Advanced Microeconomic Theory

Chapter 1: Preferences and Utility

Outline

- Preference and Choice
- Preference-Based Approach
- Utility Function
- Indifference Sets, Convexity, and Quasiconcavity
- Special and Continuous Preference Relations
- Social and Reference-Dependent Preferences
- Hyperbolic and Quasi-Hyperbolic Discounting
- Choice-Based Approach
- Weak Axiom of Revealed Preference (WARP)
- Consumption Sets and Constraints

Preference and Choice

Preference and Choice

- We begin our analysis of individual decision-making in an abstract setting.
- Let $X \subseteq \mathbb{R}_+^N$ be a set of possible alternatives for a particular decision maker.
  - It might include the consumption bundles that an individual is considering to buy.
  - *Example*:

$$X = \{x, y, z, \dots\}$$
$$X = \{\text{Apple}, \text{Orange}, \text{Banana}, \dots\}$$

Preference and Choice

- Two ways to approach the decision making process:
  1) ***Preference-based approach***: analyzing how the individual uses his preferences to choose an element(s) from the set of alternatives X.
  2) ***Choice-based approach***: analyzing the actual choices the individual makes when he is called to choose element(s) from the set of possible alternatives.

Preference and Choice

- Advantages of the Choice-based approach:
  – It is based on observables (actual choices) rather than on unobservables (individual preferences)
- Advantages of Preference-based approach:
  – More tractable when the set of alternatives X has many elements.

Preference and Choice

- After describing both approaches, and the assumptions on each approach, we want to understand:

$$\text{Rational Preferences} \implies \text{Consistent Choice behavior}$$

$$\text{Rational Preferences} \impliedby \text{Consistent Choice behavior}$$

Preference-Based Approach

Preference-Based Approach

- ***Preferences***: "attitudes" of the decision-maker towards a set of possible alternatives X.

- For any $x, y \in X$, how do you compare x and y?
  - ❑ I prefer x to y $(x \succ y)$
  - ❑ I prefer y to x $(y \succ x)$
  - ❑ I am indifferent $(x \sim y)$

Preference-Based Approach

By asking:	We impose the assumption:
Check one box (i.e., not refrain from answering)	*Completeness*: individuals must compare any two alternatives, even the ones they don't know.
Check only one box	The individual is capable of comparing any pair of alternatives.
Don't add any new box in which the individual says, "I love x and hate y"	We don't allow the individual to specify the intensity of his preferences.

Preference-Based Approach

- ***Completeness***:
  - For any pair of alternatives $x, y \in X$, the individual decision maker:
    - ❑ $x \succ y$, or
    - ❑ $y \succ x$, or
    - ❑ both, i.e., $x \sim y$
  - (The decision maker is allowed to choose one, and only one, of the above boxes).

Preference-Based Approach

- ***Not all binary relations satisfy Completeness***.
- *Example*:
  - "Is the brother of": John $\nsucc$ Bob and Bob $\nsucc$ John if they are not brothers.
  - "Is the father of": John $\nsucc$ Bob and Bob $\nsucc$ John if the two individuals are not related.
- Not all pairs of alternatives are comparable according to these two relations.

Preference-Based Approach

- ***Weak preferences***:
  - Consider the following questionnaire:
  - For all $x, y \in X$, where x and y are not necessarily distinct, is x at least as preferred to y?
    - ❑ Yes $(x \succsim y)$
    - ❑ No $(y \succsim x)$
  - Respondents must answer yes, no, or both
    - Checking both boxes reveals that the individual is indifferent between x and y.
    - Note that the above statement relates to completeness, but in the context of weak preference $\succsim$ rather than strict preference $\succ$.

Preference-Based Approach

- ***Reflexivity***: every alternative x is weakly preferred to, at least, one alternative: itself.
- A preference relation satisfies reflexivity if for any alternative $x \in X$, we have that:
  1) $x \sim x$: any bundle is indifferent to itself.
  2) $x \succsim x$: any bundle is preferred or indifferent to itself.
  3) $x \nsucc x$: any bundle belongs to at least one indifference set, namely, the set containing itself if nothing else.

Preference-Based Approach

- The preference relation $\succsim$ is ***rational*** if it possesses the following two properties:

  a) *Completeness*: for all $x, y \in X$,
  either $x \succsim y$, or $y \succsim x$, or both.

  b) *Transitivity*: for all $x, y, z \in X$,
  if $x \succsim y$ and $y \succsim z$, then it must be that $x \succsim z$.

Preference-Based Approach

- ***Example 1.1*.**

Consider the preference relation

$$x \succsim y \quad \text{if and only if} \quad \sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$$

In words, the consumer prefers bundle x to y if the total number of goods in bundle x is larger than in bundle y.

Graphical interpretation in $\mathbb{R}^2$ (diagonal above another diagonal). Hyperplanes for $N > 2$.

Preference-Based Approach

- ***Example 1.1*** (continues).
- *Completeness*:
  - either $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$ (which implies $x \succsim y$), or
  - $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} x_i$ (which implies $y \succsim x$), or
  - both, $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$ (which implies $x \sim y$).

- *Transitivity:*
  - If $x \succsim y$, $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$, and
  - $y \succsim z$, $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} z_i$,
  - Then it must be that $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} z_i$ (which implies $x \succsim z$, as required).

Preference-Based Approach

- The assumption of transitivity is understood as that preferences should not cycle.
- Example violating transitivity:

$$\underbrace{apple \succsim banana \quad banana \succsim orange}_{apple \succsim orange \text{ (by transitivity)}}$$

  but $orange \succ apple$.

- Otherwise, we could start the cycle all over again, and extract infinite amount of money from individuals with intransitive preferences.

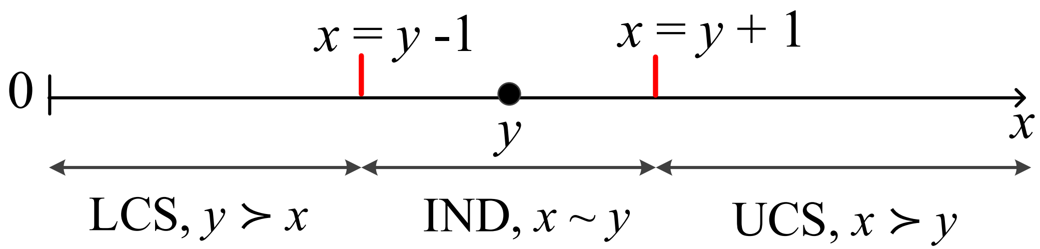
Preference-Based Approach

- Sources of intransitivity:
  a) Indistinguishable alternatives
  b) Framing effects
  c) Aggregation of criteria
  d) Change in preferences

Preference-Based Approach

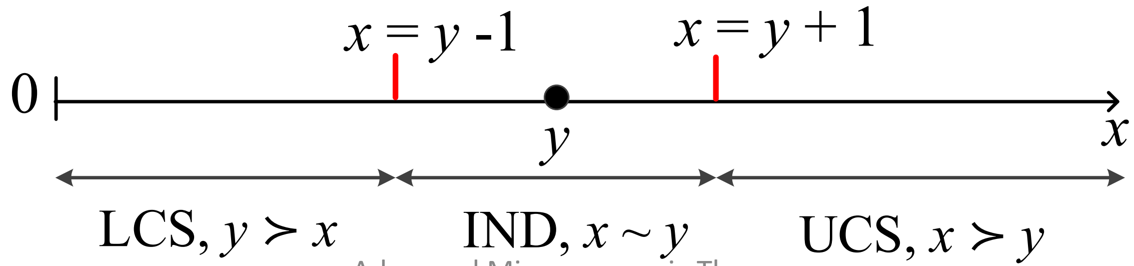
- ***Example 1.2*** (Indistinguishable alternatives):
  - Take $X = \mathbb{R}$, such as a piece of pie.
  - An individual is indifferent between x and y when $|x - y| < 1$.
    - This means that $-1 < x - y < 1$ or, after rearranging, $y - 1 < x < y + 1$.
    - In other words, when x satisfies both:
      - $x > y - 1$ and
      - $x < y + 1$,

      the individual is indifferent between x and y.
    - Intuitively, when alternatives are relatively similar (see figure), the individual cannot tell them apart.



Preference-Based Approach

- ***Example 1.2*** (Indistinguishable alternatives):
  - However, he strictly prefers x to y when $x \geq y + 1$, meaning that x is at least one unit larger than y.
    - See UCS at the right-hand of the figure.
  - In contrast, he strictly prefers y to x when $x \leq y - 1$, which means that y is at least one unit larger than x.
    - See LCS at the left-hand of the figure.



Preference-Based Approach

- ***Example 1.2*** (Indistinguishable alternatives):
  - **Completeness.** The above preference relation is complete:
    - For a given bundle x, another bundle y must lie in the UCS, IND, or LCS of x (see figure).
  - **Transitivity.** It does not hold:
    - Construct a counterexample, such as:

$$1.5 \sim 0.8 \ \text{ since } 1.5 - 0.8 = 0.7 < 1$$

$$0.8 \sim 0.3 \ \text{ since } 0.8 - 0.3 = 0.5 < 1$$

    - By transitivity, we would have $1.5 \sim 0.3$, but in fact $1.5 \succ 0.3$ (intransitive preference relation).

Preference-Based Approach

- *Other examples*:
  - similar shades of gray paint
  - milligrams of sugar in your coffee

Preference-Based Approach

- ***Example 1.3*** (Framing effects):
  - Transitivity might be violated because of the way in which alternatives are presented to the individual decision-maker.
  - What holiday package do you prefer?
    - a) A weekend in Paris for \$574 at a four-star hotel.
    - b) A weekend in Paris at the four-star hotel for \$574.
    - c) A weekend in Rome at the five-star hotel for \$612.
  - By transitivity, we should expect that if $a \sim b$ and $b \succ c$, then $a \succ c$.

Preference-Based Approach

- ***Example 1.3*** (continued):
  - However, this did not happen!
  - More than 50% of the students responded $c \succ a$.
  - Such intransitive preference relation is induced by the framing of the options.

Preference-Based Approach

- ***Example 1.4*** (Aggregation of criteria):
  - Aggregation of several individual preferences might violate transitivity.
  - Consider $X = \{MIT, WSU, Home\ University\}$
  - When considering which university to attend, you might compare:
    a) Academic prestige (criterion #1)
       $\succ_1$: $MIT \succ_1 WSU \succ_1 Home\ Univ.$
    b) City size/congestion (criterion #2)
       $\succ_2$: $WSU \succ_2 Home\ Univ. \succ_2 MIT$
    c) Proximity to family and friends (criterion #3)
       $\succ_3$: $Home\ Univ. \succ_3 MIT \succ_3 WSU$

Preference-Based Approach

- ***Example 1.4*** (continued):
  - By majority of these considerations:
    $$MIT \underset{\text{criteria 1 \& 3}}{\succ} WSU \underset{\text{criteria 1 \& 2}}{\succ} Home\ Univ \underset{\text{criteria 2 \& 3}}{\succ} MIT$$
  - Transitivity is violated due to a cycle.
  - A similar argument can be used for the aggregation of individual preferences in *group decision-making*:
    - Every person in the group has a different (transitive) preference relation but the group preferences are not necessarily transitive ("***Condorcet paradox***").

Preference-Based Approach

- Intransitivity due to a *change in preferences*
  - When you start smoking

    $$\text{One cigarette} \succsim \text{No smoking} \succsim \text{Smoking heavily}$$

    By transitivity,

    $$\text{One cigarette} \succsim \text{Smoking heavily}$$

  - Once you started

    $$\text{Smoking heavily} \succsim \text{One cigarette} \succsim \text{No smoking}$$

    By transitivity,

    $$\text{Smoking heavily} \succsim \text{One cigarette}$$

  - But this contradicts the individual's past preferences when he started to smoke.

Utility Function

Utility Function

- A function $u\colon X \rightarrow \mathbb{R}$ is a ***utility function*** representing preference relations $\succsim$ if, for every pair of alternatives $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y)$$

Utility Function

- Two points:
  1) Only the ranking of alternatives matters.
     – That is, it does not matter if

$$u(x) = 14 \text{ or if } u(x) = 2000$$
$$u(y) = 10 \text{ or if } u(y) = 3$$

     – We do not care about *cardinality* (the number that the utility function associates with each alternative) but instead care about *ordinality* (ranking of utility values among alternatives).

Utility Function

2) If we apply any strictly increasing function $f(\cdot)$ on $u(x)$, i.e.,

$$f\colon \mathbb{R} \to \mathbb{R} \text{ such that } v(x) = f(u(x))$$

the new function keeps the ranking of alternatives intact and, therefore, the new function still represents the same preference relation.

– *Example*:

$$v(x) = 3u(x)$$
$$v(x) = 5u(x) + 8$$

Desirability

Desirability

- We can express desirability in different ways.
  - Monotonicity
  - Strong monotonicity
  - Non-satiation
  - Local non-satiation
- In all the above definitions, consider that x is an n-dimensional bundle

$$x \in \mathbb{R}^n, \text{ i.e., } x = (x_1, x_2, \dots, x_N)$$

where its k^{th} component represents the amount of good (or service) k, $x_k \in \mathbb{R}_+$.

Desirability

- ***Monotonicity***:
  - A preference relation satisfies monotonicity if, for all $x, y \in X$, where $x \neq y$,
    - *a)* $x_k \geq y_k$ for every good k implies $x \succsim y$
    - *b)* $x_k > y_k$ for every good k implies $x \succ y$
  - That is,
    - increasing the amounts of some commodities (without reducing the amount of any other commodity) cannot hurt, $x \succsim y$; and
    - increasing the amounts of all commodities is strictly preferred, $x \succ y$.

Desirability

- ***Strong Monotonicity***:
  - A preference relation satisfies strong monotonicity if, for all $x, y \in X$, where $x \neq y$,

$$x_k \geq y_k \text{ for every good } k \text{ implies } x \succ y$$
$$\text{and } x_l \geq y_l \text{ for at least one good } l$$

  - That is, even if we increase the amounts of only one of the commodities, we make the consumer strictly better off.

Desirability

- Relationship between **monotonicity** and utility function:
  - Monotonicity in preferences implies that the utility function is weakly monotonic (weakly increasing) in its arguments
    - That is, increasing some of its arguments weakly increases the value of the utility function, and increasing all its arguments strictly increases its value.
  - For any scalar $\alpha > 1$,

$$u(\alpha x_1, x_2) \geq u(x_1, x_2)$$
$$u(\alpha x_1, \alpha x_2) > u(x_1, x_2)$$

Desirability

- Relationship between **strong monotonicity** and utility function:
  - Strong monotonicity in preferences implies that the utility function is strictly monotonic (strictly increasing) in all its arguments.
    - That is, increasing some of its arguments strictly increases the value of the utility function.
  - For any scalar $\alpha > 1$,

$$u(\alpha x_1, x_2) > u(x_1, x_2)$$

Desirability

- ***Example 1.5***: $u(x_1, x_2) = \min\{x_1, x_2\}$
  - Monotone, since
    $$\min\{x_1 + \delta, x_2 + \delta\} > \min\{x_1, x_2\}$$
    for all $\delta > 0$.
  - Not strongly monotone, since
    $$\min\{x_1 + \delta, x_2\} \ngtr \min\{x_1, x_2\}$$
    if $\min\{x_1, x_2\} = x_2$.

Desirability

- ***Example 1.6***: $u(x_1, x_2) = x_1 + x_2$
  - Monotone, since
    $$(x_1 + \delta) + (x_2 + \delta) > x_1 + x_2$$
    for all $\delta > 0$.
  - Strongly monotone, since
    $$(x_1 + \delta) + x_2 > x_1 + x_2$$
- Hence, strong monotonicity implies monotonicity, but the converse is not necessarily true.

Desirability

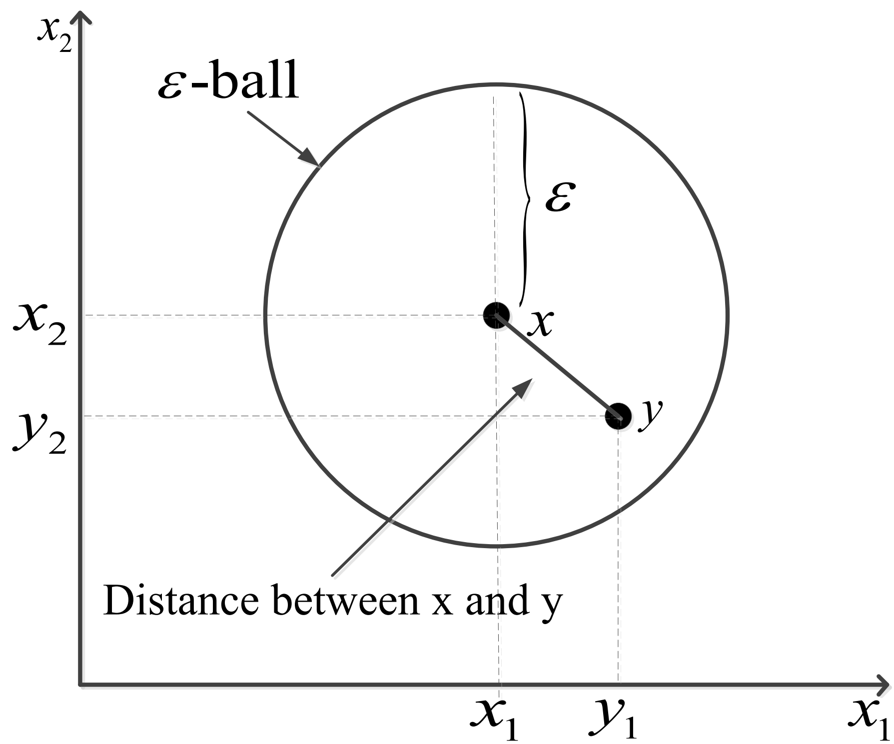
- ***Non-satiation*** (NS):
  - A preference relation satisfies NS if, for every $x \in X$, there is another bundle in set X, $y \in X$, which is strictly preferred to x, i.e., $y \succ x$.
    - NS is too general, since we could think about a bundle y containing extremely larger amounts of some goods than x.
    - How far away are y and x?

Desirability

- ***Local non-satiation*** (LNS):
  - A preference relation satisfies LNS if, for every bundle $x \in X$ and every $\varepsilon > 0$, there is another bundle $y \in X$ which is less than ε-away from x, $\|y - x\| < \varepsilon$, and for which $y \succ x$.
    - $\|y - x\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$ is the Euclidean distance between x and y, where $x, y \in \mathbb{R}_+^2$.
    - In words, for every bundle x, and for ***every*** distance ε from x, we can find a more preferred bundle y.

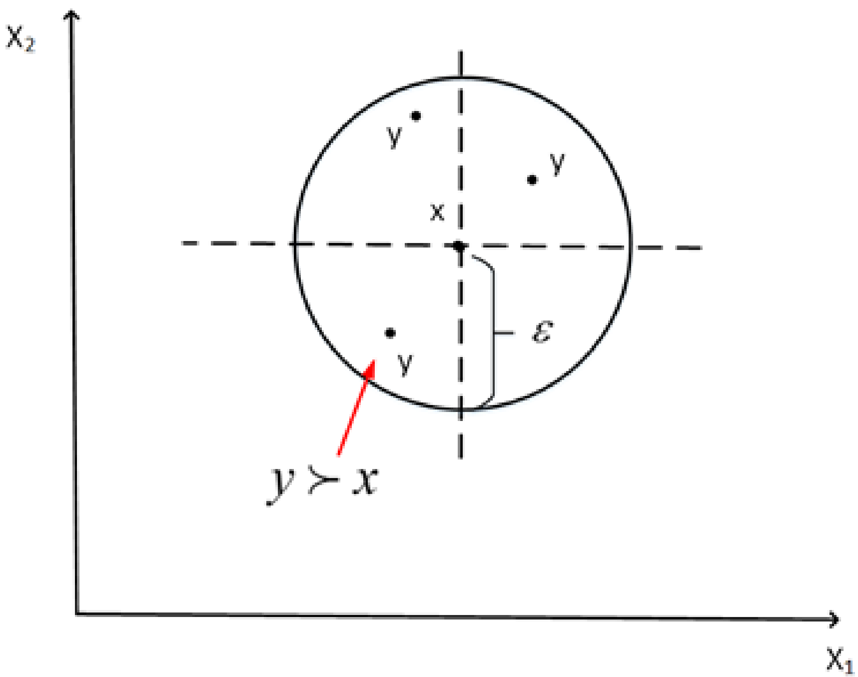
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of good 2 (but more of good 1) than bundle x.



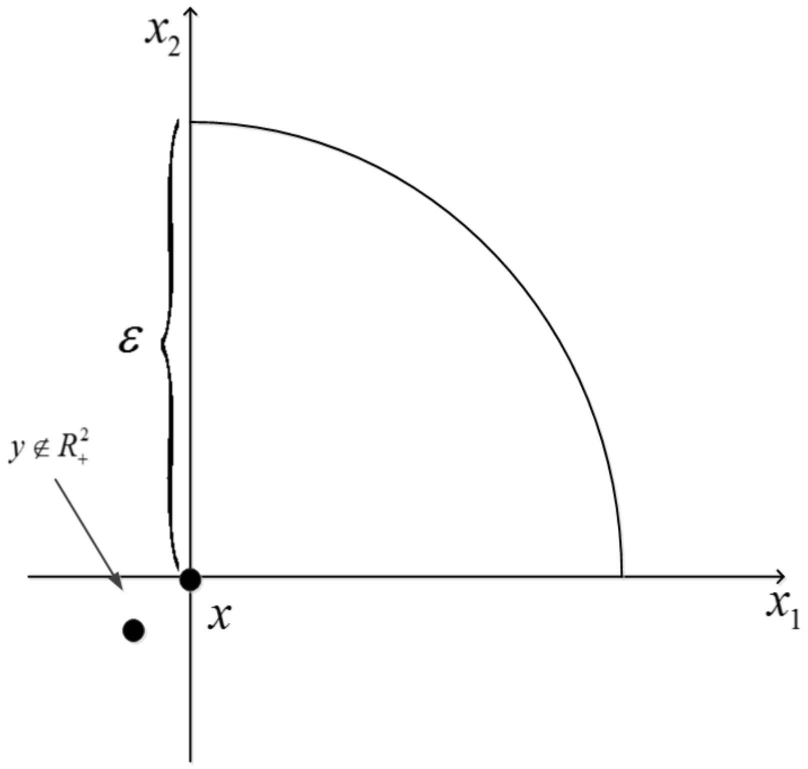
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of *both* goods than bundle x.



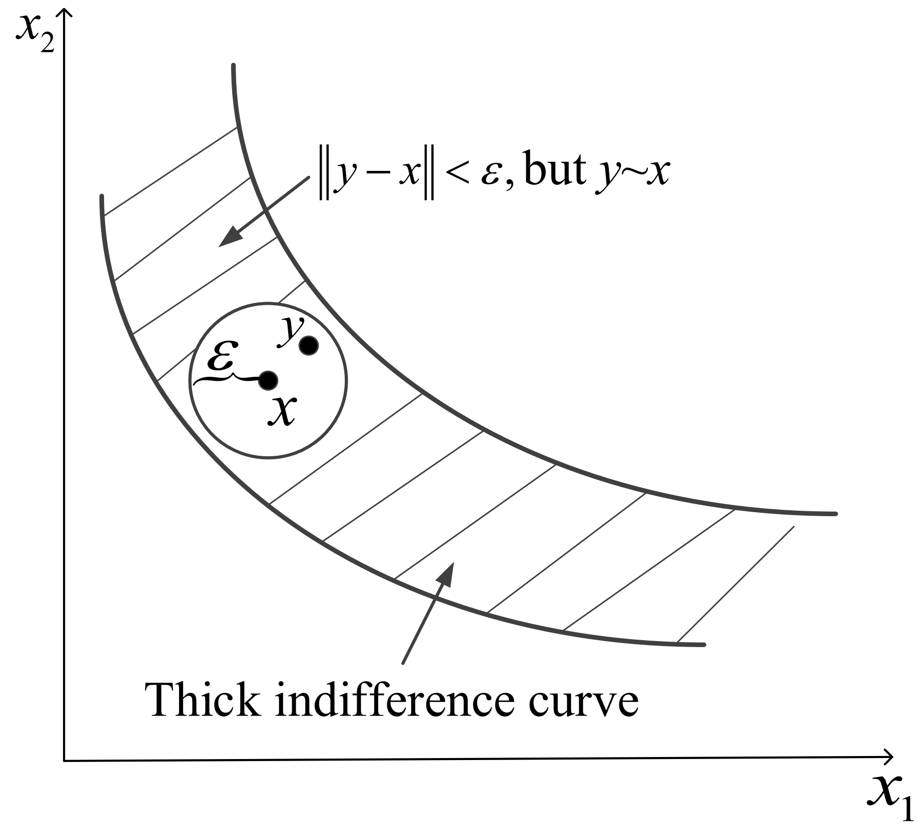
Desirability

- *Violation of LNS*
  - LNS rules out the case in which the decision-maker regards all goods as bads.
  - Although $y \succ x$, y is unfeasible given that it lies away from the consumption set, i.e., $y \notin \mathbb{R}^2_+$.



Desirability

- *Violation of LNS*
  - LNS also rules out "thick" indifference sets.
  - Bundles y and x lie on the same indifference curve.
  - Hence, decision maker is indifferent between x and y, i.e., $y \sim x$.



Desirability

- *Note*:
  - If a preference relation satisfies monotonicity, it must also satisfy LNS.
    - Given a bundle $x = (x_1, x_2)$, increasing all of its components yields a bundle $(x_1 + \delta, x_2 + \delta)$, which is strictly preferred to bundle (x_1, x_2) by monotonicity.
    - Hence, there is a bundle $y = (x_1 + \delta, x_2 + \delta)$ such that $y \succ x$ and $\|y - x\| < \varepsilon$.

Indifference sets

Indifference sets

- The indifference set of a bundle $x \in X$ is the set of all bundles $y \in X$, such that $y \sim x$.

$$IND(x) = \{y \in X: y \sim x\}$$

- The upper-contour set of bundle x is the set of all bundles $y \in X$, such that $y \succsim x$.

$$UCS(x) = \{y \in X: y \succsim x\}$$

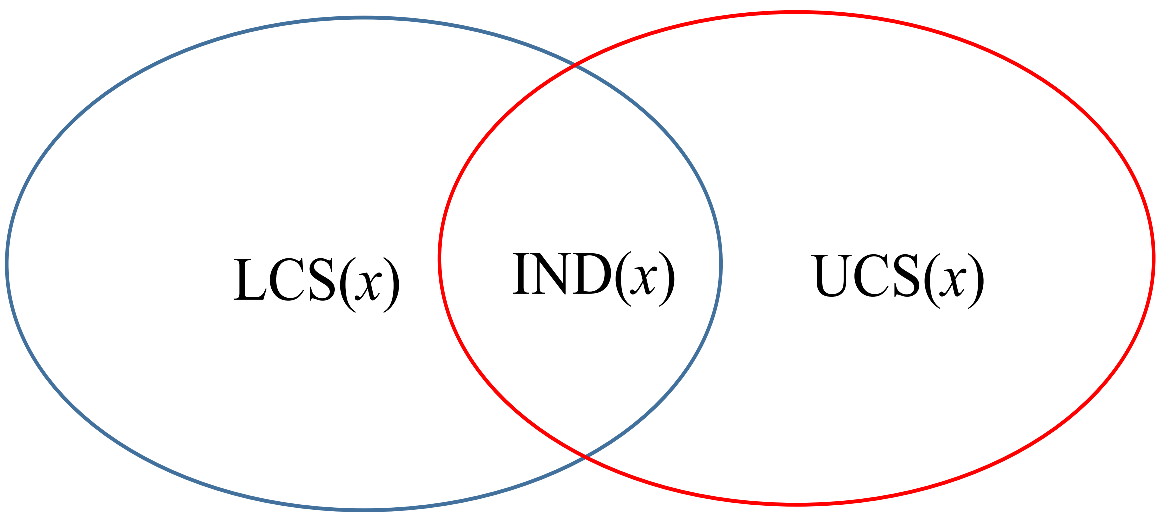
- The lower-contour set of bundle x is the set of all bundles $y \in X$, such that $x \succsim y$.

$$LCS(x) = \{y \in X: x \succsim y\}$$

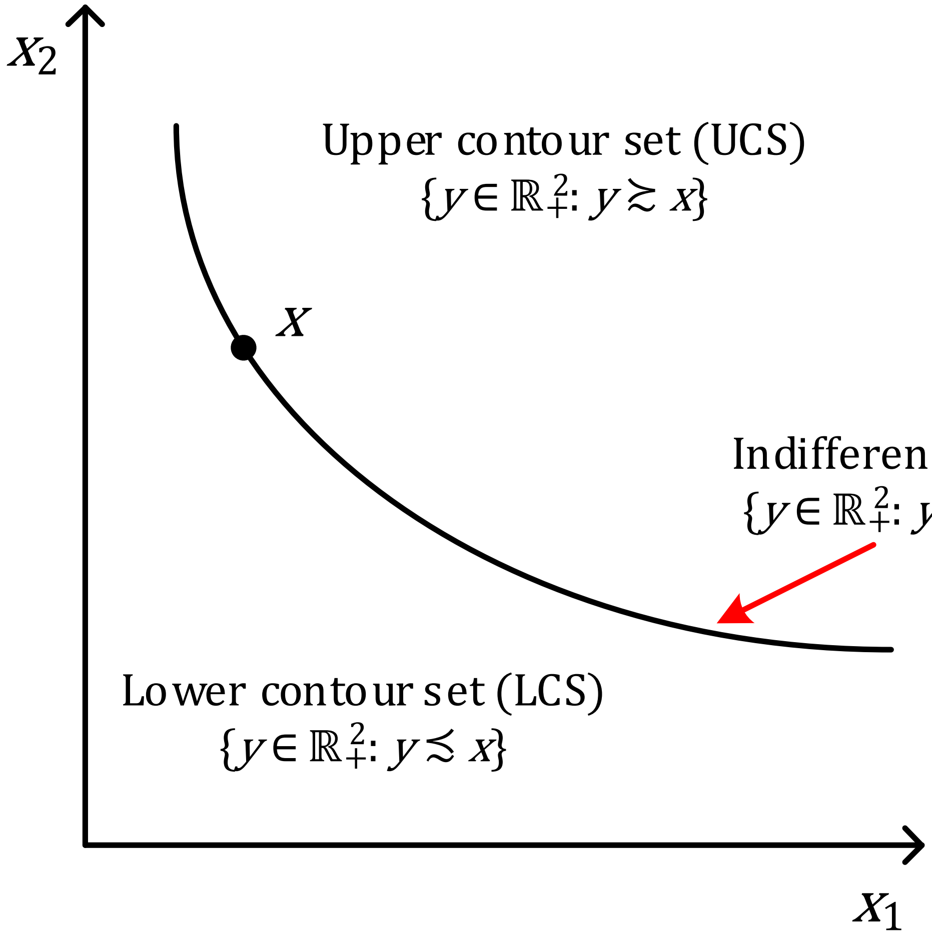
Indifference sets

- Therefore, $IND(x)$ is the intersection of $UCS(x)$ and $LCS(x)$, that is,

$$IND(x) = UCS(x) \cap LCS(x).$$

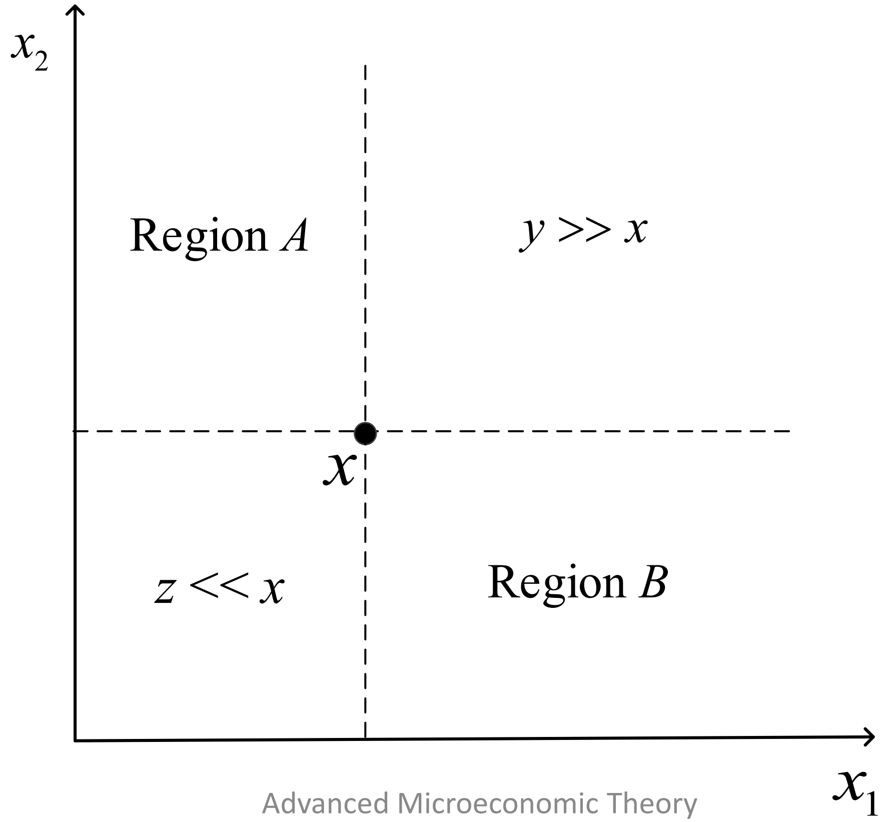


Indifference sets



Indifference sets

- Strong monotonicity implies that indifference curves must be negatively sloped.



Indifference sets

- *Note*:
  - Strong monotonicity implies that indifference curves must be negatively sloped.
  - In contrast, if an individual preference relation satisfies LNS, indifference curves can be upward sloping.
    - This can happen if, for instance, the individual regards good 2 as desirable but good 1 as a bad.

Convexity of Preferences

Convexity of Preferences

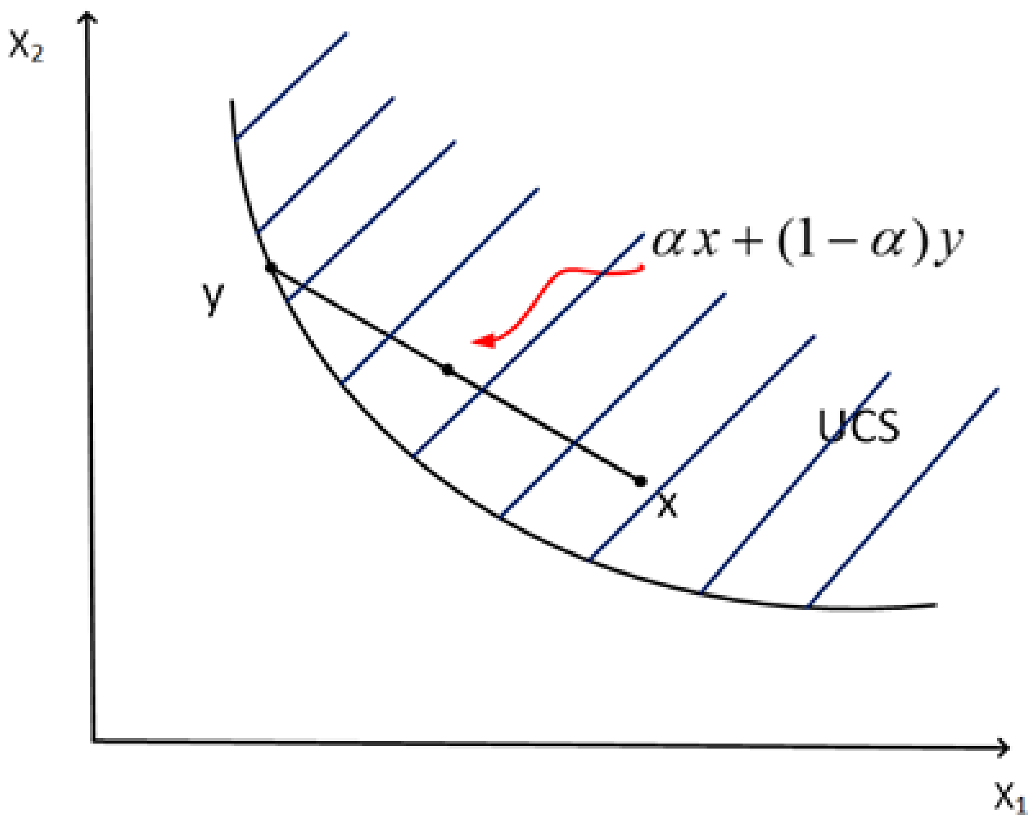
- ***Convexity 1***: A preference relation satisfies convexity if, for all $x, y \in X$,

$$x \succsim y \implies \alpha x + (1-\alpha)y \succsim y$$

for all $\alpha \in (0,1)$.

Convexity of Preferences

- Convexity 1



Convexity of Preferences

- ***Convexity 2***: A preference relation satisfies convexity if, for every bundle x, its upper contour set is convex.

$$UCS(x) = \{y \in X: y \succsim x\} \text{ is convex}$$

- That is, for every two bundles y and z,

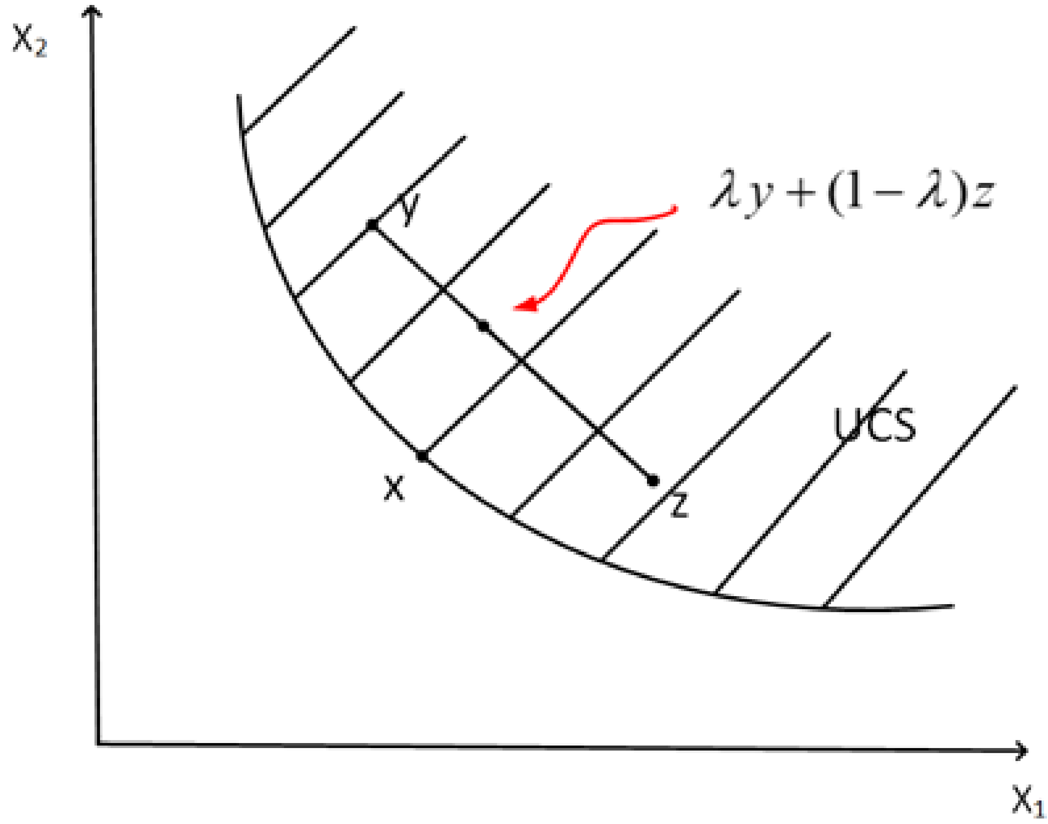
$$\begin{cases} y \succsim x \\ z \succsim x \end{cases} \implies \lambda y + (1-\lambda)z \succsim x$$

for any $\lambda \in (0,1)$.

- Hence, points y, z, and their convex combination belong to the UCS of x.

Convexity of Preferences

- Convexity 2



Convexity of Preferences

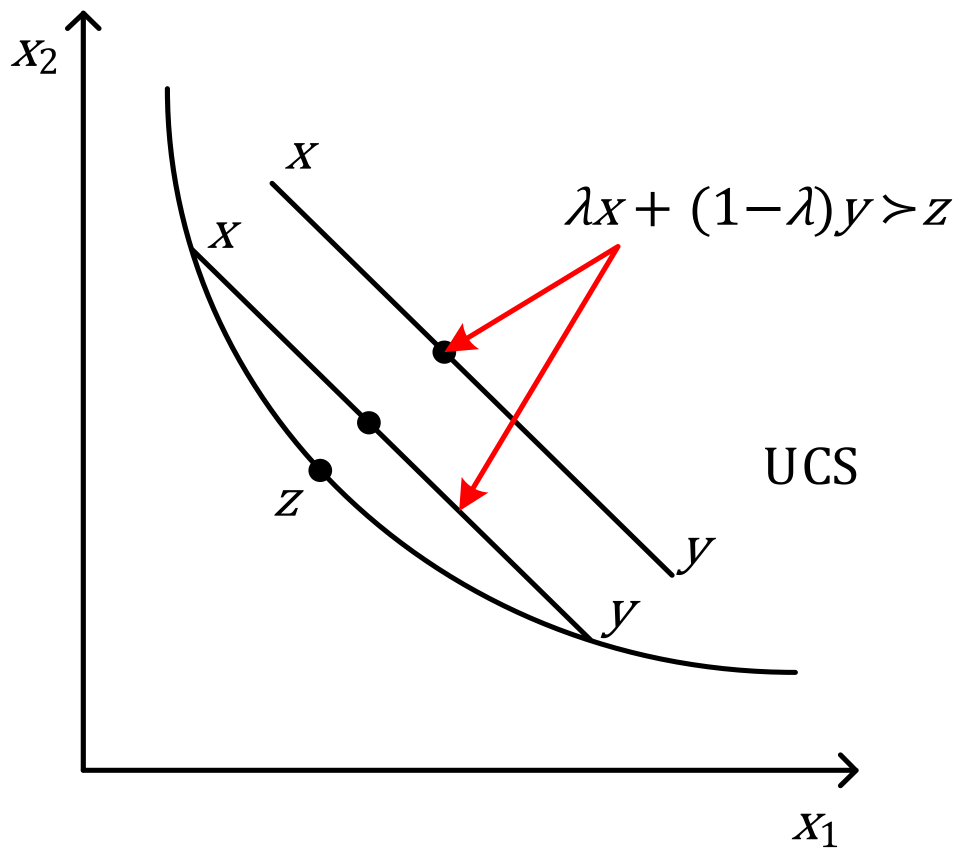
- ***Strict convexity***: A preference relation satisfies strict convexity if, for every $x, y \in X$ where $x \neq y$,

$$\begin{cases} x \succsim z \\ y \succsim z \end{cases} \Longrightarrow \lambda x + (1 - \lambda) y \succ z$$

for all $\lambda \in [0,1]$.

Convexity of Preferences

- Strictly convex preferences

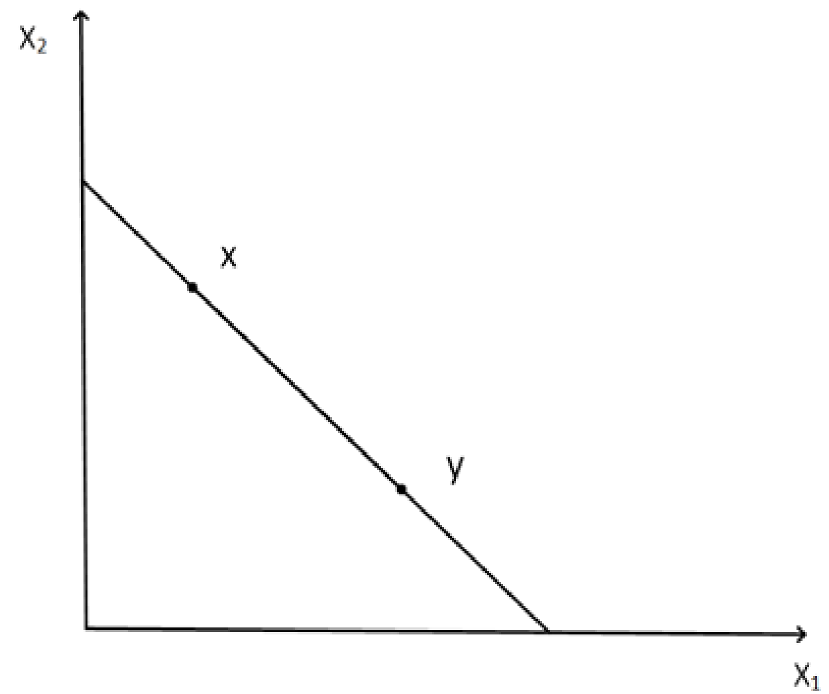


Convexity of Preferences

- **Convex but not strict convex preferences**
  - $\lambda x + (1 - \lambda)y \sim z$
  - This type of preference relation is represented by linear utility functions such as

    $$u(x_1, x_2) = ax_1 + bx_2$$

    where x_1 and x_2 are regarded as substitutes.

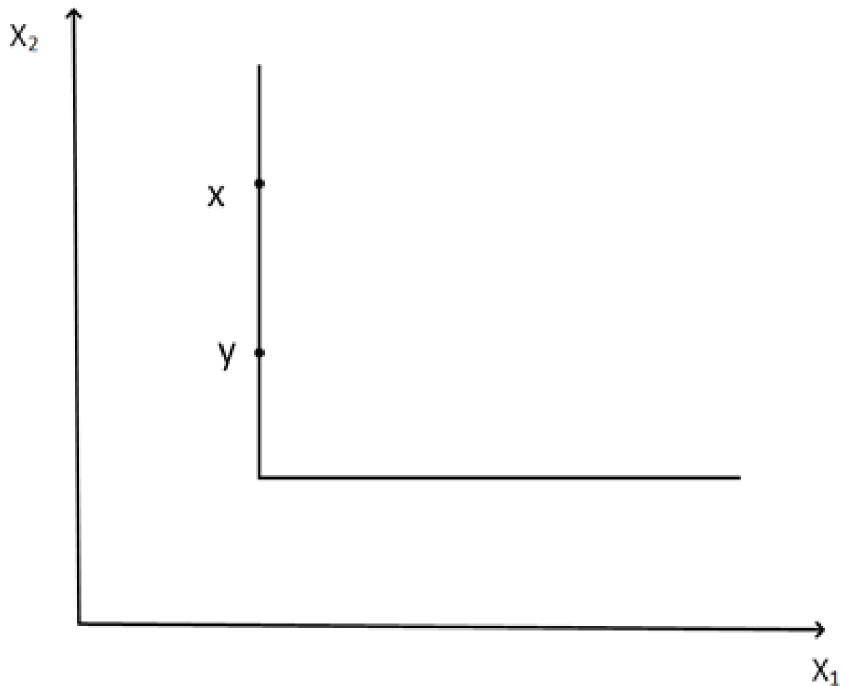


Convexity of Preferences

- **Convex but not strict convex preferences**
  - *Other example*: If a preference relation is represented by utility functions such as

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

where $a, b > 0$, then the pref. relation satisfies convexity, but not strict convexity.



Convexity of Preferences

- ***Example 1.7***

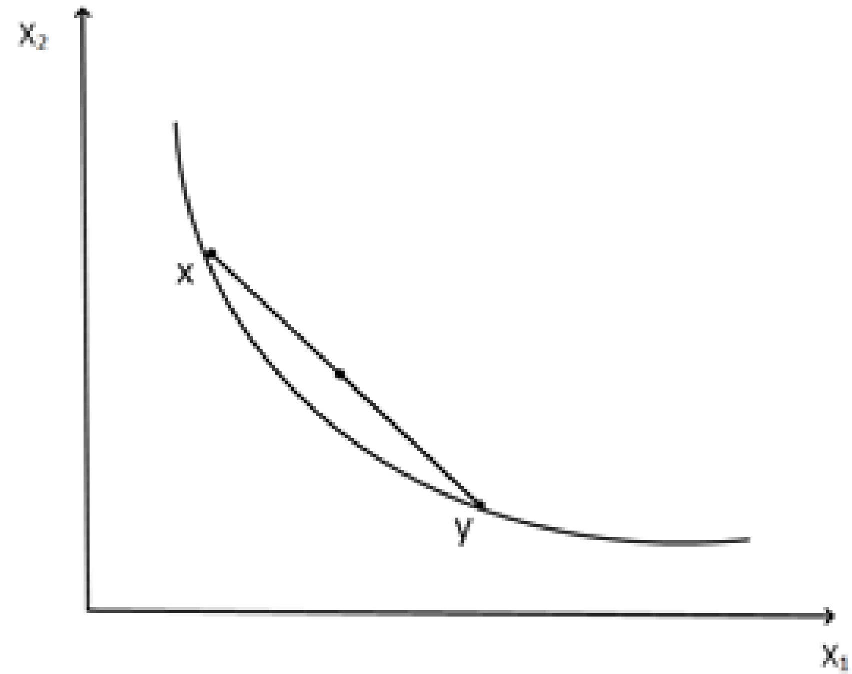
$u(x_1, x_2)$	Satisfies convexity	Satisfies strict convexity
$ax_1 + bx_2$	√	X
$\min\{ax_1, bx_2\}$	√	X
$ax_1^{\frac{1}{2}} \times bx_2^{\frac{1}{2}}$	√	√
$ax_1^2 \times bx_2^2$	√	√
$ax_1^{\frac{1}{2}} + bx_2^{\frac{1}{2}}$	√	√
$ax_1^2 + bx_2^2$	X	X

Convexity of Preferences

- ***Interpretation of convexity***

  1) Taste for diversification:

  – An individual with convex preferences prefers the convex combination of bundles x and y, than either of those bundles alone.



Convexity of Preferences

- ***Interpretation of convexity***

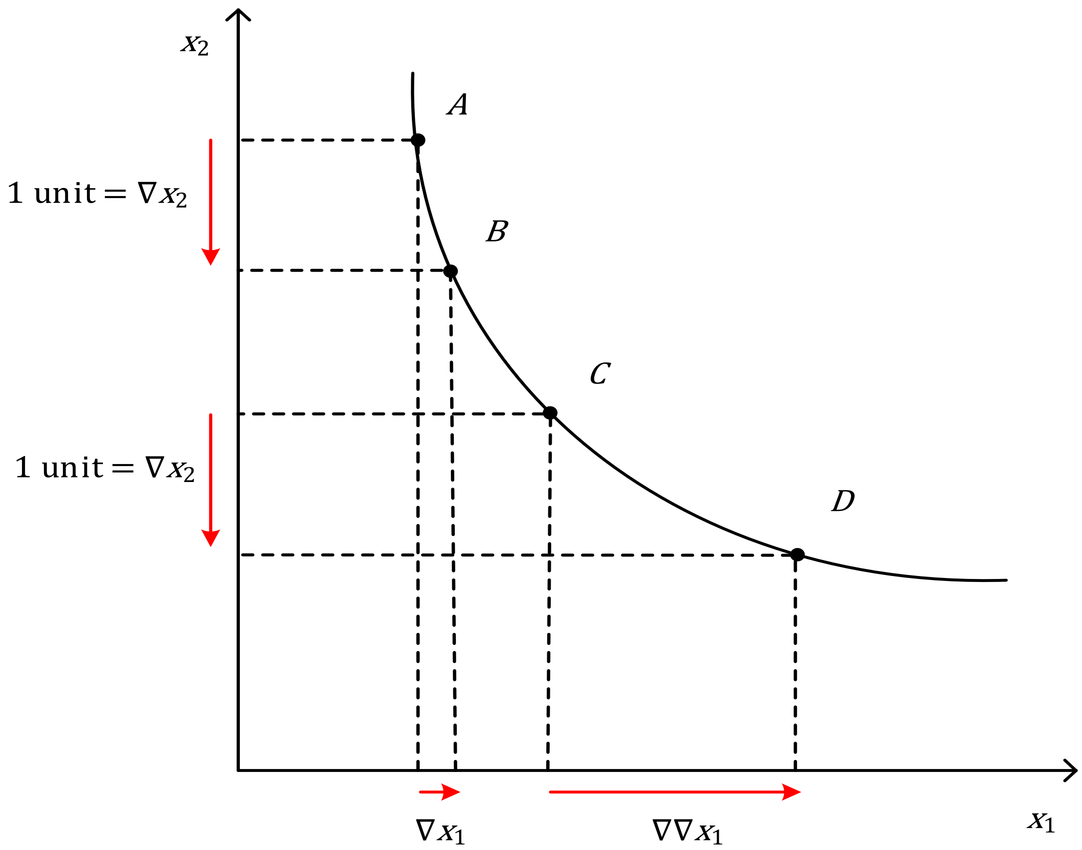
  2) Diminishing marginal rate of substitution:

$$MRS_{1,2} \equiv \frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

  – *MRS* describes the additional amount of good 1 that the consumer needs to receive in order to keep her utility level unaffected, when the amount of good 2 is reduced by one unit.
  – Hence, a *diminishing MRS* implies that the consumer needs to receive increasingly larger amounts of good 1 in order to accept further reductions of good 2.

Convexity of Preferences

- Diminishing marginal rate of substitution



Convexity of Preferences

- *Remark*:
  - Let us show that the slope of the indifference curve is given by the MRS.
  - Consider a continuous and differentiable utility function $u(x_1, x_2, \dots, x_n)$.
  - Totally differentiating, we obtain

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \cdots + \frac{\partial u}{\partial x_n} dx_n$$

  - But since we move along the same indifference curve, $du = 0$.

Convexity of Preferences

– Inserting $du = 0$,

$$0 = \frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial x_j} dx_j$$

$$\text{or} \quad -\frac{\partial u}{\partial x_i} dx_i = \frac{\partial u}{\partial x_j} dx_j$$

– If we want to analyze the rate at which the consumer substitutes units of good i for good j, we must solve for $\frac{dx_j}{dx_i}$, to obtain

$$-\frac{dx_j}{dx_i} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} \equiv MRS_{i,j}$$

Quasiconcavity

Quasiconcavity

- A utility function $u(\cdot)$ is ***quasiconcave*** if, for every bundle $x \in X$, the set of all bundles for which the consumer experiences a higher utility, i.e., the $UCS(x) = \{y \in X \mid u(y) \geq u(x)\}$ is convex.
- The following three properties are equivalent:

$$\text{Convexity of preferences} \Leftrightarrow UCS(x) \text{ is convex} \Leftrightarrow u(\cdot) \text{ is quasiconcave}$$

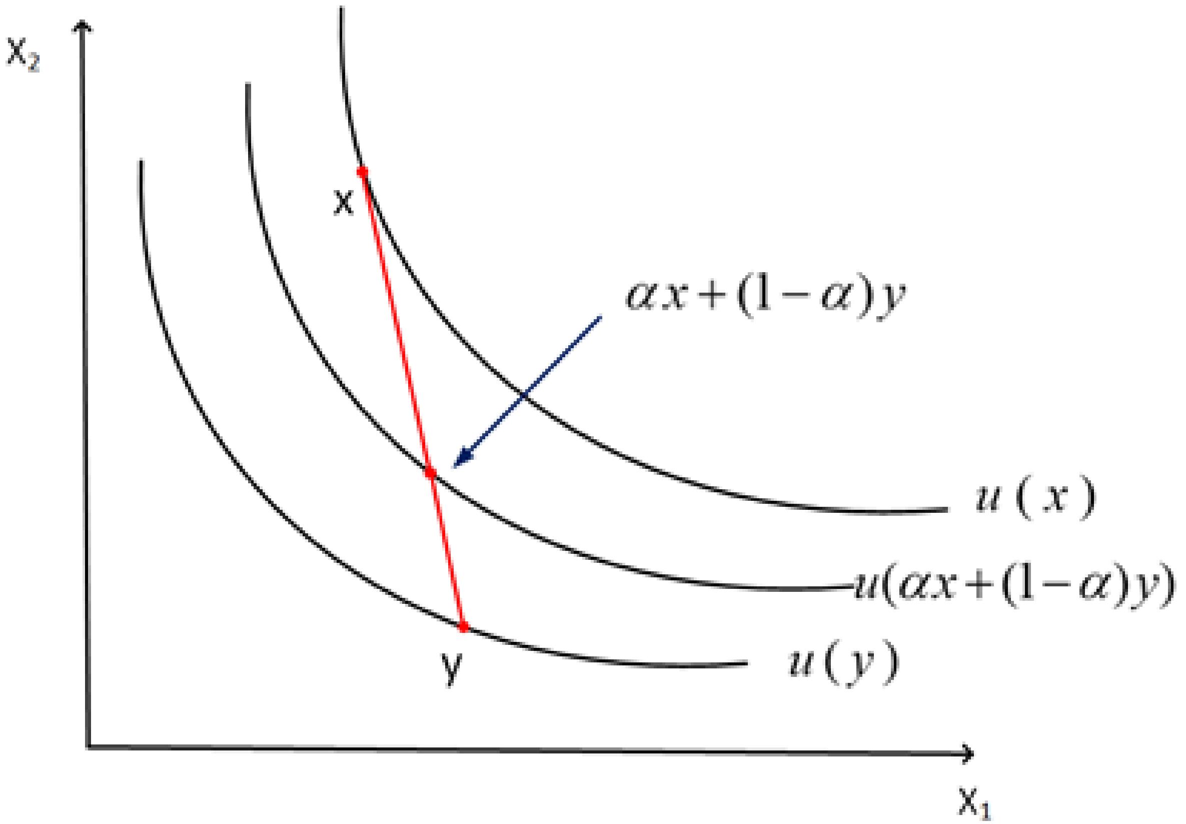
Quasiconcavity

- ***Alternative definition of quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *weakly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$$

Quasiconcavity

- Quasiconcavity (second definition)



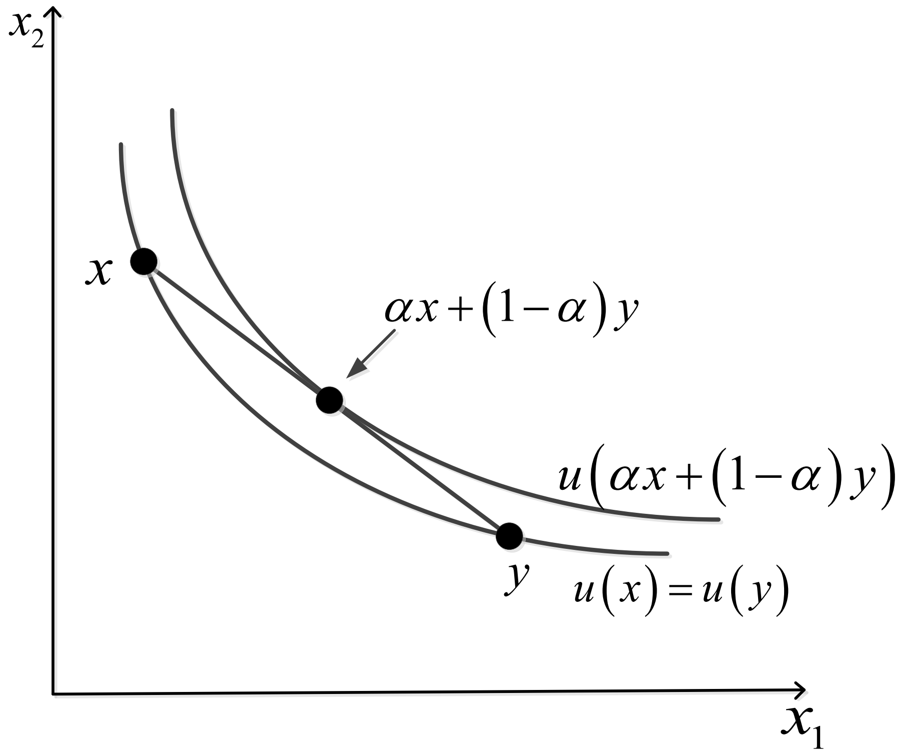
Quasiconcavity

- ***Strict quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *strict quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *strictly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$$

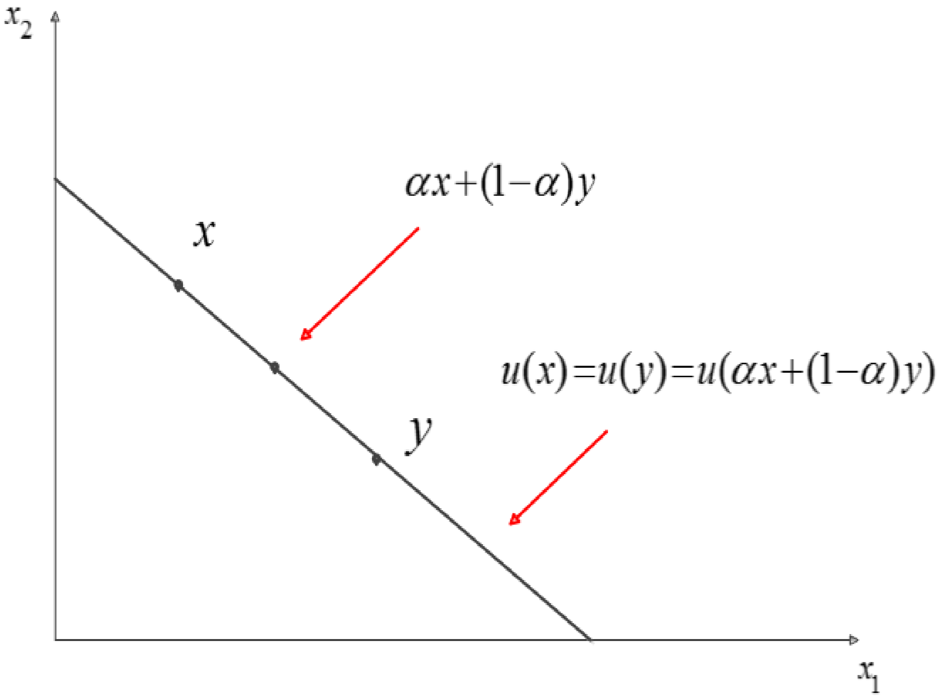
Quasiconcavity

- *What if bundles x and y lie on the same indifference curve?*
- Then, $u(x) = u(y)$.
- Since indifference curves are strictly convex, $u(\cdot)$ satisfies quasiconcavity.



Quasiconcavity

- *What if indifference curves are linear?*
- $u(\cdot)$ satisfies the definition of a quasiconcavity since
$$u(\alpha x + (1-\alpha)y) = \min\{u(x), u(y)\}$$
- But $u(\cdot)$ does not satisfy *strict* quasiconcavity.



Quasiconcavity

- ***Relationship between concavity and quasiconcavity***:

$$\text{Concavity} \underset{\not\Leftarrow}{\overset{\Longrightarrow}{}} \text{Quasiconcavity}$$

  – If a function $f(\cdot)$ is *concave*, then for any two points $x, y \in X$,

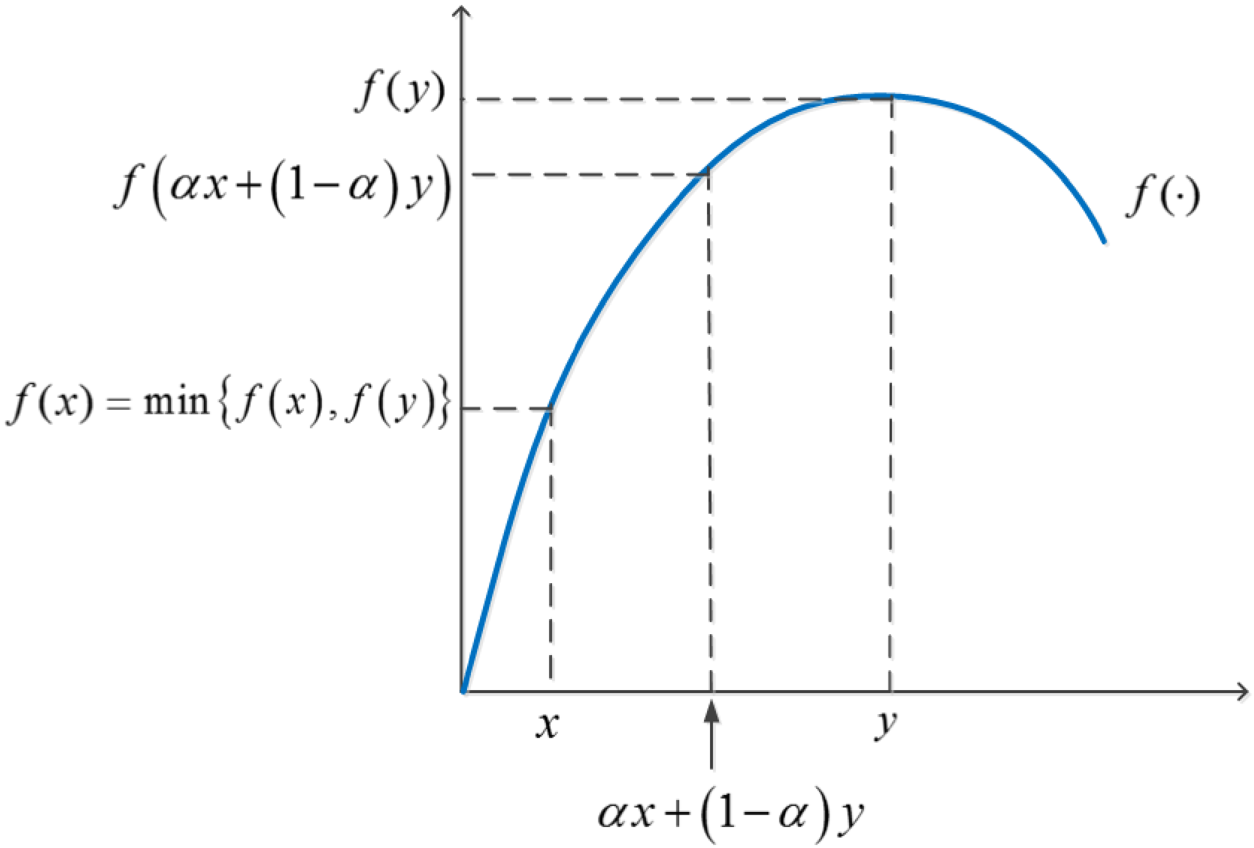
$$\begin{aligned} f(\alpha x + (1-\alpha)y) &\geq \alpha f(x) + (1-\alpha)f(y) \\ &\geq \min\{f(x), f(y)\} \end{aligned}$$

  for all $\alpha \in (0,1)$.

    - The first inequality follows from the definition of concavity, while the second holds true for all concave functions.
    - Hence, quasiconcavity is a weaker condition than concavity.

Quasiconcavity

- Concavity implies quasiconcavity



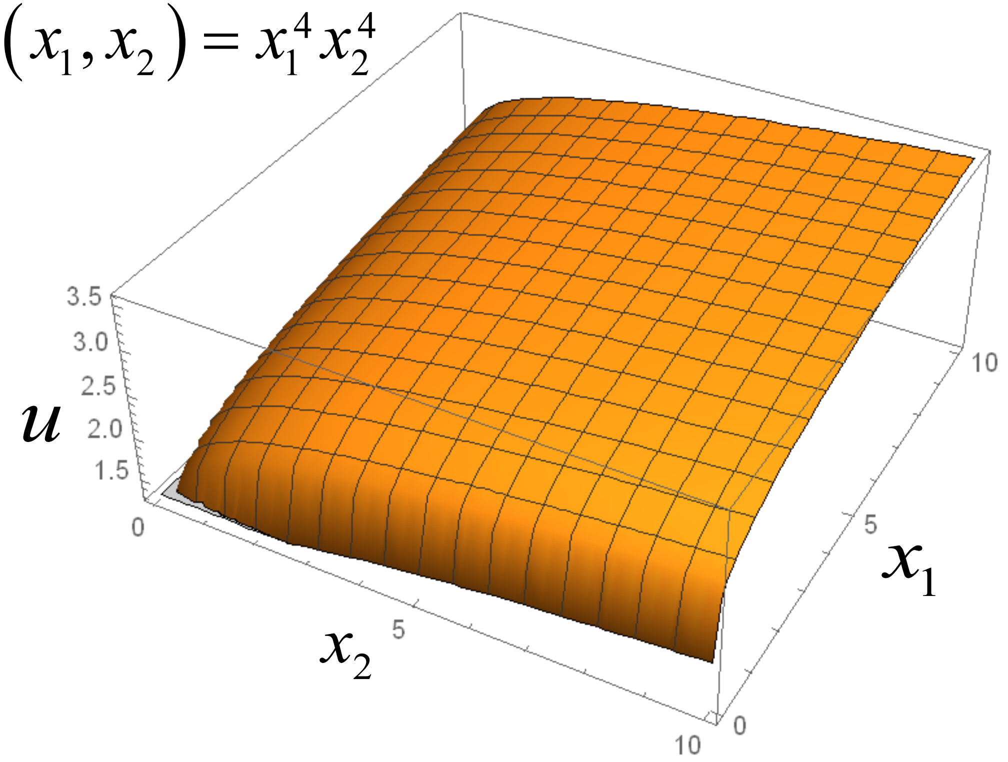
Quasiconcavity

- A concave $u(\cdot)$ exhibits diminishing marginal utility.
  - That is, for an increase in the consumption bundle, the increase in utility is *smaller* as we move away from the origin.
- The "jump" from one indifference curve to another requires:
  - a slight increase in the amount of x_1 and x_2 when we are close to the origin
  - a large increase in the amount of x_1 and x_2 as we get further away from the origin

Quasiconcavity

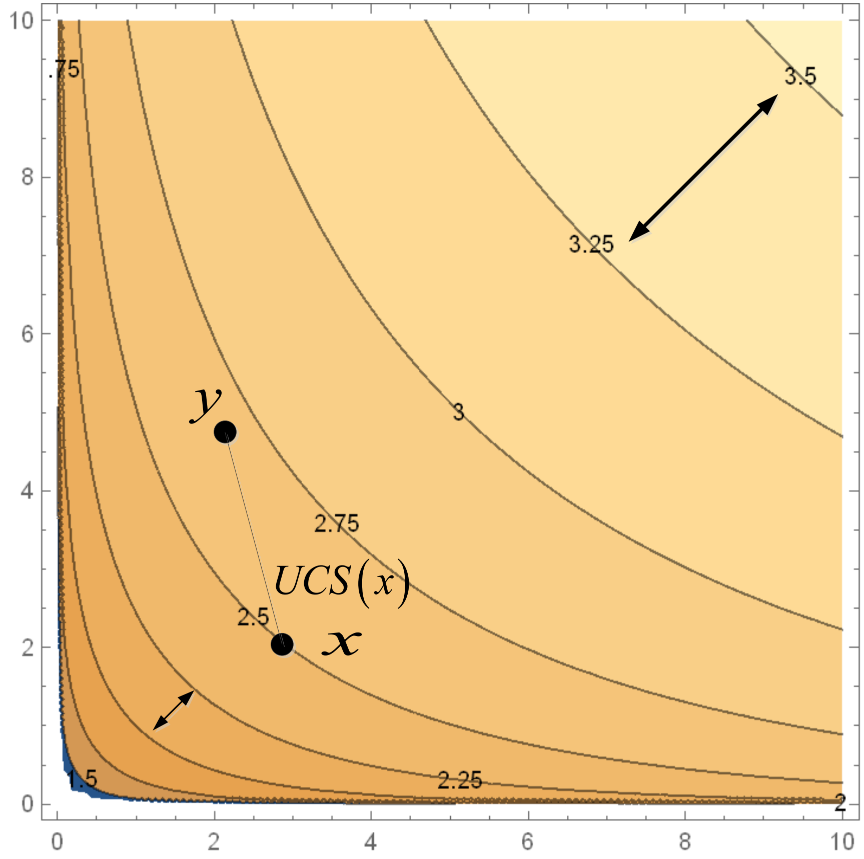
- Concave and quasiconcave utility function (3D)

$$u(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$$



Quasiconcavity

- Concave and quasiconcave utility function (2D)



Quasiconcavity

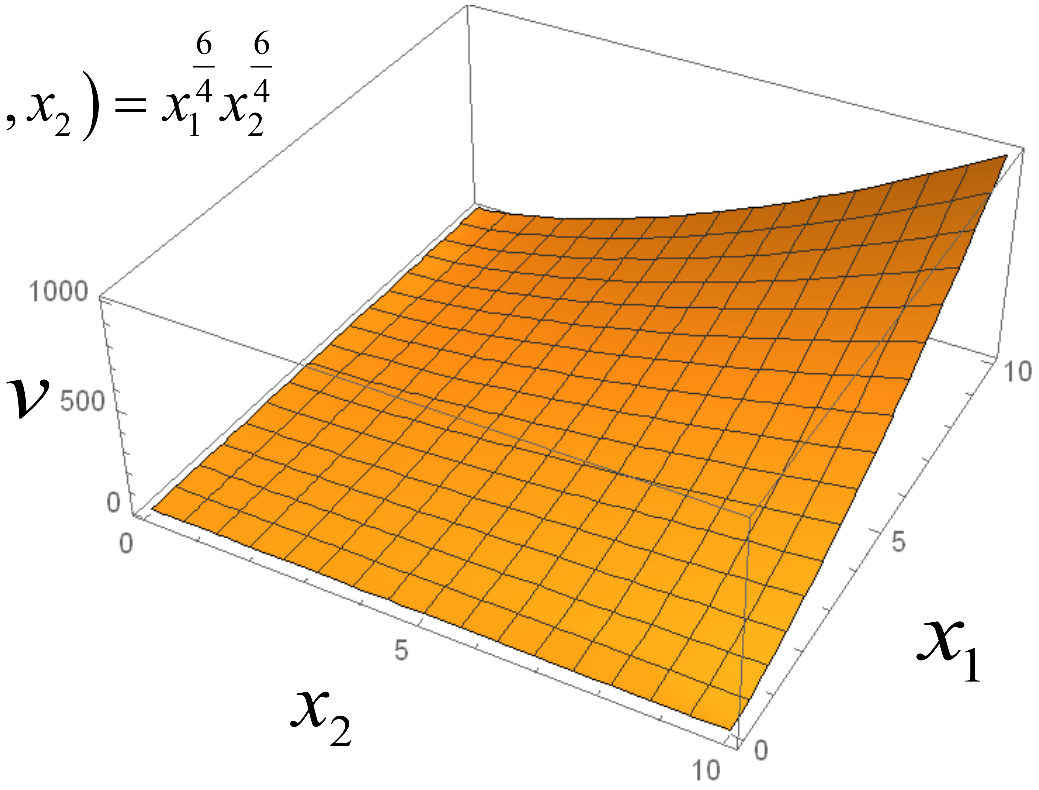
- A convex $u(\cdot)$ exhibits increasing marginal utility.
  - That is, for an increase in the consumption bundle, the increase in utility is *larger* as we move away from the origin.

- The "jump" from one indifference curve to another requires:
  - a large increase in the amount of x_1 and x_2 when we are close to the origin, but…
  - a small increase in the amount of x_1 and x_2 as we get further away from the origin

Quasiconcavity

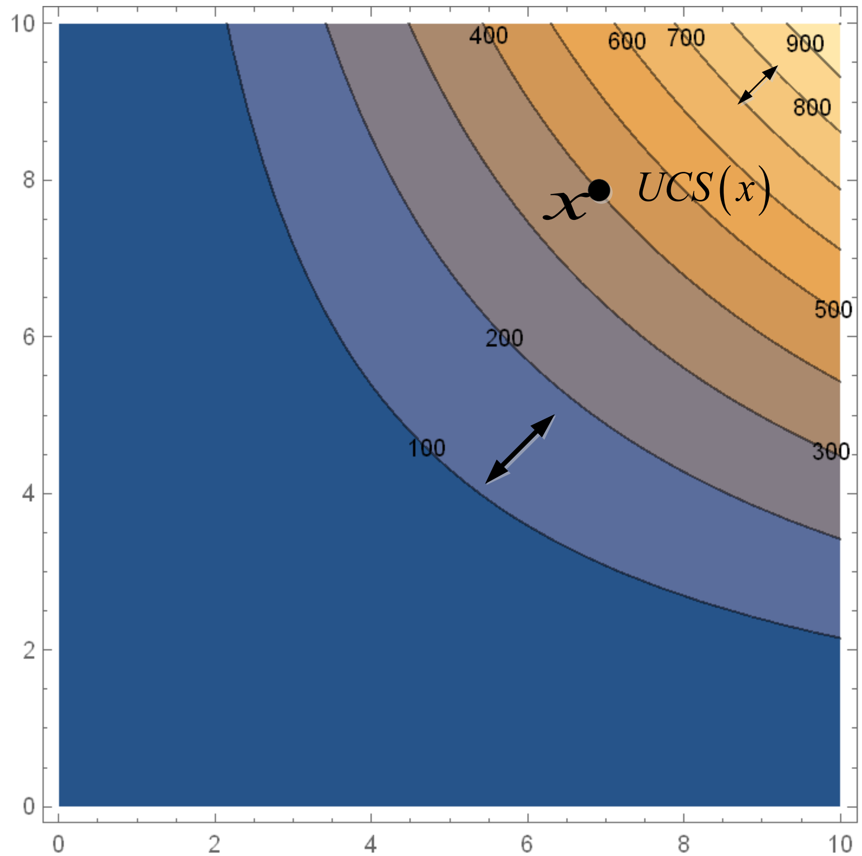
- Convex but quasiconcave utility function (3D)

$$v(x_1, x_2) = x_1^{\frac{6}{4}} x_2^{\frac{6}{4}}$$



Quasiconcavity

- Convex but quasiconcave utility function (2D)



Quasiconcavity

- *Note*:
  - Utility function $v(x_1, x_2) = x_1^{\frac{6}{4}} x_2^{\frac{6}{4}}$ is a strictly monotonic transformation of $u(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$,
    - That is, $v(x_1, x_2) = f(u(x_1, x_2))$, where $f(u) = u^6$.
  - Therefore, utility functions $u(x_1, x_2)$ and $v(x_1, x_2)$ represent the same preference relation.
  - Both utility functions are quasiconcave although one of them is concave and the other is convex.
  - Hence, we normally require utility functions to satisfy quasiconcavity alone.

Quasiconcavity

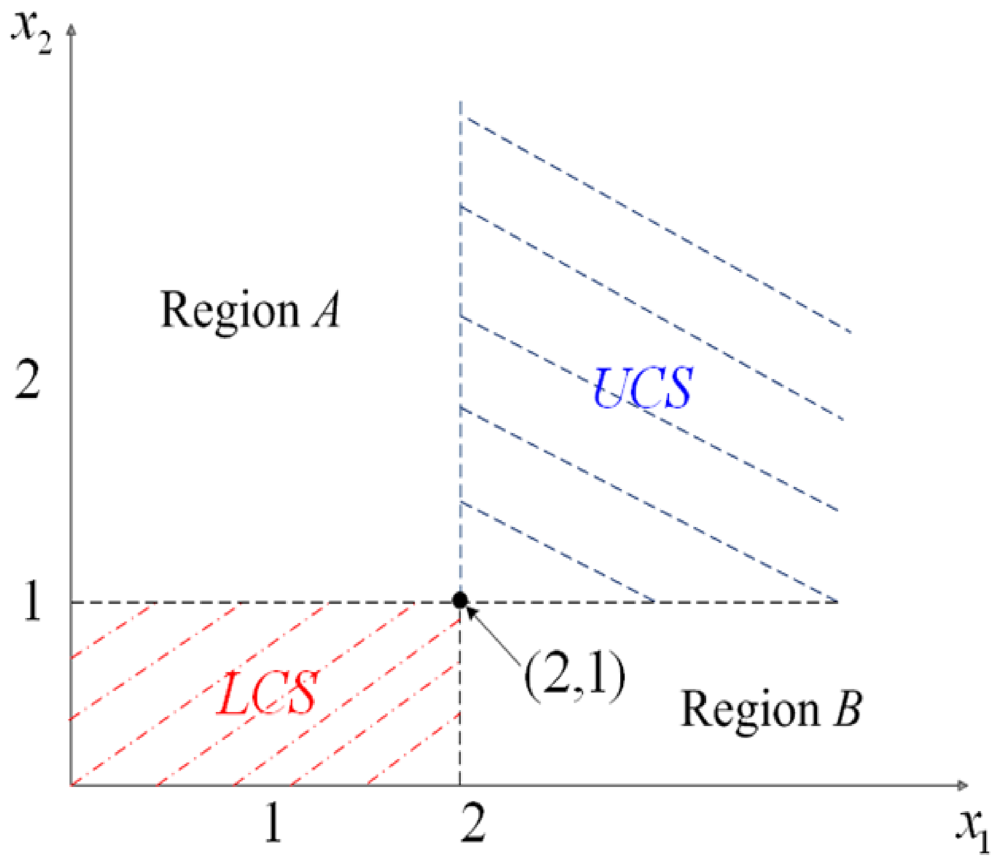
- ***Example 1.8*** (Testing properties of preference relations):
  - Consider an individual decision maker who consumes bundles in $\mathbb{R}_+^L$.
  - Informally, he "prefers more of everything"
  - Formally, for two bundles $x, y \in \mathbb{R}_+^L$, bundle x is weakly preferred to bundle y, $x \succsim y$, iff bundle x contains more units of every good than bundle y does, i.e., $x_k \geq y_k$ for every good k.
  - Let us check if this preference relation satisfies: (a) completeness, (b) transitivity, (c) strong monotonicity, (d) strict convexity, and (e) local non-satiation.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us consider the case of only two goods, $L = 2$.
  - Then, an individual prefers a bundle $x = (x_1, x_2)$ to another bundle $y = (y_1, y_2)$ iff x contains more units of both goods than bundle y, i.e., $x_1 \geq y_1$ and $x_2 \geq y_2$.
  - For illustration purposes, let us take bundle such as $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):



Quasiconcavity

- ***Example 1.8*** (continued):

  1) UCS:

  – The upper contour set of bundle $(2,1)$ contains bundles (x_1, x_2) with weakly more than 2 units of good 1 and/or weakly more than 1 unit of good 2:

  $$UCS(2,1) = \{(x_1, x_2) \succsim (2,1) \Leftrightarrow x_1 \geq 2, x_2 \geq 1\}$$

  – The frontiers of the UCS region also represent bundles preferred to $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):

  2) LCS:

  – The bundles in the lower contour set of bundle $(2,1)$ contain fewer units of both goods:

  $$LCS(2,1) = \{(2,1) \succsim (x_1, x_2) \ \Leftrightarrow x_1 \leq 2, x_2 \leq 1\}$$

  – The frontiers of the LCS region also represent bundles with fewer unis of either good 1 or good 2.

Quasiconcavity

- ***Example 1.8*** (continued):

  3) IND:

  – The indifference set comprising bundles (x_1, x_2) for which the consumer is indifferent between (x_1, x_2) and $(2{,}1)$ is a singleton (itself):

  $$IND(2{,}1) = \{(2{,}1) \sim (x_1, x_2)\} = \{(2{,}1)\}$$

  – There is no other bundle making the consumer indifferent between (2,1) and such a bundle.

  – There is no region for which the UCS and LCS overlap. These sets only “touch” at bundle (2,1).

Quasiconcavity

- ***Example 1.8*** (continued):

  4) Regions A and B:
  - Region A contains bundles with more units of good 2 but fewer units of good 1 (the opposite argument applies to region B).
  - The consumer cannot compare bundles in either of these regions against bundle $(2,1)$.
  - For him to be able to rank one bundle against another, one of the bundles must contain the same or more units of all goods.

Quasiconcavity

- ***Example 1.8*** (continued):

  5) Preference relation is not complete:

  – Completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both).

  – Consider two bundles $x, y \in \mathbb{R}^2_+$ with bundle x containing more units of good 1 than bundle y but fewer units of good 2, i.e., $x_1 > y_1$ and $x_2 < y_2$ (as in Region B)

  – Then, we have neither $x \succsim y$ (UCS) nor $y \succsim x$ (LCS).

Quasiconcavity

- ***Example 1.8*** (continued):

  6) Preference relation is transitive:

  – Transitivity requires that, for any three bundles x, y and z, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

  – Now $x \succsim y$ and $y \succsim z$ means $x_k \geq y_k$ and $y_k \geq z_k$ for all k goods.

  – Then, $x_k \geq z_k$ implies $x \succsim z$.

Quasiconcavity

- ***Example 1.8*** (continued):

  7) Preference relation is strongly monotone:

  – Strong monotonicity requires that if we increase one of the goods in a given bundle y, then the newly created bundle x must be strictly preferred to the original bundle.

  – Now $x \geq y$ and $x \neq y$ implies that $x_l \geq y_l$ for all good l and $x_k > y_k$ for at least one good k.

  – Thus, $x \geq y$ and $x \neq y$ implies $x \succsim y$ and not $y \succsim x$.

  – Thus, we can conclude that $x \succ y$.

Quasiconcavity

- ***Example 1.8*** (continued):

  8) Preference relation is strictly convex:

  – Strict convexity requires that if $x \succsim z$ and $y \succsim z$ and $x \neq y$, then $\alpha x + (1-\alpha)y \succ z$ for all $\alpha \in (0,1)$.

  – Now $x \succsim z$ and $y \succsim z$ implies that $x_l \geq y_l$ and $y_l \geq z_l$ for all good l.

  – $x \neq z$ implies, for some good k, we must have $x_k > z_k$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Hence, for any $\alpha \in (0,1)$, we must have that
    $$\alpha x_l + (1-\alpha)y_l \geq z_l \text{ for every good } l$$
    $$\alpha x_k + (1-\alpha)y_k > z_k \text{ for some } k$$
  - Thus, we have that $\alpha x + (1-\alpha)y \geq z$ and $\alpha x + (1-\alpha)y \neq z$, and so
    $$\alpha x + (1-\alpha)y \succsim z$$
    $$\text{and not } z \succsim \alpha x + (1-\alpha)y$$
  - Therefore, $\alpha x + (1-\alpha)y \succ z$.

Quasiconcavity

- ***Example 1.8*** (continued):

  9) Preference relation satisfies LNS:

  – Take any bundle (x_1, x_2) and a scalar $\varepsilon > 0$.

  – Let us define a new bundle (y_1, y_2) where

  $$(y_1, y_2) \equiv \left(x_1 + \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2}\right)$$

  so that $y_1 > x_1$ and $y_2 > x_2$.

  – Hence, $y \gtrsim x$ but not $x \gtrsim y$, which implies $y \succ x$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us know check if bundle y is within an ε-ball around x.
  - The Cartesian distance between x and y is

$$\|x-y\| = \sqrt{\left[x_1 - \left(x_1 + \frac{\varepsilon}{2}\right)\right]^2 + \left[x_2 - \left(x_2 + \frac{\varepsilon}{2}\right)\right]^2} = \frac{\varepsilon}{\sqrt{2}}$$

which is smaller than ε for all $\varepsilon > 0$.

Common Utility Functions

Common Utility Functions

- ***Cobb-Douglas utility functions***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$$
  where $A, \alpha, \beta > 0$.
  - Applying logs on both sides
  $$\log u = \log A + \alpha \log x_1 + \beta \log x_2$$
  - Hence, the exponents in the original $u(\cdot)$ can be interpreted as *elasticities*:

$$\varepsilon_{u,x_1} = \frac{\partial u(x_1,x_2)}{\partial x_1} \cdot \frac{x_1}{u(x_1,x_2)} = \alpha A x_1^{\alpha-1} x_2^{\beta} \cdot \frac{x_1}{Ax_1^{\alpha} x_2^{\beta}} = \alpha$$

Common Utility Functions

– Intuitively, a one-percent increase in the amount of good x_1 increases individual utility by α percent.

– Similarly, $\varepsilon_{u,x_2} = \beta$.

– Special cases:
  - $\alpha + \beta = 1$: $u(x_1, x_2) = Ax_1^{\alpha} x_2^{1-\alpha}$
  - $A = 1$: $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$
  - $A = \alpha = \beta = 1$: $u(x_1, x_2) = x_1 x_2$

Common Utility Functions

– Marginal utilities:

$$\frac{\partial u}{\partial x_1} > 0 \text{ and } \frac{\partial u}{\partial x_2} > 0$$

– Diminishing MRS, since

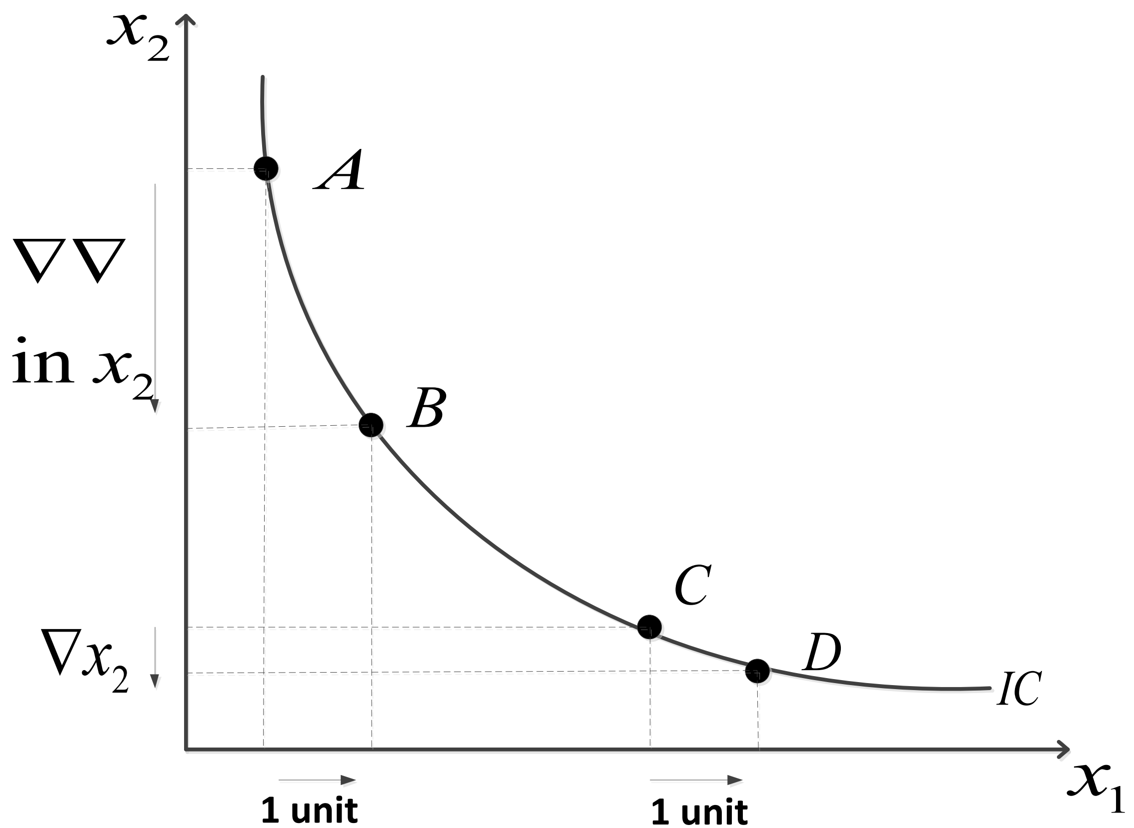
$$MRS_{x_1,x_2} = \frac{\alpha A x_1^{\alpha-1} x_2^{\beta}}{\beta A x_1^{\alpha} x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1}$$

which is decreasing in x_1.

- Hence, indifference curves become flatter as x_1 increases.

Common Utility Functions

- Cobb-Douglas preference

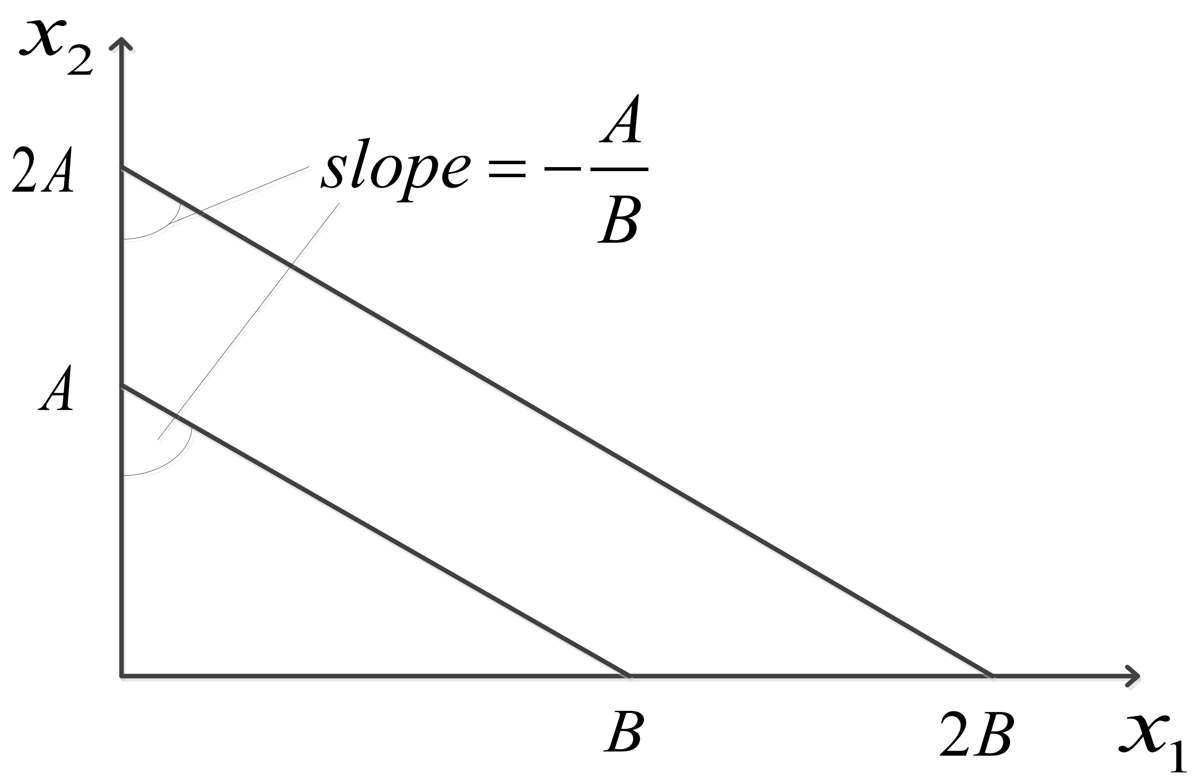


Common Utility Functions

- ***Perfect substitutes***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = Ax_1 + Bx_2$$
  where $A, B > 0$.
  - Hence, the marginal utility of every good is constant:
  $$\frac{\partial u}{\partial x_1} = A \text{ and } \frac{\partial u}{\partial x_2} = B$$
  - MRS is also constant, i.e., $MRS_{x_1,x_2} = \frac{A}{B}$
    - Therefore, indifference curves are straight lines with a slope of $-\frac{A}{B}$.

Common Utility Functions

- Perfect substitutes



Common Utility Functions

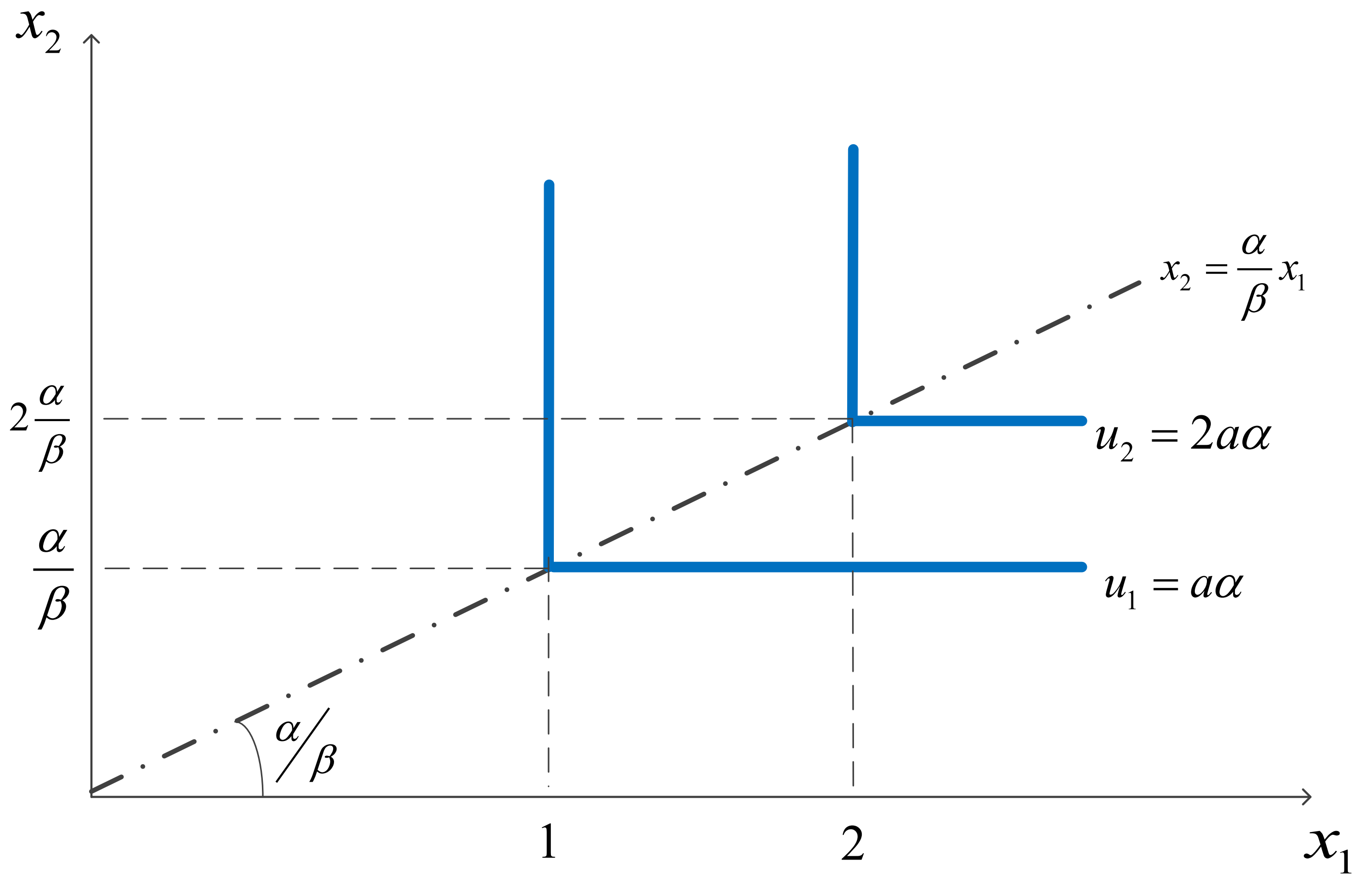
- Intuitively, the individual is willing to give up $\frac{A}{B}$ units of x_2 to obtain one more unit of x_1 and keep his utility level unaffected.
- Unlike in the Cobb-Douglas case, such willingness is independent in the relative abundance of the two goods.
- *Examples*: butter and margarine, coffee and black tea, or two brands of unflavored mineral water

Common Utility Functions

- ***Perfect Complements***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = A \cdot \min\{\alpha x_1, \beta x_2\}$$
  where $A, \alpha, \beta > 0$.
  - Intuitively, increasing one of the goods without increasing the amount of the other good entails *no* increase in utility.
    - The amounts of *both* goods must increase for the utility to go up.
  - The indifference curve is right angle with a kink at $\alpha x_1 = \beta x_2$.

Common Utility Functions

- Perfect complements



Common Utility Functions

– The slope of a ray $x_2 = \frac{\beta}{\alpha} x_1$, $\frac{\beta}{\alpha}$, indicates the rate at which goods x_1 and x_2 must be consumed in order to achieve utility gains.

– Special case: $\alpha = \beta$

$$\begin{aligned} u(x_1, x_2) &= A \cdot \min\{\alpha x_1, \alpha x_2\} \\ &= A\alpha \cdot \min\{x_1, x_2\} \\ &= B \cdot \min\{x_1, x_2\} \text{ if } B \equiv A\alpha \end{aligned}$$

– *Examples*: cars and gasoline, or peanut butter and jelly. Other food recipes, which often require the use of ingredients in a fixed proportion, are good examples as well.

Common Utility Functions

- ***CES utility function***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = \left[a x_1^{\frac{\sigma-1}{\sigma}} + b x_2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

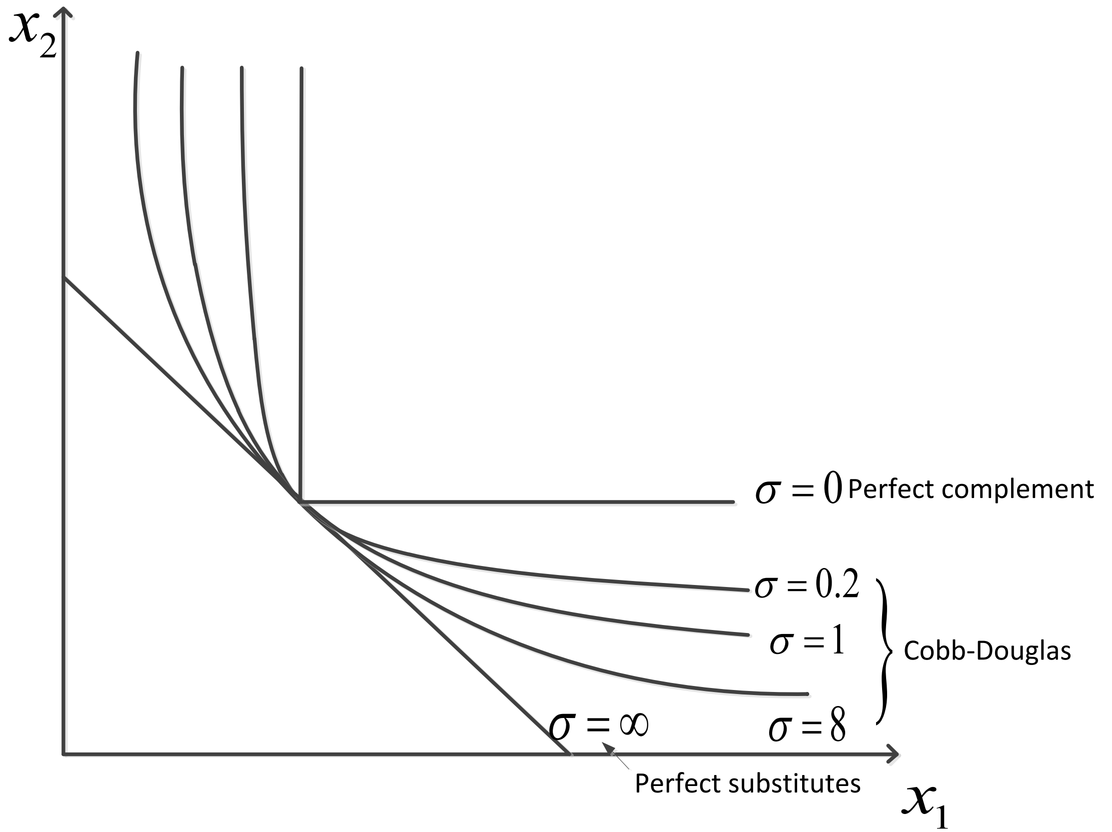
    where σ measures the elasticity of substitution between goods x_1 and x_2.

  - In particular,

$$\sigma = \frac{\partial\left(\frac{x_2}{x_1}\right)}{\partial MRS_{1,2}} \cdot \frac{MRS_{1,2}}{\frac{x_2}{x_1}}$$

Common Utility Functions

- CES preferences



Common Utility Functions

– CES utility function is often presented as

$$u(x_1, x_2) = \left[ax_1^{\rho} + bx_2^{\rho}\right]^{\frac{1}{\rho}}$$

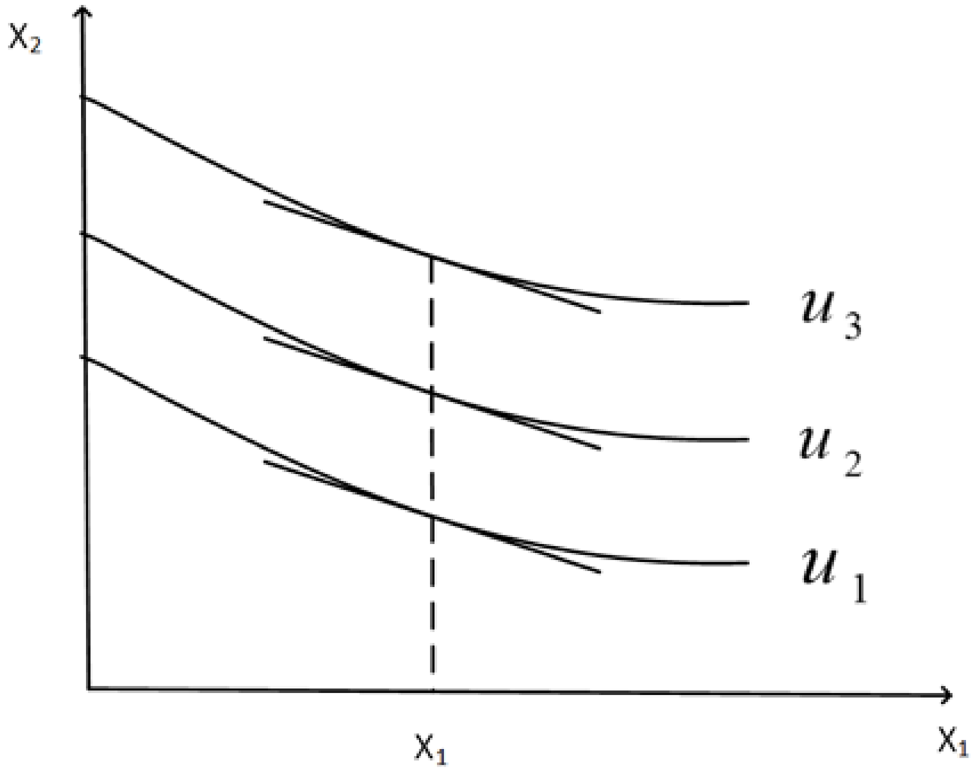
where $\rho \equiv \frac{\sigma - 1}{\sigma}$.

Common Utility Functions

- ***Quasilinear utility function***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = v(x_1) + bx_2$$
  where x_2 enters *linearly*, $b > 0$, and $v(x_1)$ is a *nonlinear* function of x_1.
    - For example, $v(x_1) = a \ln x_1$ or $v(x_1) = ax_1^{\alpha}$, where $a > 0$ and $\alpha \neq 1$.
  - The MRS is constant in the good that enters linearly in the utility function (x_2 in our case).

Common Utility Functions

- MRS of quasilinear preferences



Common Utility Functions

– For $u(x_1, x_2) = v(x_1) + bx_2$, the marginal utilities are

$$\frac{\partial u}{\partial x_2} = b \text{ and } \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1}$$

which implies

$$MRS_{x_1,x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

which is constant in the good entering linearly, x_2

– Quasilinear preferences are often used to represent the consumption of goods that are relatively insensitive to income.

– *Examples*: garlic, toothpaste, etc.

Properties of Preference Relations

Properties of Preference Relations

- ***Homogeneity***:
  - A utility function is *homogeneous of degree* k if varying the amounts of all goods by a common factor $\alpha > 0$ produces an increase in the utility level by α^k.
  - That is, for the case of two goods,
$$u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$$
where $\alpha > 0$. This allows for:
    - $\alpha > 1$ in the case of a common increase
    - $0 < \alpha < 1$ in the case of a common decrease

Properties of Preference Relations

– Three properties:

1) The first-order derivative of a function $u(x_1, x_2)$ which is homogeneous of degree k is homogeneous of degree $k-1$.

- Given $u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$, we can show that

$$\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_i} \cdot \alpha = \alpha^k \cdot \frac{\partial u(x_1, x_2)}{\partial x_i}$$

or re-arranging

$$u'(\alpha x_1, \alpha x_2) = \alpha^{k-1} u'(x_1, x_2)$$

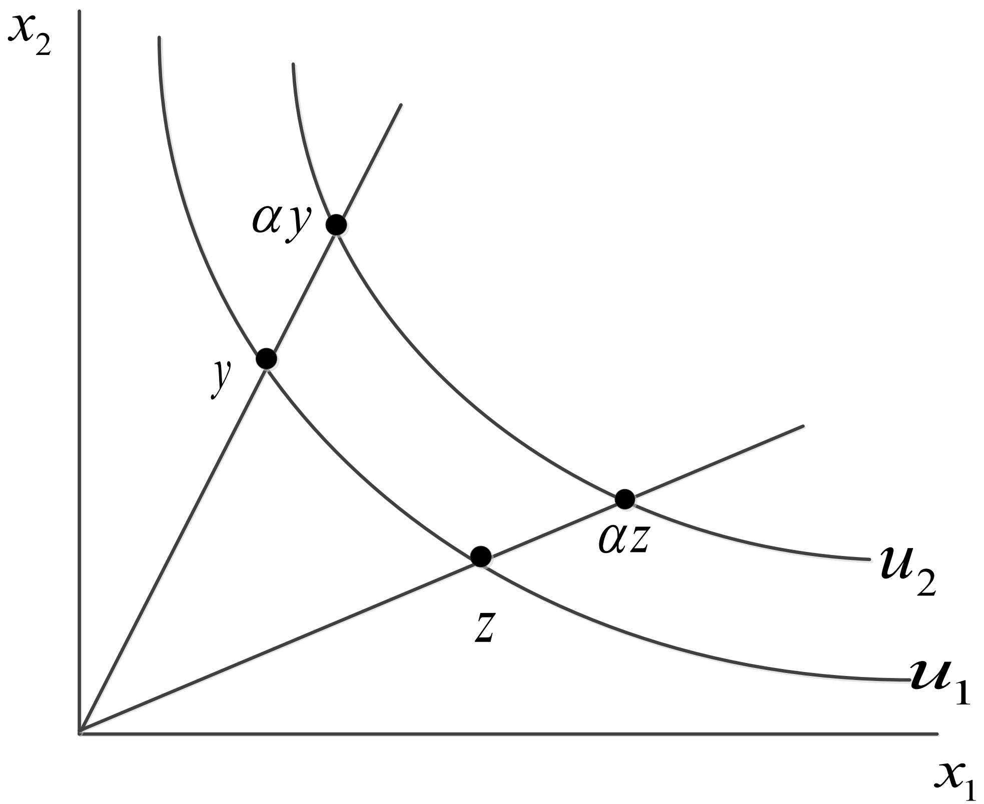
Properties of Preference Relations

2) The indifference curves of homogeneous functions are radial expansions of one another.

- That is, if two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$.

Properties of Preference Relations

- Homogenous preference



Properties of Preference Relations

3) *The MRS of a homogeneous function is constant for all points along each ray from the origin.*

- That is, the slope of the indifference curve at point y coincides with the slope at a "scaled-up version" of point y, αy, where $\alpha > 1$.
- The MRS at bundle $x = (x_1, x_2)$ is

$$MRS_{1,2}(x_1, x_2) = \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

- The MRS at $(\alpha x_1, \alpha x_2)$ is

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

$$= \frac{\alpha^{k-1} \frac{\partial u(x_1, x_2)}{\partial x_1}}{\alpha^{k-1} \frac{\partial u(x_1, x_2)}{\partial x_2}} = \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

where the second equality uses the first property.

- Hence, the MRS is unaffected along all the points crossed by a ray from the origin.

Properties of Preference Relations

- ***Homotheticity***:
  - A utility function $u(x)$ is homothetic if it is a monotonic transformation of a homogeneous function.
  - That is, $u(x) = g(v(x))$, where
    - $g: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function, and
    - $v: \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree k.

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
    - In particular,

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k \, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k \, v(z))$$

Properties of Preference Relations

- The MRS of a homothetic function is homogeneous of degree zero.
- In particular,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial v} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial g}{\partial v} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

- Canceling the $\frac{\partial g}{\partial v}$ terms yields

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_1}}{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

– In summary,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} =$$

$$= \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = MRS_{1,2}(x_1, x_2)$$

Properties of Preference Relations

- ***Homotheticity (graphical interpretation)***
  - A preference relation on $X = \mathbb{R}^L_+$ is homothetic if all indifference sets are related to proportional expansions along the rays.
  - That is, if the consumer is indifferent between bundles x and y, i.e., $x \sim y$, he must also be indifferent between a common scaling in these two bundles, i.e., $\alpha x \sim \alpha y$, for every scalar $\alpha > 0$.

Properties of Preference Relations

– For a given ray from the origin, the slope of the indifference curves (i.e., the MRS) that the ray crosses coincides.
  - The ratio between the two goods x_1/x_2 remains constant along all points in the ray.

– Intuitively, the rate at which a consumer is willing to substitute one good for another (his MRS) only depends on:
  - the rate at which he consumes the two goods, i.e., x_1/x_2, but does not depend on the utility level he obtains.

– But it is independent in the volume of goods he consumes, and in the utility he achieves.

Properties of Preference Relations

- ***Homogeneity and homotheticity***:
  - Homogeneous functions are homothetic.
    - We only need to apply a monotonic transformation $g(\cdot)$ on $v(x_1, x_2)$, i.e., $u(x_1, x_2) = g(v(x_1, x_2))$.
  - But homothetic functions are not necessarily homogeneous.
    - Take a homogeneous (of degree two) function $v(x_1, x_2) = x_1 x_2$.
    - Apply a monotonic transformation $g(y) = y + a$, where $a > 0$, to obtain homothetic function

$$u(x_1, x_2) = x_1 x_2 + a$$

Properties of Preference Relations

- This function is not homogeneous, since increasing all arguments by α yields

$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) + a$$
$$= \alpha^2 v(x_1, x_2) + a$$

- Other monotonic transformations yielding non-homogeneous utility functions are

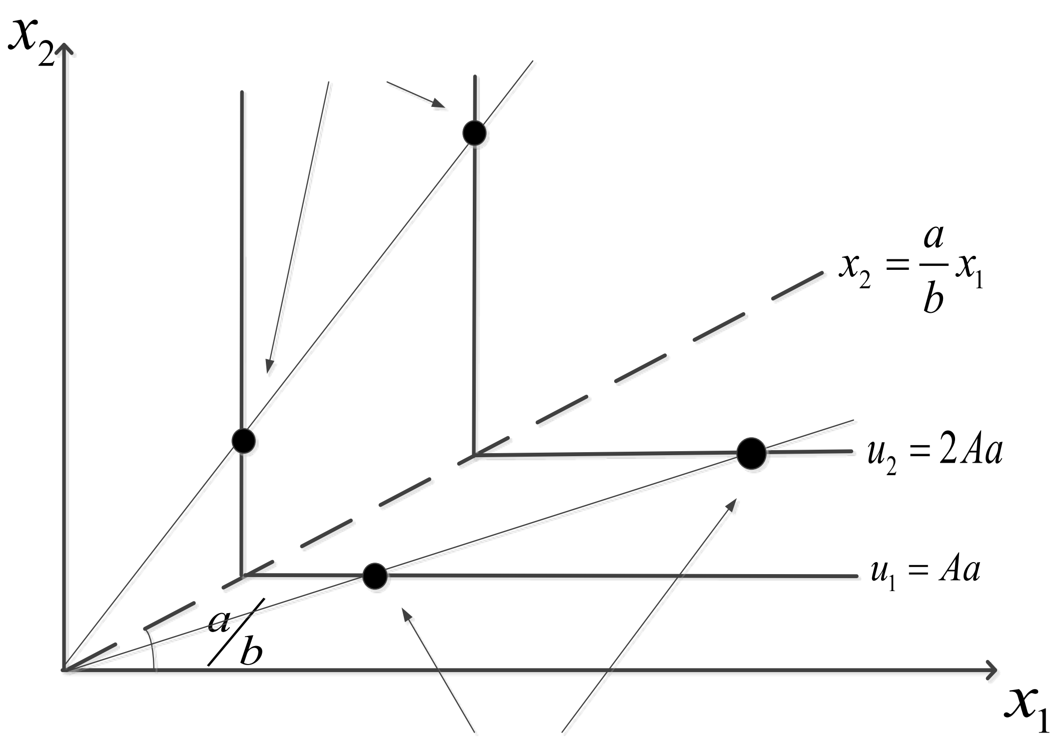
$$g(y) = ay^\gamma + by, \text{ where } a, b, \gamma > 0, \quad \text{or}$$
$$g(y) = a \ln y, \text{ where } a > 0$$

Properties of Preference Relations

- Utility functions that satisfy homotheticity:
  - Linear utility function $u(x_1, x_2) = ax_1 + bx_2$, where $a, b > 0$
    - Goods x_1 and x_2 are perfect substitutes
    - $MRS(x_1, x_2) = \frac{a}{b}$ and $MRS(tx_1, tx_2) = \frac{at}{bt} = \frac{a}{b}$
  - The Leontief utility function $u(x_1, x_2) = A \cdot \min\{ax_1, bx_2\}$, where $A > 0$
    - Goods x_1 and x_2 are perfect complements
    - We cannot define the MRS along all the points of the indifference curves
    - However, the slope of the indifference curves coincide for those points where these curves are crossed by a ray from the origin.

Properties of Preference Relations

- Perfect complements and homotheticity



Properties of Preference Relations

- ***Example 1.9*** (Testing for quasiconcavity and homotheticity):
  - Let us determine if $u(x_1, x_2) = \ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave, homothetic, both or neither.
  - *Quasiconcavity*:
    - Note that $\ln(x_1^{0.3} x_2^{0.6})$ is a monotonic transformation of the Cobb-Douglas function $x_1^{0.3} x_2^{0.6}$.
    - Since $x_1^{0.3} x_2^{0.6}$ is a Cobb-Douglas function, where $\alpha + \beta = 0.3 + 0.6 < 1$, it must be a concave function.
    - Hence, $x_1^{0.3} x_2^{0.6}$ is also quasiconcave, which implies $\ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave (as quasiconcavity is preserved through a monotonic transformation).

Properties of Preference Relations

- ***Example 1.9*** (continued):
  - *Homogeneity*:
    - Increasing all arguments by a common factor α,

$$(\alpha x_1)^{0.3}(\alpha x_2)^{0.6} = \alpha^{0.3}x_1^{0.3}\alpha^{0.6}x_2^{0.6} = \alpha^{0.9}x_1^{0.3}x_2^{0.6}$$

    - Hence, $x_1^{0.3}x_2^{0.6}$ is homogeneous of degree 0.9
  - *Homotheticity:*
    - Therefore, $x_1^{0.3}x_2^{0.6}$ is also homothetic.
    - As a consequence, its transformation, $\ln(x_1^{0.3}x_2^{0.6})$, is also homothetic (as homotheticity is preserved through a monotonic transformation).

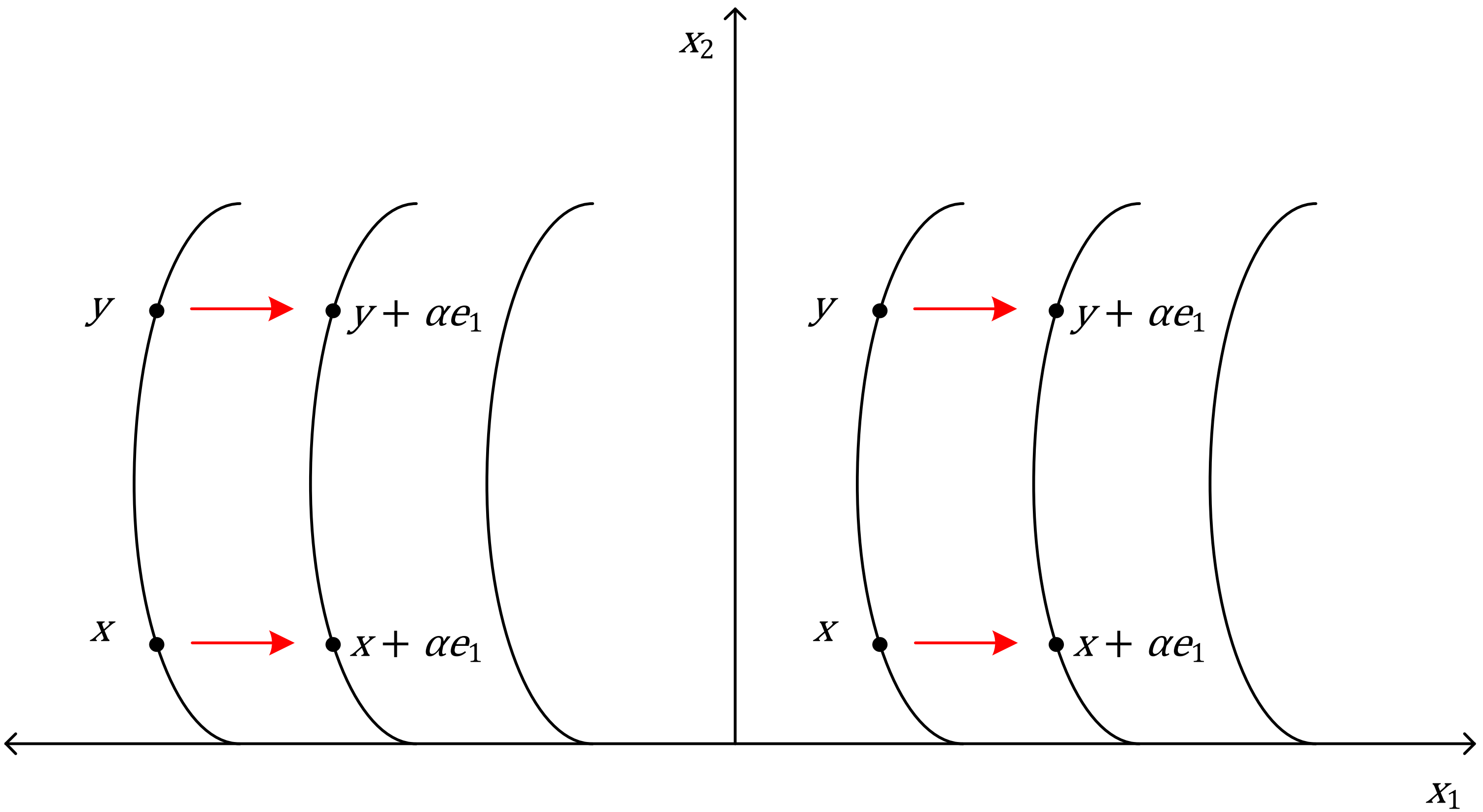
Properties of Preference Relations

- ***Quasilinear preference relations***:
  - The preference relation on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is *quasilinear* with respect to good 1 if:

    1) All indifference sets are parallel displacements of each other along the axis of good 1.
       - That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$, where $e_1 = (1,0, \dots, 0)$.
    2) Good 1 is desirable.
       - That is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Properties of Preference Relations

- Quasilinear preference-I

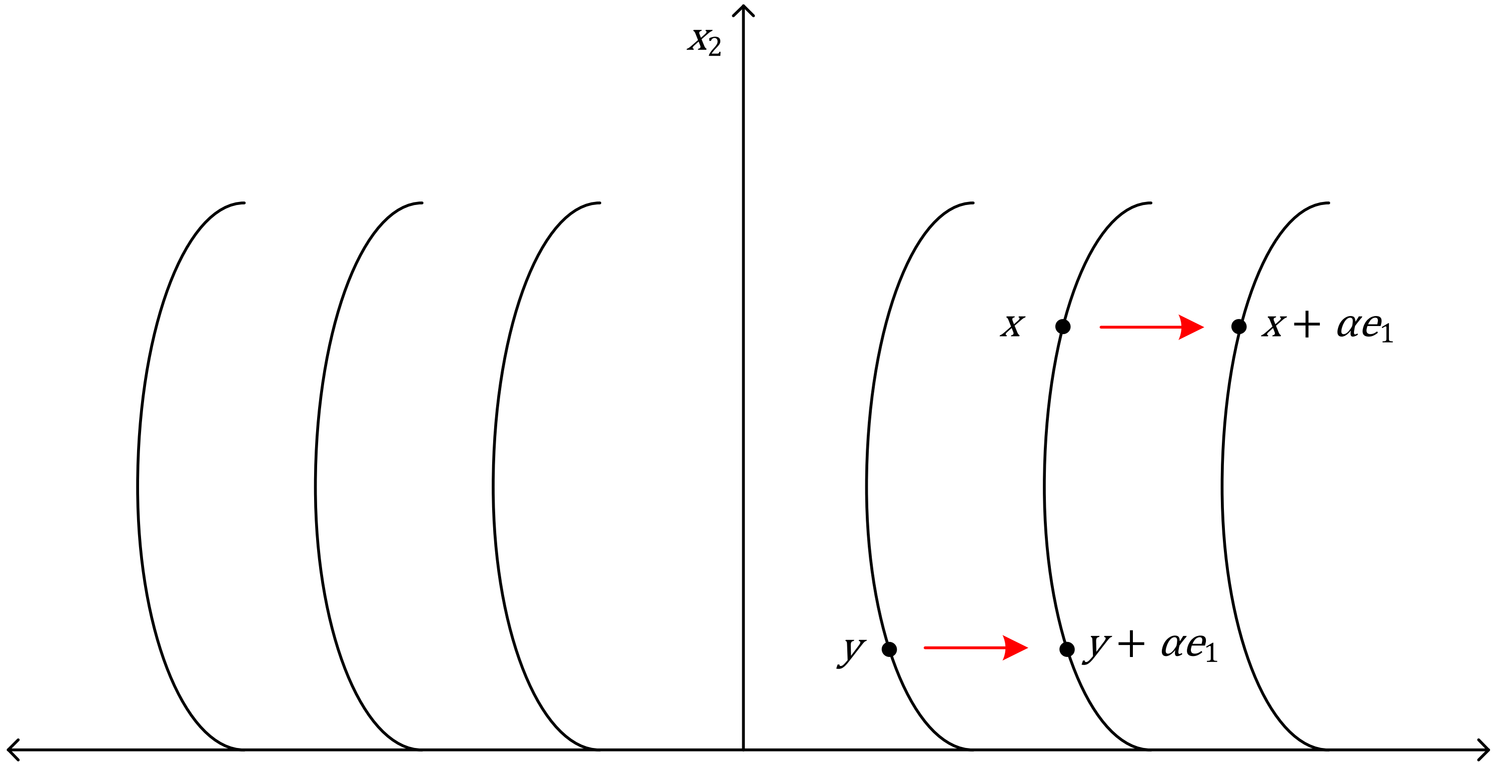


Properties of Preference Relations

- *Notes*:
  - No lower bound on the consumption of good 1, i.e., $x_1 \in (-\infty, \infty)$.
  - If $x \succ y$, then $(x + \alpha e_1) \succ (y + \alpha e_1)$.

Properties of Preference Relations

- Quasilinear preference-II



Properties of Preference Relations

- The properties we considered so far are not enough to guarantee that a preference relation can be represented by a utility function.
- *Example*:
  - Lexicographic preferences cannot be represented by a utility function.

Lexicographic Preferences

– A bundle $x = (x_1, x_2)$ is weakly preferred to another bundle $y = (y_1, y_2)$, i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\begin{cases} x_1 > y_1, & \text{or if} \\ x_1 = y_1 \text{ and } x_2 > y_2 \end{cases}$$

– *Intuition*:

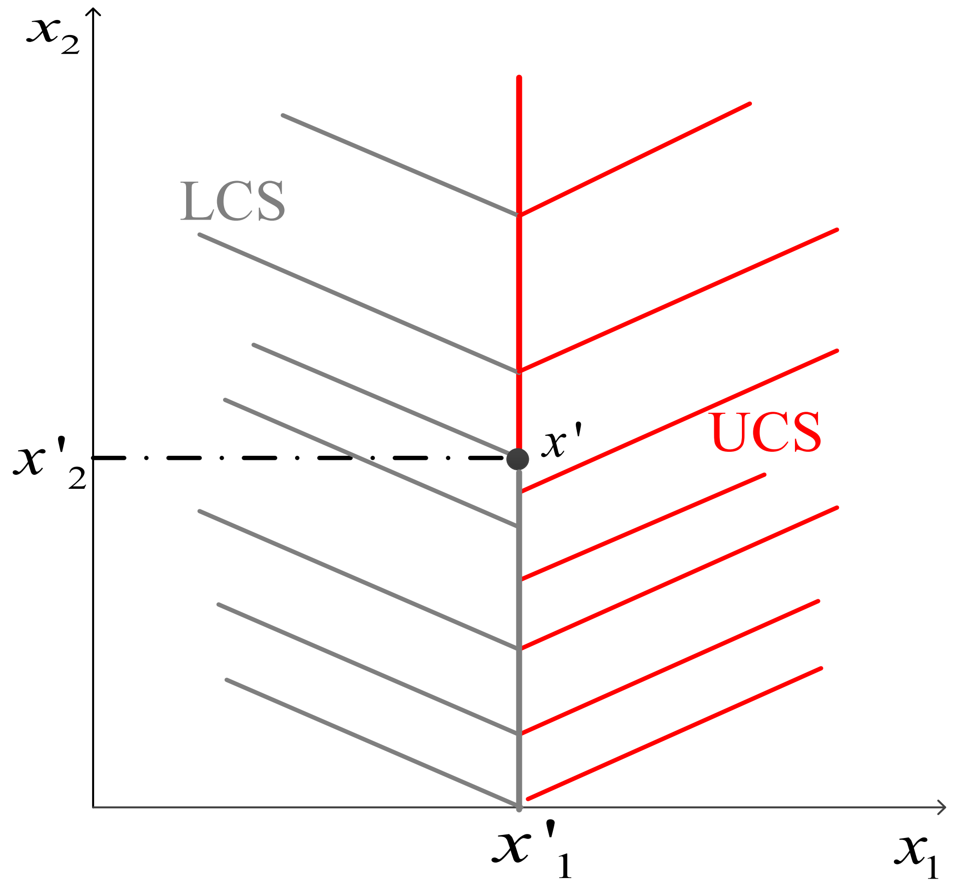
- The individual prefers bundle x if it contains more of good 1 than bundle y, i.e., $x_1 > y_1$.
- If, however, both bundles contain the same amount of good 1, $x_1 = y_1$, then the individual prefers bundle x if it contains more of the second good, i.e., $x_2 > y_2$.

Lexicographic Preferences

- Indifference set cannot be drawn as an indifference curve.
  - For a given bundle $x' = (x_1', x_2')$, there are no more bundles for which the consumer is indifferent.
  - Indifference sets are then *singletons* (sets containing only one element).

Lexicographic Preferences

- Lexicographic preference relation



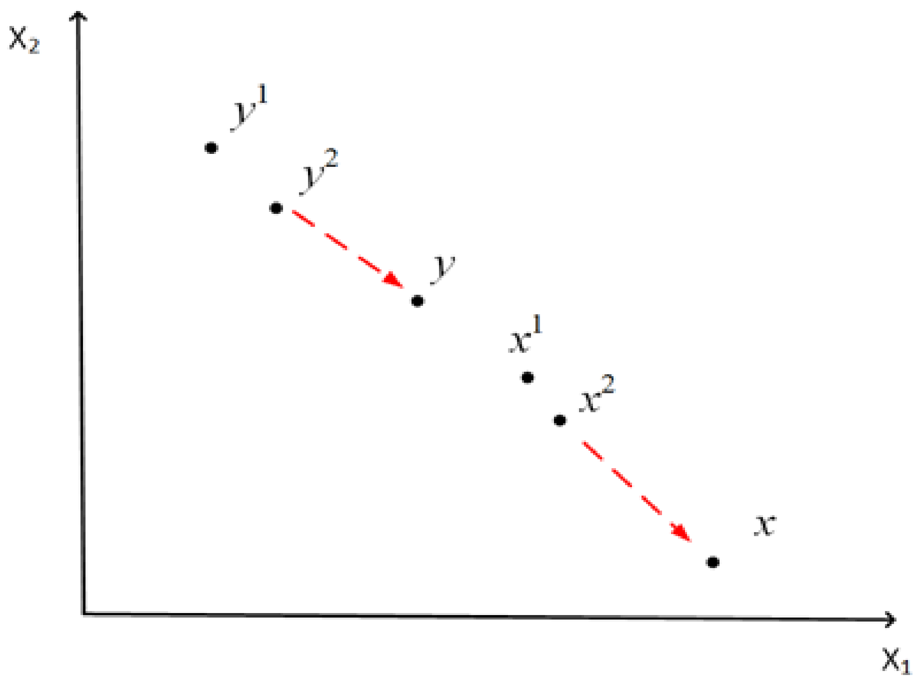
Continuous Preferences

Continuous Preferences

- In order to guarantee that preference relations can be represented by a utility function we need *continuity*.
- ***Continuity***: A preference relation defined on X is continuous if it is preserved under limits.
  - That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \gtrsim y^n$ for all n and where $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$, the preference relation is maintained in the limiting points, i.e., $x \gtrsim y$.

Continuous Preferences

– Intuitively, there can be no sudden jumps (i.e., preference reversals) in an individual preference over a sequence of bundles.



Continuous Preferences

- *Lexicographic preferences are not continuous*
  - Consider the sequence $x^n = \left(\frac{1}{n}, 0\right)$ and $y^n = (0,1)$, where $n = \{1,2,3, \ldots\}$.
  - The sequence $y^n = (0,1)$ is constant in n.
  - The sequence $x^n = \left(\frac{1}{n}, 0\right)$ is not:
    - It starts at $x^1 = (1,0)$, and moves leftwards to $x^2 = \left(\frac{1}{2}, 0\right)$, $x^3 = \left(\frac{1}{3}, 0\right)$, etc.

Continuous Preferences

- Thus, the individual prefers:

$$x^1 = (1,0) \succ (0,1) = y^1$$

$$x^2 = \left(\frac{1}{2}, 0\right) \succ (0,1) = y^2$$

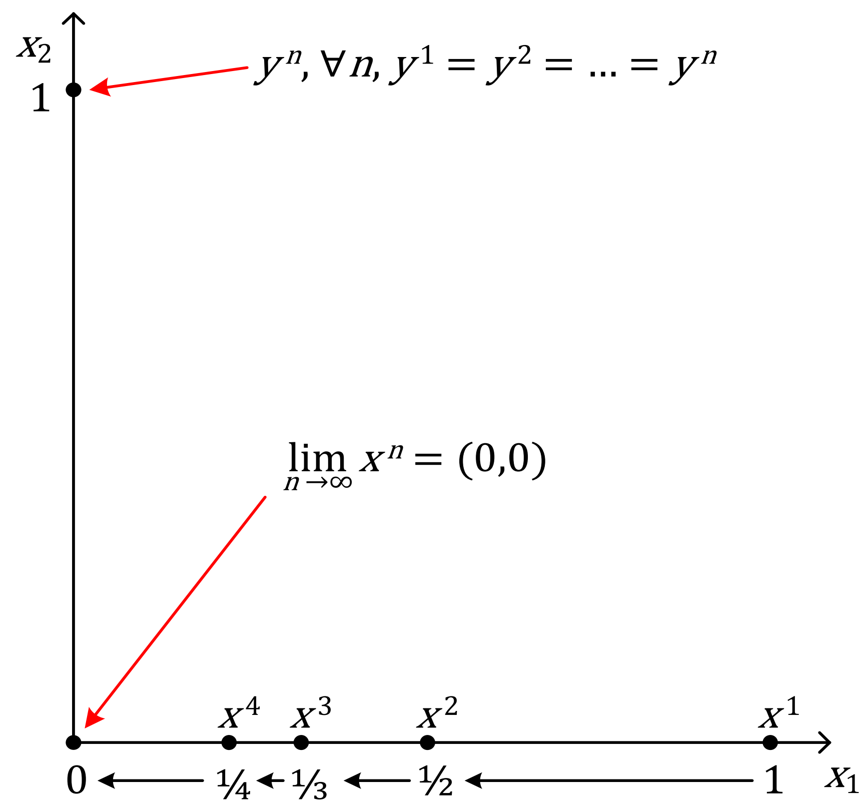
$$x^3 = \left(\frac{1}{3}, 0\right) \succ (0,1) = y^3$$

$$\vdots$$

- But,

$$\lim_{n\to\infty} x^n = (0,0) \prec (0,1) = \lim_{n\to\infty} y^n$$

- Preference reversal!



Existence of Utility Function

Existence of Utility Function

- *If a preference relation satisfies monotonicity and continuity, then there exists a utility function $u(\cdot)$ representing such preference relation.*

- *Proof*:
  - Take a bundle $x \neq 0$.
  - By monotonicity, $x \gtrsim 0$, where $0 = (0,0,\ldots,0)$.
    - That is, if bundle $x \neq 0$, it contains positive amounts of at least one good and, it is preferred to bundle 0.

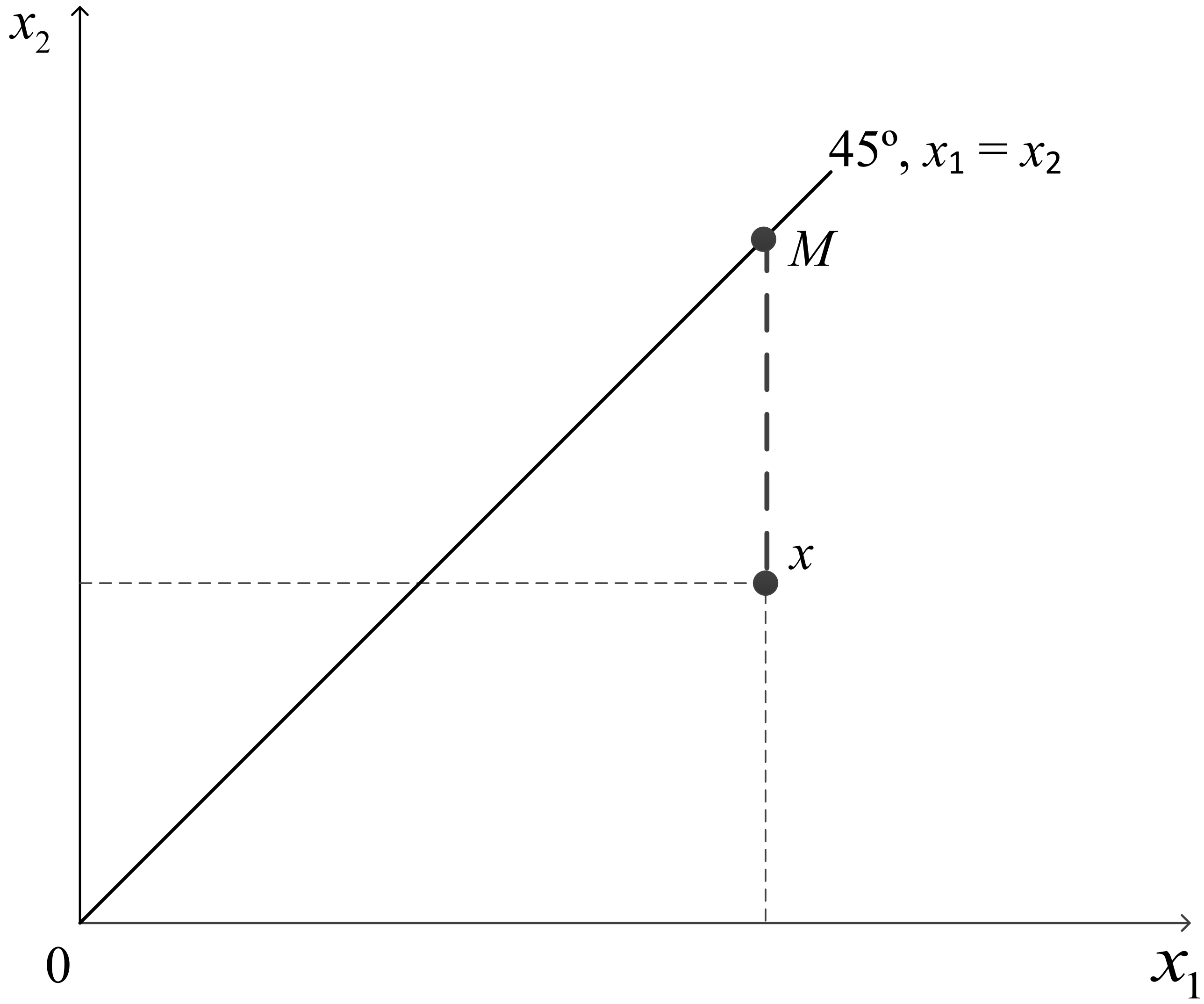
Existence of Utility Function

- Let $m \equiv \max\{x_1, x_2, \ldots, x_N\}$ be the number of units of the most abundant good in bundle x.
- Define bundle M as the bundle where all components coincide with the highest component of bundle x. That is,

$$M = (m, m, \ldots, m).$$

- Hence, by monotonicity, $M \gtrsim x$.
- Bundles 0 and M are both on the main diagonal, since each of them contains the same amount of good x_1 and x_2.

Existence of Utility Function



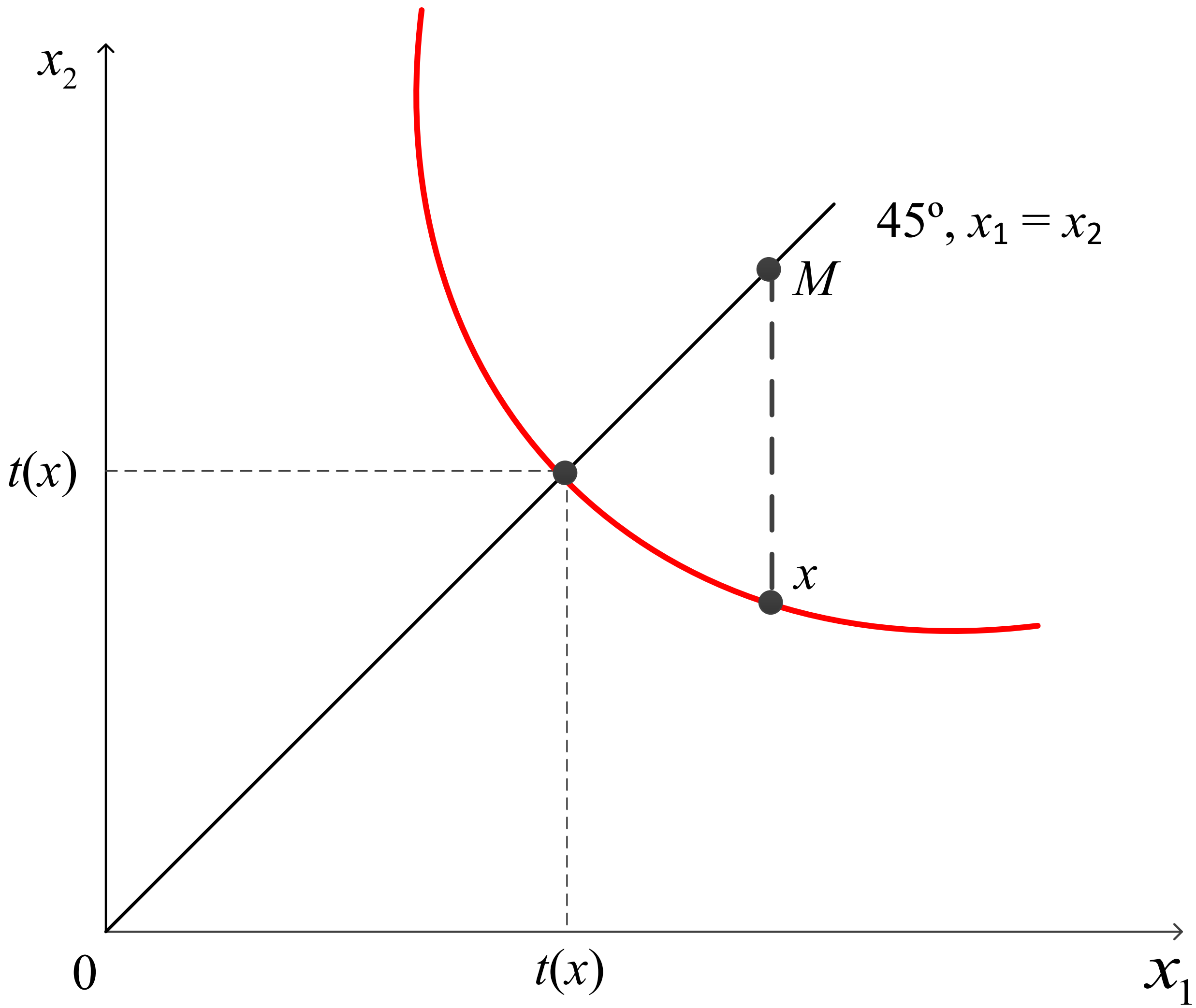
Existence of Utility Function

– By continuity and monotonicity, there exists a bundle that is indifferent to x and which lies on the main diagonal.
– By monotonicity, this bundle is unique
  - Otherwise, modifying any of its components would lead to higher/lower indifference curves.
– Denote such bundle as

$$(t(x), t(x), \dots, t(x))$$

– Let $u(x) = t(x)$, which is a real number.

Existence of Utility Function



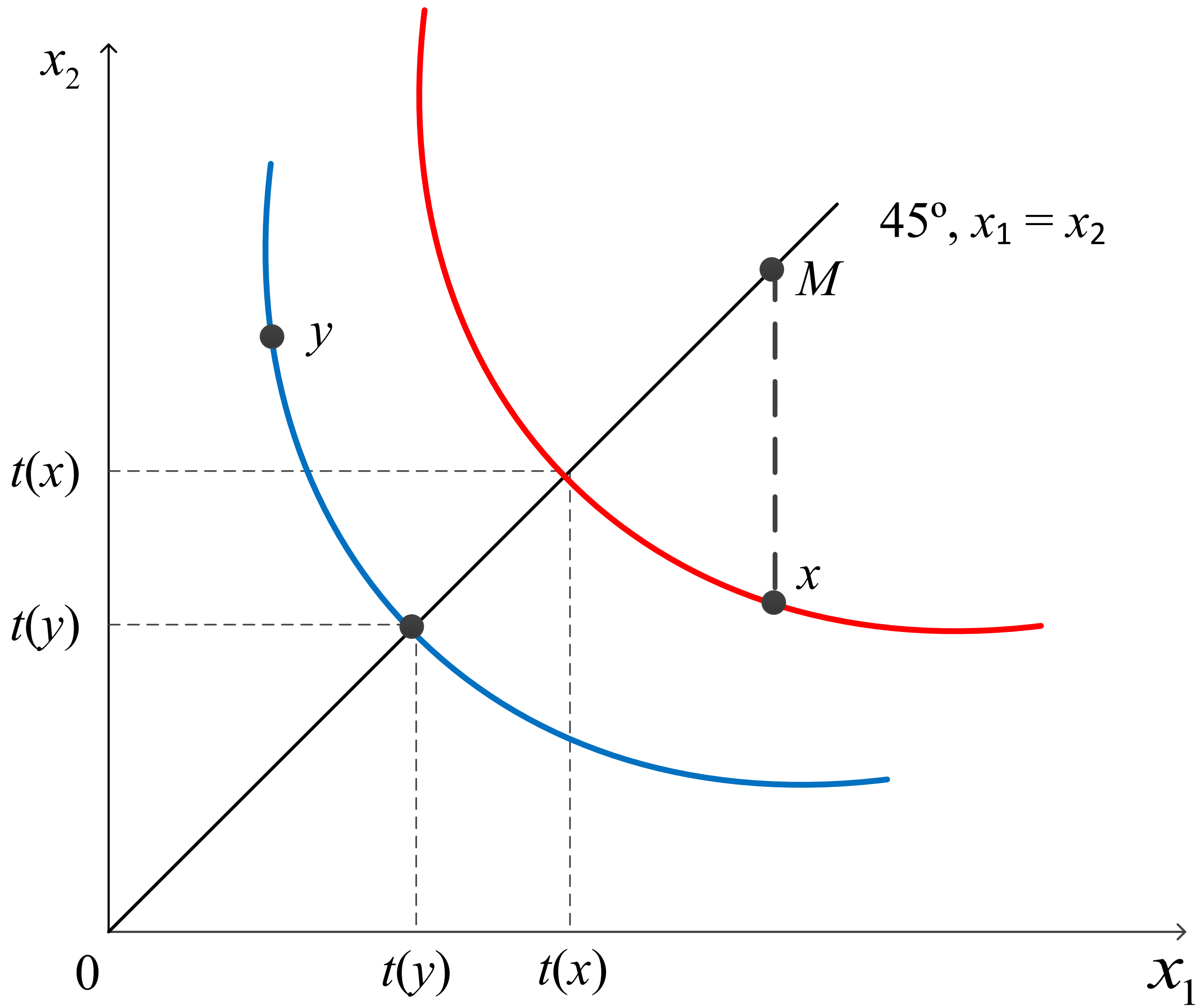
Existence of Utility Function

– Applying the same steps to another bundle $y \neq x$, we obtain

$$(t(y), t(y), \ldots, t(y))$$

and let $u(y) = t(y)$, which is also a real number.

Existence of Utility Function



Existence of Utility Function

– We know that

$$x \sim (t(x), t(x), \dots, t(x))$$
$$y \sim (t(y), t(y), \dots, t(y))$$
$$x \succsim y$$

– Hence, by transitivity, $x \succsim y$ iff

$$x \sim (t(x), t(x), \dots, t(x)) \succsim \big(t(y), t(y), \dots, t(y)\big) \sim y$$

– And by monotonicity,

$$x \succsim y \iff t(x) \geq t(y) \iff u(x) \geq u(y)$$

Existence of Utility Function

– *Note*: A utility function can satisfy continuity but still be non-differentiable.
  - For instance, the Leontief utility function, $A\min\{ax_1, bx_2\}$, is continuous but cannot be differentiated at the kink.

Social and Reference-Dependent Preferences

Social Preferences

- We now examine social, as opposed to individual, preferences.
- Consider additively separable utility functions of the form

$$u_i(x_i, x) = f(x_i) + g_i(x)$$

where
  - $f(x_i)$ captures individual i's utility from the monetary amount that he receives, x_i;
  - $g_i(x)$ measures the utility/disutility he derives from the distribution of payoffs $x = (x_1, x_2, \ldots, x_N)$ among all N individuals.

Social Preferences

- **Fehr and Schmidt (1999)**:
  - For the case of two players,

$$u_i(x_i, x_j) = x_i - \alpha_i \max\{x_j - x_i, 0\} - \beta_i \max\{x_i - x_j, 0\}$$

    where x_i is player i's payoff and $j \neq i$.

  - Parameter $\alpha_i \geq 0$ represents player i's disutility from envy
    - When $x_i < x_j$, $\max\{x_j - x_i, 0\} = x_j - x_i > 0$ but $\max\{x_i - x_j, 0\} = 0$.
    - Hence, $u_i(x_i, x_j) = x_i - \alpha_i(x_j - x_i)$.

Social Preferences

– Parameter $\beta_i \geq 0$ captures player i's disutility from guilt
  - When $x_i > x_j$, $\max\{x_i - x_j, 0\} = x_i - x_j > 0$ but $\max\{x_j - x_i, 0\} = 0$.
  - Hence, $u_i(x_i, x_j) = x_i - \beta_i(x_i - x_j)$.

– Players' envy is stronger than their guilt, i.e., $\alpha_i \geq \beta_i$ for $0 \leq \beta_i < 1$.
  - Intuitively, players (weakly) suffer more from inequality directed at them than inequality directed at others.

Social Preferences

– Thus players exhibit "concerns for fairness" (or "social preferences") in the distribution of payoffs.

– If $\alpha_i = \beta_i = 0$ for every player i, individuals only care about their material payoff $u_i(x_i, x_j) = x_i$.

  - Preferences coincide with the individual preferences.

Social Preferences

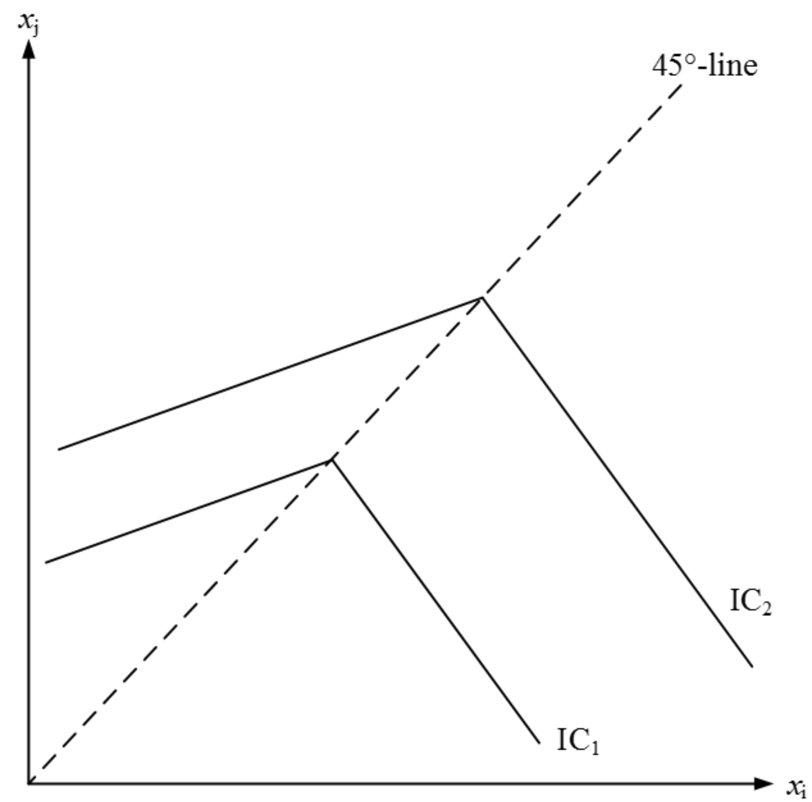
- Let's depict the indifference curves of this utility function by fixing the utility level at $u = \bar{u}$.
- When $x_i > x_j$,

  $\bar{u} = x_i - \beta_i(x_i - x_j)$

  which, solving for x_j, yields

  $x_j = \frac{\bar{u}}{\beta_i} - \frac{1-\beta_i}{\beta_i} x_i$

  with slope $-\frac{1-\beta_i}{\beta_i}$ for all points below the 45-degree line.
- See downward sloping segment of ICs.

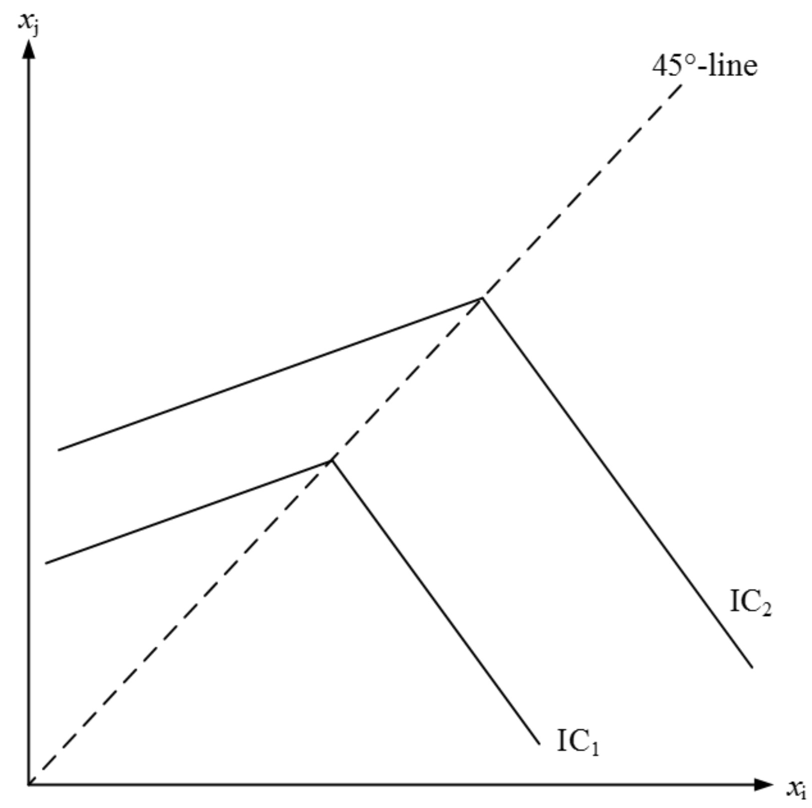


Social Preferences

- Similarly, when $x_i < x_j$, $\bar{u} = x_i - \alpha_i(x_j - x_i)$, which, solving for x_j, yields

  $$x_j = -\frac{\bar{u}}{\alpha_i} + \frac{1+\alpha_i}{\alpha_i} x_i$$

  with slope $\frac{1+\alpha_i}{\alpha_i}$ above 45-degree line.
- See upward sloping segment of the ICs.
- Note that (x_i, x_j)-pairs to the northeast yield larger utility levels for individual i.



Social Preferences

– ***Remark 1***: If
  - the disutility from envy is positive, $\alpha_i \in [0,1]$;
  - the disutility from guilt is negative, $\beta_i \in (-1,0]$; and
  - the former dominates the latter in absolute value, $|\alpha_i| \geq |\beta_i|$;

  then Fehr and Schmidt's (1999) specification would capture *concerns for status acquisition*.

Social Preferences

– ***Remark 2***: If
  - the disutility from envy is negative, $\alpha_i \in \left(-\frac{1}{2}, 0\right]$;
  - the disutility from guilt is positive, $\beta_i \in \left[0, \frac{1}{2}\right)$; and
  - the latter dominates the former in absolute value, $|\alpha_i| < |\beta_i|$;

then Fehr and Schmidt's (1999) specification would now capture a *preference for efficiency*.

That is, a reduction in my own payoff is acceptable only if the payoff other individuals receive increases by a larger amount.

Social Preferences

- **Bolton and Ockenfels (2000)**:
  - Similar to Fehr and Schmidt (1999), but they allow for nonlinearities

$$u_i\left(x_i, \frac{x_i}{x_i + x_j}\right)$$

    where $u_i(\cdot)$

    - increases in x_i (i.e., selfish component)
    - decreases in the share of total payoffs that individual i enjoys, $\frac{x_i}{x_i+x_j}$ (i.e., social preferences)

Social Preferences

– For instance,

$$u_i\left(x_i, \frac{x_i}{x_i+x_j}\right) = x_i - \alpha\left(\frac{x_i}{x_i+x_j}\right)^{\frac{1}{2}}$$

– Letting $u = \overline{u}$ and solving for x_j yields the indifference curve

$$x_j = \frac{x_i[\alpha^2 - (\overline{u} - x_i)^2]}{(\overline{u} - x_i)^2}$$

which produces nonlinear indifference curves (nonlinear in x_i).

Social Preferences

- **Charness and Rabin (2002)**:
  - Fehr and Schmidt's (1999) preferences might not explain individuals' reactions in strategic settings.
    - *Example*: inferring certain intentions from individuals who acted before them.
  - Utility function that rationalizes such behavior

$$u_i(x_i, x_j) = x_i - (\alpha_i - \theta\gamma_j) \max\{x_j - x_i, 0\}$$
$$-(\beta_i + \theta\gamma_j) \max\{x_i - x_j, 0\}$$

where parameter γ_j only takes two possible values, i.e., $\gamma_j = \{-1,0\}$.

Social Preferences

– If $\gamma_j = -1$, individual i interprets that j misbehaved, and thus increases its envy parameter by θ, or reduces his guilt parameter by θ.

– If $\gamma_j = 0$, individual i interprets that j is well behaved, implying that the utility function coincides with that in Fehr and Schmidt's (1999) specification.

– Intuitively, when individuals interpret that others misbehaved, the envy (guilt) concerns analyzed above are emphasized (attenuated, respectively).

Social Preferences

- **Andreoni and Miller (2002)**:
  - A CES utility function

$$u_i(x_i, x_j) = \left(\alpha x_i^{\rho} + (1-\alpha) x_j^{\rho}\right)^{\frac{1}{\rho}}$$

where x_i and x_j are the monetary payoff of individual i rather than the amount of goods.

  - If individual i is completely selfish, i.e., $\alpha = 1$, $u(x_i) = x_i$

Social Preferences

– If $\alpha \in (0,1)$, parameter ρ captures the elasticity of substitution between individual i's and j's payoffs.

- That is, if x_j decreases by one percent, x_i needs to be increased by ρ percent for individual i to maintain his utility level unaffected.

Hyperbolic and Quasi-Hyperbolic Discounting

Exponential discounting (standard)

- The discounted value of an amount of money $\$x$ received t periods from today is

$$\frac{1}{(1+r)^t}x$$

- We can find the "subjective discount rate" which measures how $\frac{1}{(1+r)^t}x$ varies along time, relative to its initial value,

$$\frac{\frac{\partial\left(\frac{1}{(1+r)^t}x\right)}{\partial t}}{\frac{1}{(1+r)^t}x} = \frac{-\ln(1+r)\frac{1}{(1+r)^t}x}{\frac{1}{(1+r)^t}x} = -\ln(1+r)$$

which is constant in the time period t when it is evaluated.

- In other words, exponential discounting assumes that the comparison of $\$x$ between period 0 and k coincides with the comparison between period t and $t+k$ since k periods mediated.

Exponential discounting (standard)

- Not generally confirmed in controlled experiments.
- In particular, individuals exhibit *present bias*:
  - When asked to choose between \$100 today or \$110 tomorrow, most individuals prefer \$100 today.
  - However, when the same individuals are asked between \$100 in, for instance, 60 days or \$110 in 61 days, some reveal a preference for the \$110 in 61 days.
- Individuals show a large discount of future payoffs
- Preferences are time-inconsistent

Hyperbolic Discounting

- This approach assumes that the discounted value of an amount of money $\$x$ received t periods from today is

$$\frac{1}{(1+rt)^{\gamma/\alpha}}x$$

where $\gamma, \alpha > 0$. In this setting, the subjective discount rate is

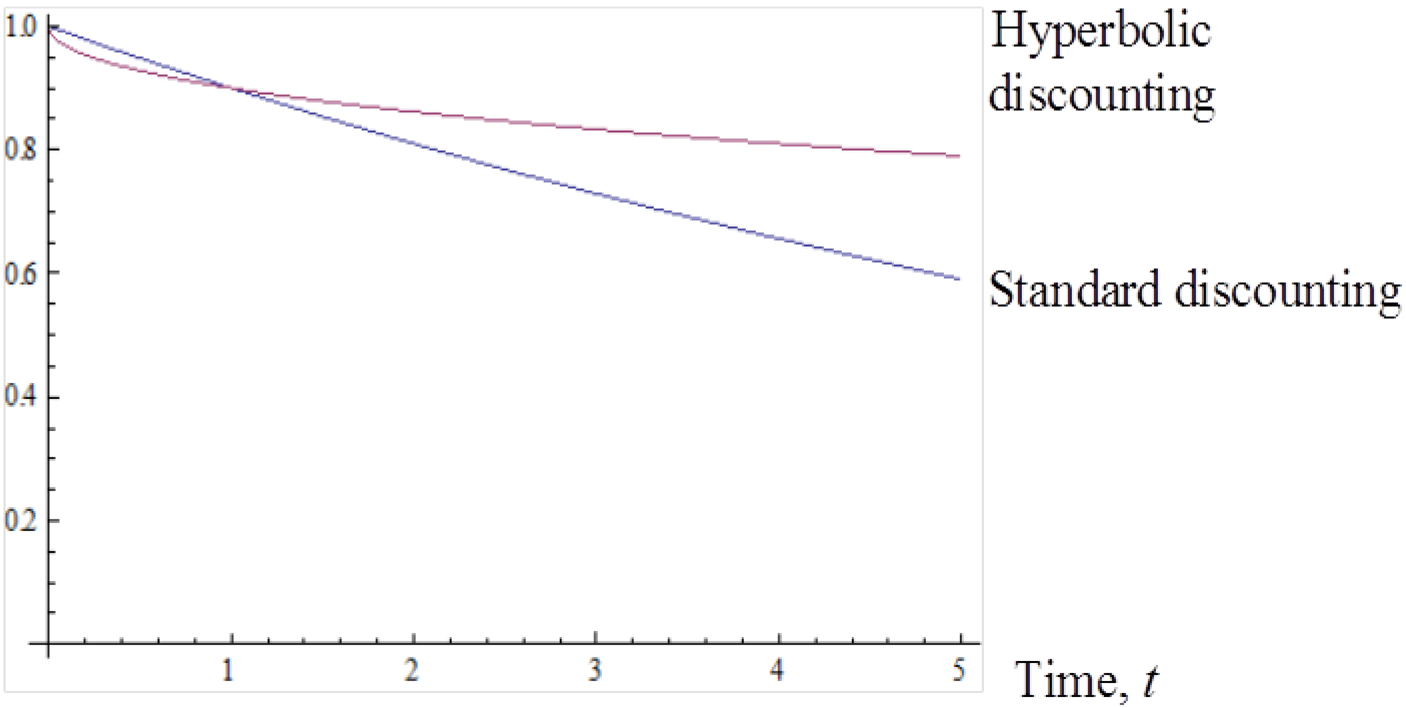
$$\frac{\dfrac{\partial\left(\dfrac{1}{(1+rt)^{\gamma/\alpha}}x\right)}{\partial t}}{\dfrac{1}{(1+rt)^{\gamma/\alpha}}x} = \frac{-\gamma r}{\alpha(1+rt)}$$

which is decreasing in t.

In most applications, $\gamma = \alpha$, yielding a subjective discount rate of

$$\frac{-r}{(1+rt)}$$

Hyperbolic Discounting



- Individuals with hyperbolic discounting exhibit present bias: relative to standard (exponential) discounting
  - They strongly discount payoffs in the nearby future, but
  - They do not significantly discount two distant payoffs that are close to each other.

Quasi-Hyperbolic Discounting

- In a discrete time context, individuals discount future payoffs according to

$$\beta\delta^t$$

for all $t \geq 1$, where parameter $\beta \leq 1$.

  - When $\beta = 1$, Quasi-hyperbolic discounting embodies exponential discounting.
  - The subjective discount rate is

$$\frac{\frac{\Delta(\beta\delta^t x)}{\Delta t}}{\beta\delta^t x} = \frac{\beta\delta^{t+1}x - \beta\delta^t x}{\beta\delta^t x} = \delta - 1$$

which is constant in time, but still allows for present bias to arise.

Quasi-Hyperbolic Discounting

- Consider an individual evaluating today whether to invest in a firm.
- He will need to incur a cost $c > 0$ in period t, and obtain a benefit $b > 0$ with certainty n periods into the future (in period $t + n$).
- Under exponential discounting, he would invest if

$$\delta^t c < \delta^{t+n} b \text{ or } c < \delta^n b$$

- If this individual is given the opportunity to reconsider his investment when period t arrives, he will ***not*** reconsider his decision since $c < \delta^n b$ still holds.

Quasi-Hyperbolic Discounting

– Under quasi-hyperbolic discounting, he would invest if

$$\beta\delta^t c < \beta\delta^{t+n} b \text{ or } c < \delta^n b$$

which is same decision rule as the above time-consistent individual.

– However, if this individual is given the opportunity to reconsider his investment when period t arrives, he will invest only if

$$c < \beta\delta^n b$$

which does not coincide with his decision rule t periods ago.

– Hence, preference reversal occurs if

$$\beta\delta^n b < c < \delta^n b$$

Choice Based Approach

Choice Based Approach

- We now focus on the actual choice behavior rather than individual preferences.
  - From the alternatives in set B, which one would you choose?

- A choice structure $(\mathcal{B}, c(\cdot))$ contains two elements:

  1) $\mathcal{B}$ is a family of nonempty subsets of X, so that every element of $\mathcal{B}$ is a set $B \subset X$.

Choice Based Approach

– *Example 1*: In consumer theory, B is a particular set of all the affordable bundles for a consumer, given his wealth and market prices.

– *Example 2*: B is a particular list of all the universities where you were admitted, among all universities in the scope of your imagination X, i.e., $B \subset X$.

Choice Based Approach

2) $c(\cdot)$ is a choice rule that selects, for each budget set B, a subset of elements in B, with the interpretation that $c(B)$ are the chosen elements from B.

- *Example 1*: In consumer theory, $c(B)$ would be the bundles that the individual chooses to buy, among all bundles he can afford in budget set B;
- *Example 2*: In the example of the universities, $c(B)$ would contain the university that you choose to attend.

Choice Based Approach

– *Note*:
  - If $c(B)$ contains a single element, $c(\cdot)$ is a function;
  - If $c(B)$ contains more than one element, $c(\cdot)$ is a correspondence.

Choice Based Approach

- ***Example 1.11*** (Choice structures):
  - Define the set of alternatives as
  $$X = \{x, y, z\}$$
  - Consider two different budget sets
  $$B_1 = \{x, y\} \text{ and } B_2 = \{x, y, z\}$$
  - Choice structure one $(\mathcal{B}, c_1(\cdot))$
  $$c_1(B_1) = c_1(\{x, y\}) = \{x\}$$
  $$c_1(B_2) = c_1(\{x, y, z\}) = \{x\}$$

Choice Based Approach

- ***Example 1.11*** (continued):
  - Choice structure two $(\mathcal{B}, c_2(\cdot))$

$$c_2(B_1) = c_2(\{x, y\}) = \{x\}$$
$$c_2(B_2) = c_2(\{x, y, z\}) = \{y\}$$

  - Is such a choice rule consistent?
    - We need to impose a consistency requirement on the choice-based approach, similar to rationality assumption on the preference-based approach.

Consistency on Choices: the Weak Axiom of Revealed Preference (WARP)

WARP

- ***Weak Axiom of Revealed Preference (WARP)***: The choice structure $(\mathcal{B}, c(\cdot))$ satisfies the WARP if:
  1) for some budget set $B \in \mathcal{B}$ with $x, y \in B$, we have that element x is chosen, $x \in c(B)$, then
  2) for any other budget set $B' \in \mathcal{B}$ where alternatives x and y are also available, $x, y \in B'$, and where alternative y is chosen, $y \in c(B')$, then we must have that alternative x is chosen as well, $x \in c(B')$.

WARP

- ***Example 1.12*** (Checking WARP in choice structures):
  - Take budget set $B = \{x, y\}$ with the choice rule of $c(\{x, y\}) = x$.
  - Then, for budget set $B' = \{x, y, z\}$, the "legal" choice rules are either:

$$c(\{x, y, z\}) = \{x\}, \text{ or}$$

$$c(\{x, y, z\}) = \{z\}, \text{ or}$$

$$c(\{x, y, z\}) = \{x, z\}$$

WARP

- ***Example 1.12*** (continued):
  - This implies that the individual decision-maker cannot select

$$c(\{x, y, z\}) \neq \{y\}$$
$$c(\{x, y, z\}) \neq \{y, z\}$$
$$c(\{x, y, z\}) \neq \{x, y\}$$
$$c(\{x, y, z\}) \neq \{x, y, z\}$$

WARP

- ***Example 1.13*** (More on choice structures satisfying/violating WARP):
  - Take budget set $B = \{x, y\}$ with the choice rule of $c(\{x, y\}) = \{x, y\}$.
  - Then, for budget set $B' = \{x, y, z\}$, the "legal" choices according to WARP are either:

$$c(\{x, y, z\}) = \{x, y\}, \text{ or}$$

$$c(\{x, y, z\}) = \{z\}, \text{ or}$$

$$c(\{x, y, z\}) = \{x, y, z\}$$

WARP

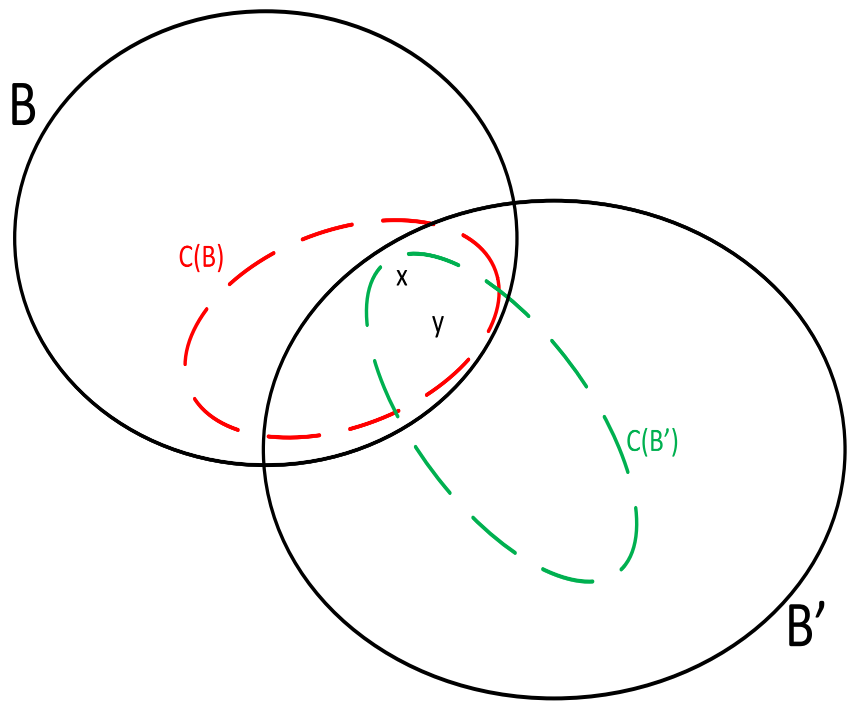
- ***Example 1.13*** (continued):
  - This implies that the decision-maker cannot select:

$$c(\{x,y,z\}) \neq \{x\}$$
$$c(\{x,y,z\}) \neq \{y\}$$
$$c(\{x,y,z\}) \neq \{x,z\}$$
$$c(\{x,y,z\}) \neq \{y,z\}$$

  - In summary, when both x and y are available in B and B', as long as they are chosen in B, both of them must be chosen in B'.

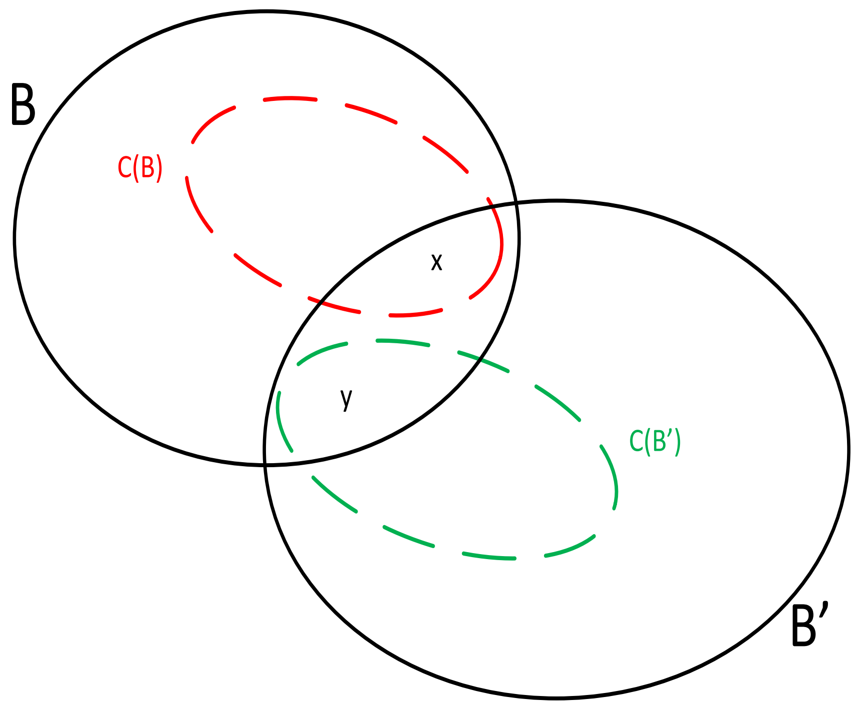
WARP

- ***Example 1.13*** (continued):
  - Choice rule satisfying WARP



WARP

- ***Example 1.13*** (continued):
  - Choice rule violating WARP



Consumption Sets

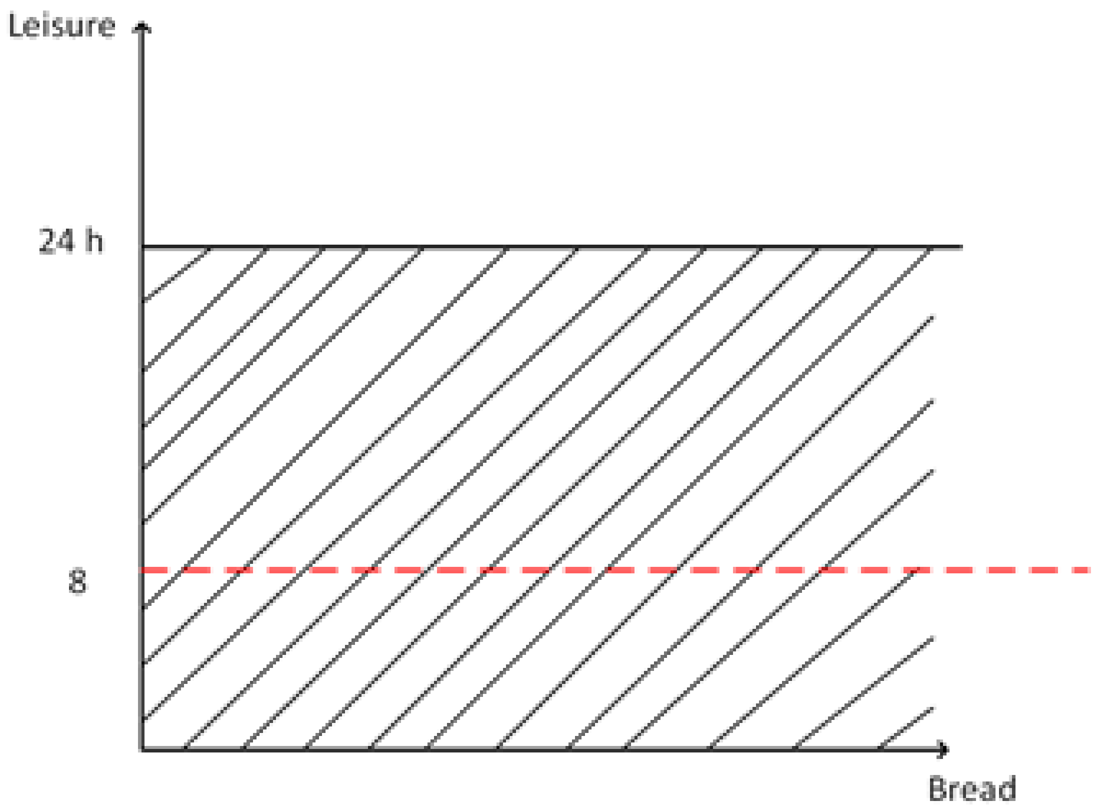
Consumption Sets

- ***Consumption set***: a subset of the commodity space $\mathbb{R}^N$, denoted by $x \subset \mathbb{R}^N$, whose elements are the consumption bundles that the individual can conceivably consume, given the physical constraints imposed by his environment.

- Let us denote a commodity bundle x as a vector of L components.

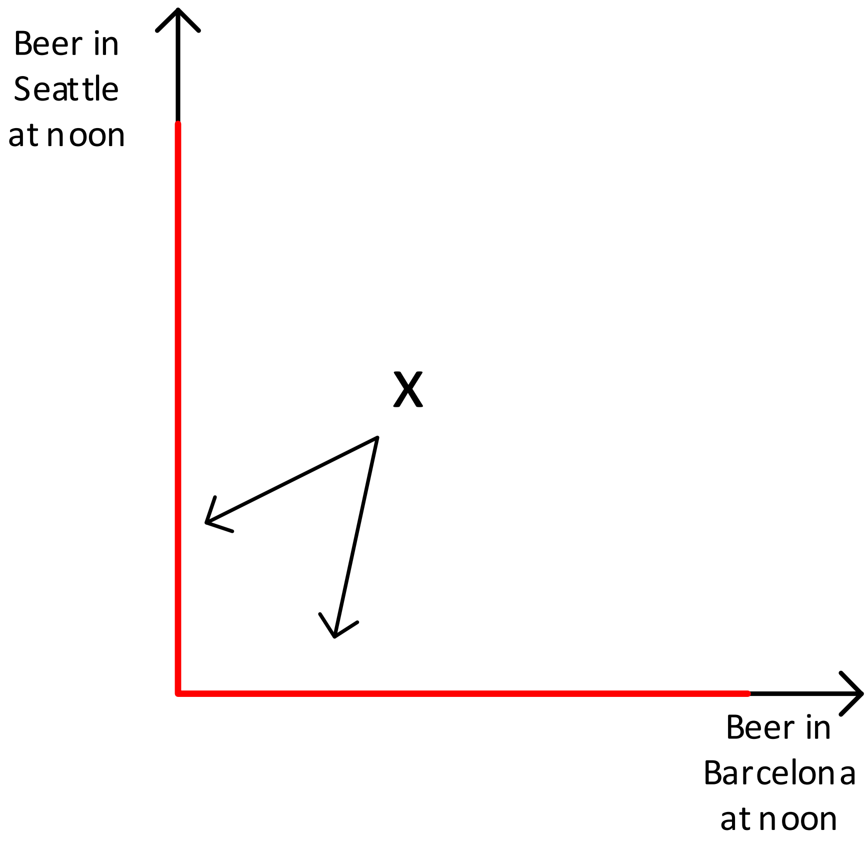
Consumption Sets

- Physical constraint on the labor market



Consumption Sets

- Consumption at two different locations



Consumption Sets

- ***Convex consumption sets***:
  - A consumption set X is convex if, for two consumption bundles $x, x' \in X$, the bundle
$$x'' = \alpha x + (1 - \alpha)x'$$
is also an element of X for any $\alpha \in (0,1)$.
  - Intuitively, a consumption set is convex if, for any two bundles that belong to the set, we can construct a straight line connecting them that lies completely within the set.

Consumption Sets: Economic Constraints

- Assumptions on the price vector in $\mathbb{R}^N$:
  1) All commodities can be traded in a market, at prices that are publicly observable.
     - This is the principle of completeness of markets
     - It discards the possibility that some goods cannot be traded, such as pollution.
  2) Prices are strictly positive for all N goods, i.e., $p \gg 0$ for every good k.
     - Some prices could be negative, such as pollution.

Consumption Sets: Economic Constraints

3) Price taking assumption: a consumer's demand for all N goods represents a small fraction of the total demand for the good.

Consumption Sets: Economic Constraints

- Bundle $x \in \mathbb{R}_+^N$ is affordable if
$$p_1 x_1 + p_2 x_2 + \cdots + p_N x_N \leq w$$
or, in vector notation, $p \cdot x \leq w$.

- Note that $p \cdot x$ is the total cost of buying bundle $x = (x_1, x_2, \ldots, x_N)$ at market prices $p = (p_1, p_2, \ldots, p_N)$, and w is the total wealth of the consumer.

- When $x \in \mathbb{R}_+^N$ then the set of feasible consumption bundles consists of the elements of the set:
$$B_{p,w} = \{x \in \mathbb{R}_+^N : p \cdot x \leq w\}$$

Consumption Sets: Economic Constraints

- ***Example for two goods***:

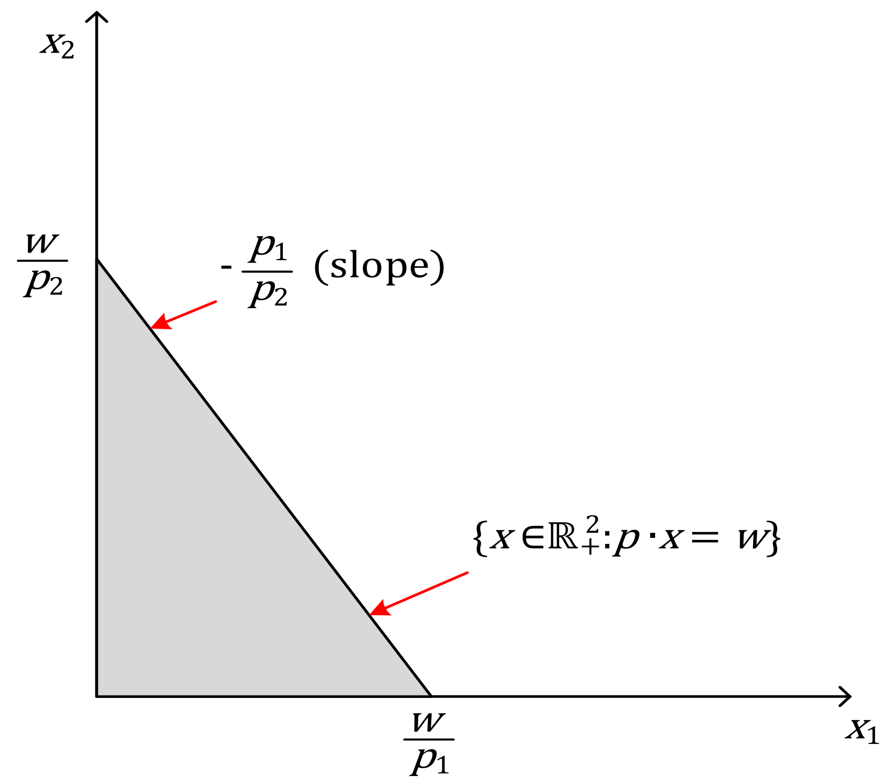
$$B_{p,w} = \{x \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq w\}$$

The budget line is

$p_1 x_1 + p_2 x_2 = w$.

Hence, solving for the good on the vertical axis, x_2, we obtain

$$x_2 = \frac{w}{p_2} - \frac{p_1}{p_2} x_1$$

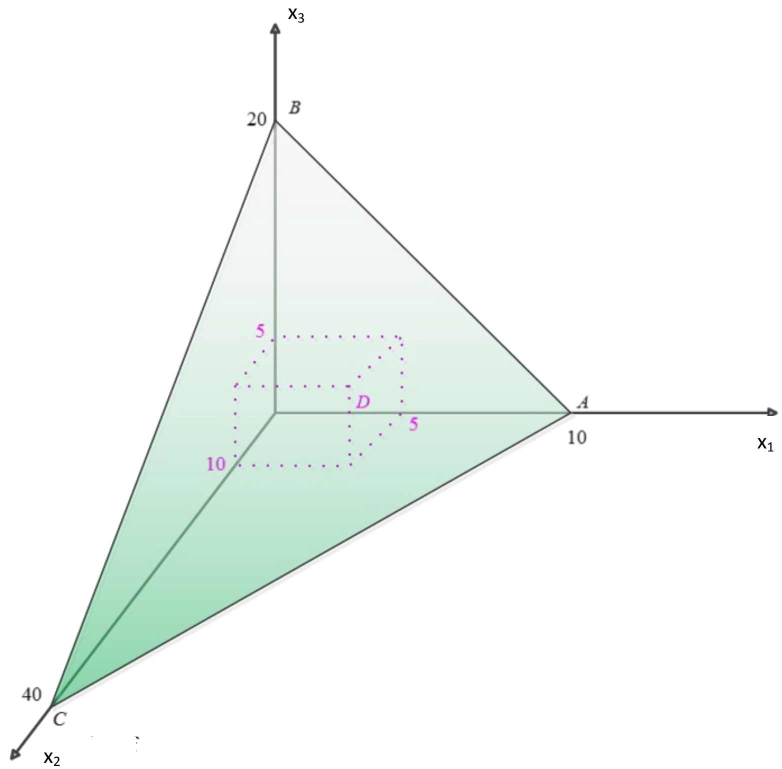


Consumption Sets: Economic Constraints

- ***Example for three goods***:

$$B_{p,w} = \{x \in \mathbb{R}^3_+ : p_1x_1 + p_2x_2 + p_3x_3 \leq w\}$$

  – The surface $p_1x_1 + p_2x_2 + p_3x_3 = w$ is referred to as the "Budget hyperplane"



Consumption Sets: Economic Constraints

- *Price vector* p *is orthogonal (perpendicular) to the budget line* $B_{p,w}$.
  - Note that $p \cdot x = w$ holds for any bundle x on the budget line.
  - Take any other bundle x' which also lies on $B_{p,w}$. Hence, $p \cdot x' = w$.
  - Then,

$$p \cdot x' = p \cdot x = w$$

$$p \cdot (x' - x) = 0 \text{ or } p \cdot \Delta x = 0$$

Consumption Sets: Economic Constraints

- Since this is valid for any two bundles on the budget line, then p must be perpendicular to Δx on $B_{p,w}$.
- This implies that the price vector is perpendicular (orthogonal) to $B_{p,w}$.

Consumption Sets: Economic Constraints

- *The budget set $B_{p,w}$ is convex.*
  - We need that, for any two bundles $x, x' \in B_{p,w}$, their convex combination
    $$x'' = \alpha x + (1-\alpha)x'$$
    also belongs to the $B_{p,w}$, where $\alpha \in (0,1)$.
  - Since $p \cdot x \leq w$ and $p \cdot x' \leq w$, then
    $$p \cdot x'' = p\alpha x + p(1-\alpha)x'$$
    $$= \alpha p x + (1-\alpha)p x' \leq w$$

Appendix 1.1: Rational Preference Relations Satisfy the WARP

Rational Preferences and WARP

- We can construct the preferences that the individual "reveals" in his actual choices when he is confronted to choose an element(s) from different budget sets.

  1) If there is some budget set B for which the individual chooses $x \in c(B)$, where $x, y \in B$, then we can say that alternative x is ***revealed at least as good as*** alternative y, and denote it as $x \succsim^* y$.
  2) If there is some budget set B for which the individual chooses $x \in c(B)$ but $y \notin c(B)$, where $x, y \in B$, then we can say that alternative x is ***revealed preferred to*** alternative y, and denote it as $x \succ^* y$.

Rational Preferences and WARP

- Let $C^*(B, \succsim)$ be the set of optimal choices generated by the preference relation $\succsim$ when facing a budget set B.
- Using this notation, we can restate the WARP as follows:
  - If alternative x is revealed at least as good as y, then y cannot be revealed preferred to x.
  - That is, if $x \succsim^* y$, then we cannot have $y \succ^* x$.

Rational Preferences and WARP

- Let us next show that:

$$\text{Rational preference relation} \Longrightarrow$$
$$\text{Choice structure satisfying WARP}$$

- *Proof*:
  - Suppose that for some budget set $B \in \mathcal{B}$, we have that $x, y \in B$ and $x \in C^*(B, \succsim)$.
    - That is, x belongs to the set of optimal choices given the preference relation $\succsim$ when the decision maker faces a budget set B.
  - Hence, $x \in C^*(B', \succsim) \implies x \succsim y$ for all $y \in B$.

Rational Preferences and WARP

– In order to check WARP, assume some other budget set $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C^*(B', \succsim)$.
  - That is, y belongs to the set of optimal choices given the preference relation $\succsim$ when the decision maker faces budget set B'.

– Thus, $y \in C^*(B, \succsim) \implies y \succsim z$ for all $z \in B'$.

Rational Preferences and WARP

– Combining the conclusions from the previous two points, $x \succsim y$ and $y \succsim z$, we can apply transitivity (because the preference relation is rational), and we obtain $x \succsim z$.

– Then $x \in \ C^*(B', \succsim)$, and we find that

$$x, y \in \ C^*(B', \succsim)$$

which proves that WARP is satisfied.

Rational Preferences and WARP

- Regarding:

$$\text{Choice structure satisfying WARP} \Longrightarrow \text{Rational preference relation}$$

- It is only true if the budget set B contains three or fewer elements (See MWG for a proof based on Arrow (1959)).