2}$ (cont.):
  - Bidder i's expected utility from participating in the auction is

$$EU_i(x|v_i) = \frac{x}{a} \times (v_i - x)^{\alpha}.$$

  - Taking first-order conditions with respect to x,

$$\frac{1}{a}(v_i - x)^{\alpha} - \frac{x}{a}\alpha(v_i - x)^{\alpha-1} = 0,$$

  and solving for x,

$$x(v_i) = \frac{1}{1+\alpha}v_i.$$

FPA with Risk-Averse Bidders

- $\boldsymbol{N = 2}$ (cont.):
  - When $\alpha = 1$ (risk-neutral bidder), $x(v_i) = \frac{v_i}{2}$.
  - When α decreases (more risk aversion), $x(v_i)$ increases. Specifically, $\frac{\partial x(v_i)}{\partial \alpha} = -\frac{v_i}{(1+\alpha)^2} < 0$.
  - When $\alpha \to 0$, $x(v_i) = v_i$.

FPA with Risk-Averse Bidders

- $\boldsymbol{N = 2}$ (cont.):
  - Optimal bidding function $\frac{1}{1+\alpha} v_i$.
    - Bid shading is ameliorated as bidders become more risk averse.
    - The bidding function approaches the 45-degree line as $\alpha \to 0$.

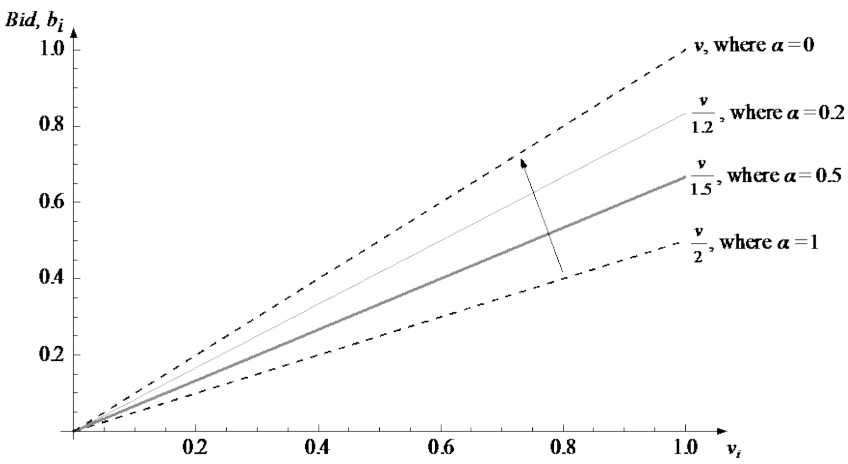

Figure 15.9

FPA with Risk-Averse Bidders

- $\boldsymbol{N = 2}$ (cont.):
  - Consider bidder i reduces her bid from b_i to $b_i - \varepsilon$,
    - if she wins the auction, she obtains an additional profit of ε because she has to pay a lower price;
    - but, lowering her bid increases her probability of losing.
  - *Intuition*: For a risk-averse bidder, the positive effect of getting the object at a cheaper price is offset by the negative effect of increasing the probability of losing the auction.

FPA with Risk-Averse Bidders

- $\boldsymbol{N \geq 2}$**:**
  - We know that the probability bidder i wins the auction is

$$prob(win) = prob\left(\frac{x}{a} > v_1\right) \cdot \cdots \cdot prob\left(\frac{x}{a} > v_{i-1}\right) \cdot prob\left(\frac{x}{a} > v_{i+1}\right) \cdot \cdots \cdot prob\left(\frac{x}{a} > v_N\right)$$

$$= \frac{x}{a} \cdot \cdots \frac{x}{a} \cdot \frac{x}{a} \cdot \cdots \frac{x}{a} = \left(\frac{x}{a}\right)^{N-1},$$

  - Bidder i's expected utility from participating in the auction is

$$EU_i(x|v_i) = \left(\frac{x}{a}\right)^{N-1} \times (v_i - x)^{\alpha}.$$

  - Differentiating with respect to x,

$$\left[(N-1)\left(\frac{x}{a}\right)^{N-2}(v_i - x)^{\alpha}\right]\frac{1}{a} - \left(\frac{x}{a}\right)^{N-1}\alpha(v_i - x)^{\alpha-1} = 0,$$

FPA with Risk-Averse Bidders

- $\boldsymbol{N \geq 2}$ (cont.):

$$\left(\frac{x}{\alpha}\right)^{N-1}(v_i - x)^{\alpha-1}[(N-1)v_i + (N-1+\alpha)x] = 0.$$

  - Solving for x, we find the equilibrium bidding function

$$x(v_i) = \frac{N-1}{N-1+\alpha}v_i.$$

  - When $N = 2$,

$$x(v_i) = \frac{2-1}{2-1+\alpha}v_i = \frac{1}{1+\alpha}v_i.$$

FPA with Risk-Averse Bidders

- $\boldsymbol{N \geq 2}$ (cont.):
  - When $N = 3$,
$$x(v_i) = \frac{3-1}{3-1+\alpha} v_i = \frac{2}{2+\alpha} v_i.$$
  - More generally,
$$\frac{\partial x(v_i)}{\partial N} = \frac{\alpha v_i}{(N-1+\alpha)^2} > 0.$$

    As N increases, bidders become more aggressive.

Efficiency in Auctions

Efficiency in Auctions

- Auctions are efficient if the bidder with the highest valuation for the object is the person receiving the object.
  - Otherwise, the outcome of the auction would open the door to negotiation and arbitrage.
- FPA and SPA are efficient because the bidder with the highest valuation submits the highest bid, winning the auction and receiving the object.

Efficiency in Auctions

- Chinese (or lottery) auctions are no necessarily efficient.
- For an auction to satisfy efficiency:
  - Bids must be increasing in a player's valuation.
  - The probability of winning the auction must be 100% is a bidder submits the highest bid.

Common-Value Auctions

Common-Value Auctions

- In some auctions might assign the same value to the object (common value).
  - *Example*: Government sales of oil leases.
    - Firms cannot observe the exact volume of oil in the reservoir, or how difficult it will be to extract.
    - They can make estimations and assign a value to the object (profits from oil lease) within a narrow range, $\nu \in 10{,}11, \ldots, 20]$ in million dollars.
    - The value in profits that all firms assign to the oil lease is common.
    - The estimate e_i that each firm i receives about this common value is potentially different. It can be upward-biased, $e_i > \nu$ and downward-biased, $e_i < \nu$.

Common-Value Auctions

- Consider bidders A and B, each receiving an estimate e_A and e_B, where $e_A > v > e_B$.
- If every bidder submits a bid that shades her estimate by $1\$$,

$b_A = e_A - 1$, and $b_B = e_B - 1$, where $b_A > b_B$.

  - A submits are more aggressive bid because $e_A > e_B$.

- Bidder A wins but her payoff could be negative if her margin after paying bid b_A is negative,

$$v - b_A = v - (e_A - 1) < 0 \quad \Longrightarrow v + 1 < e_A.$$

- The winner's curse: Winning the auction means that the winner probably received an overestimated signal of the true value.

Common-Value Auctions

- To avoid the winner's curse, participants in common-value auctions must significantly shade their bid to account for over or underestimation.
- *Example*: The winner's curse in the classroom.
  - Your instructor shows up in the class with a glass full of nickels.
  - The monetary value you assign to the jar (value of the coins) coincides with that of your classmates.

Common-Value Auctions

- *Example*: The winner's curse in the classroom (cont.).
  - None can accurately estimate the number of nickels because you can look at the jar only for a few seconds, gathering imprecise information.
  - It is usual to find that the winner ends up submitting a bid above the jar's true value.

A Look at Behavioral Economics– Experiments with Auctions

Experiments with Auctions

- Controlled experiments have been developed to test whether individuals bid according to $b_i(v_i)$.
  - Individual valuations for the object are randomly distributed prior to the auction period.
  - In each period, the bidder submitting the highest bid earns a profit equal to her valuation minus the auction price, while other bidders earn zero profit.
- Most studies indicate that individuals tend to bid more aggressively than what would be expected according to $b_i(v_i)$.
- However, comparative statics remain. They tend to bid more aggressively when competing against more bidders, when their valuation is higher, and when they are risk averse.

Appendix. First-Price Auctions in More General Settings

FPA in More General Settings

- We extend the analysis of section 15.5 allowing for valuations to be drawn from a general cumulative distribution, $F(v_i)$, with positive density in all its support, $f(v_i) > 0$.
- *Writing expected utility*.
    - Bidder i's UMP is
$$\max_{b_i \geq 0} prob(win)(v_i - b_i).$$
    - Bidder i wins the auction when her bid exceeds that of bidder j, $b_j < b_i$, which is equivalent to $v_j < v_i$. This probability can be expressed as
$$prob\left(v_j < v_i\right) = F(v_i).$$

FPA in More General Settings

- *Writing expected utility* (cont.).
  - When bidder i's faces $N-1$ rivals, her probability of winning the auction is the probability that her valuation exceeds that of all other $N-1$ bidders.
  - We can write this probability as

$$prob(v_j < v_i) \times prob(v_k < v_i) \times \cdots \times prob(v_l < v_i)$$

$$= \underbrace{F(v_i) \times F(v_i) \times \cdots \times F(v_i)}_{N-1 \text{ times}} = F(v_i)^{N-1}.$$

where $j \neq k \neq l$ represents i's rivals.

FPA in More General Settings

- *Writing expected utility* (cont.).
  - As a result, the expected PMP can be written as:
  $$\max_{b_i \geq 0} F(v_i)^{N-1}(v_i - b_i)\,.$$
  - Using this bidding function, we can write $b_i(v_i) = x_i$, where $x_i \in \mathbb{R}_+$ represents bidder i's bid when her valuation is v_i.
  - Applying the inverse $b^{-1}(\cdot)$ on both sides, $v_i = b_i^{-1}(v_i)$.
  - Then, $F(v_i)^{N-1}$ can be written as $F(b_i^{-1}(x_i))^{N-1}$.
  - Ant the PMP becomes
  $$\max_{x_i \geq 0} F\left(b_i^{-1}(x_i)\right)^{N-1}(v_i - x_i).$$

FPA in More General Settings

- *Finding equilibrium bids.*
  - Differentiating with respect to x_i,

$$-\left[F\left(b_i^{-1}(x_i)\right)^{N-1}\right]$$
$$+ (N-1)F\left(b_i^{-1}(x_i)\right)^{N-2} f\left(b_i^{-1}(x_{i)}\right)\frac{\partial b_i^{-1}(x_i)}{\partial x_i}(v_i - x_i) = 0$$

FPA in More General Settings

- *Finding equilibrium bids* (cont.)
  - Because $b_i^{-1}(x_i) = v_i$ and $\frac{\partial b_i^{-1}(x_i)}{\partial x_i} = \frac{1}{b' b_i^{-1}(x_i)}$, this expression simplifies to

$$-[F(v_i)^{N-1}] + (N-1)F(v_i)^{N-2} f(v_i) \frac{1}{b' v_i}(v_i - x_i) = 0,$$

$$(N-1)F(v_i)^{N-2} f(v_i) v_i - (N-1)F(v_i)^{N-2} f(v_i) x_i = F(v_i)^{N-1} b' v_i,$$

$$F(v_i)^{N-1} b' v_i + (N-1)F(v_i)^{N-2} f(v_i) v_i = (N-1)F(v_i)^{N-2} f(v_i) x_i.$$

FPA in More General Settings

- *Finding equilibrium bids* (cont.).
  - Because the left side is $\frac{\partial\left[F(v_i)^{N-1}b_i(v_i)\right]}{\partial v_i}$,

$$\frac{\partial\left[F(v_i)^{N-1}b_i(v_i)\right]}{\partial v_i} = (N-1)F(v_i)^{N-2}f(v_i)x_i.$$

  - Integrating both sides,

$$F(v_i)^{N-1}b_i(v_i) = \int_0^{v_i}(N-1)F(v_i)^{N-2}f(v_i)v_i dv_i\,.$$

  - Applying integration by parts on the right side,

$$\int_0^{v_i}(N-1)F(v_i)^{N-2}f(v_i)v_i dv_i = F(v_i)^{N-1}v_i - \int_0^{v_i}F(v_i)^{N-1}dv_i.$$

FPA in More General Settings

- *Finding equilibrium bids* (cont.).
  - The first—order condition can be written as:

$$F(v_i)^{N-1} b_i(v_i) = \mathrm{F}(v_i)^{N-1} v_i - \int_0^{v_i} F(v_i)^{N-1} dv_i \,.$$

  - Dividing both side by $F(v_i)^{N-1}$, and solving for the equilibrium bidding function, $b_i(v_i)$,

$$b_i(v_i) = v_i \underbrace{- \frac{\int_0^{v_i} F(v_i)^{N-1} dv_i}{F(v_i)^{N-1}}}_{\text{Bid shading}} .$$

  - The bidding function $b_i(v_i)$ constitutes the BNE of the FPA when bidder's valuations are distributed according to $F(v_i)$.

FPA in More General Settings

- **Uniformly distributed valuations.**
  - When $F(v_i) = v_i$,
$$F(v_i)^{N-1} = {v_i}^{N-1},$$
$$\int_0^{v_i} F(v_i)^{N-1} dv_i = \frac{1}{N} v_i^N .$$
  - The bidding function is,
$$b_i(v_i) = v_i - \frac{\frac{1}{N} v_i^N}{{v_i}^{N-1}} = v_i \left(\frac{N-1}{N} \right).$$
    - When $N = 2, b_i(v_i) = \frac{v_i}{2}$, and when $N = 3, b_i(v_i) = \frac{2v_i}{3}$.
    - As more bidders participate in the auction, every bidder i submits a more aggressive bid because there is a higher probability that another bidder j has a higher valuation.