EconS 501 - Microeconomic Theory I[1]
Assignment #7 - Answer Key

1. **Production and Externalities.** According to some residents, a firm's production of paper at Lewiston, Idaho, generates a smelly gas as an unpleasant side product. Let $c(y, m; \mathbf{w})$ denote the (minimum) input cost of producing y tons of paper and m cubic meters of gas, where input prices are given by the vector $\mathbf{w} >> \mathbf{0}$. Let $p > 0$ denote the market price of paper. Assume that the cost function satisfies $\frac{\partial c}{\partial y} > 0$ and $\frac{\partial c}{\partial m} < 0$, and that $c(y, m; \mathbf{w})$ is strictly convex in y and m. Let stars * denote solutions and assume throughout that the firm produces positive amounts of paper $y^* > 0$.

   (a) Show that the cost function $c(y, m; \mathbf{w})$ is concave in input prices, $\mathbf{w}$.

   - Fix two input price vectors $\mathbf{w}$ and $\mathbf{w}'$ and consider their linear combination $\mathbf{w}'' = \alpha\mathbf{w} + (1 - \alpha)\mathbf{w}'$, for any $\alpha \in (0, 1)$. Let $\mathbf{x}$ (respectively, $\mathbf{x}'$ and $\mathbf{x}''$) be the minimum cost bundle for input prices $\mathbf{w}$ (respectively, $\mathbf{w}'$ and $\mathbf{w}''$). By cost minimization we have

   $$\begin{aligned} c(y, m; \mathbf{w}'') &= \alpha\mathbf{w}\mathbf{x}'' + (1 - \alpha)\mathbf{w}'\mathbf{x}'' \\ &\geq \alpha\mathbf{w}\mathbf{x} + (1 - \boldsymbol{\alpha})\mathbf{w}'\mathbf{x}' \\ &= \alpha c(y, m; \mathbf{w}) + (1 - \alpha)c(y, m; \mathbf{w}') \end{aligned}$$

   So $c(y, m; \mathbf{w})$ is concave in input prices $\mathbf{w}$.

   (b) *Setting a quota.* Suppose that the government imposes a ceiling on gas emissions such that $m \leq \overline{m}$, i.e., a quota. Assuming that this constraint binds, write down the firm's profit maximization problem with respect to y, and find necessary and sufficient conditions for the firm's cost-minimizing production, y^*,

   - The profit maximization problem for the firm is that of selecting an output level y that solves

   $$\max_{y} \; py - c(y, m; \mathbf{w})$$
   $$\text{subject to } m \leqslant \overline{m}$$

   - If the constraint binds, $m = \overline{m}$, then the first order condition with respect to output, y, is

   $$p = \frac{\partial c(y^*, \overline{m}; \mathbf{w})}{\partial y}$$

   that is, price equals marginal cost at the optimum. Note that the constraint will be binding, i.e., $m = \overline{m}$. Otherwise, the firm would be able to further increase output and its associated profits.

   (c) *Comparative statics.* Under which condition on the cost function $c(y, m; \mathbf{w})$ can we guarantee that an increase in the ceiling on gas emissions, $\overline{m}$, produces a raise in the firm's cost-minimizing production, y^*, i.e., $\frac{\partial y^*}{\partial \overline{m}} > 0$?

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- Differentiating the above expression again with respect to $\overline{m}$, we obtain

$$0 = \frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial y^2} \frac{\partial y^*}{\partial \overline{m}} + \frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial m \partial y}$$

and rearranging we obtain the usual expression of the implicit function theorem,

$$\frac{\partial y^*}{\partial \overline{m}} = -\frac{\frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial m \partial y}}{\frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial y^2}}$$

- Since the cost function $c(\cdot)$ is strictly convex in output y, the denominator is positive. Hence, a necessary and sufficient condition for $\frac{\partial y^*}{\partial \overline{m}} > 0$ is that $\frac{\partial^2 c(y^*, \overline{m}; \mathbf{w})}{\partial m \partial y} < 0$, i.e., an increase in the pollution ceiling, $\overline{m}$, reduces the *marginal* cost of production. As long as this (relatively reasonable) condition holds, an increase in the pollution ceiling $\overline{m}$ would induce the firm to increase production, i.e., $\frac{\partial y^*}{\partial \overline{m}} > 0$.

(d) *Emission fee.* Suppose now that the government abandons its emissions ceiling and replaces it with a tax $t > 0$ on gas emissions. Thus, the new cost of producing $(y; m)$ is given by $c(y; m; \mathbf{w}) + tm$. Show that maximized profits are convex in t, and that the firm's choice of pollution decreases in the pollution tax, i.e., $\frac{\partial m^*}{\partial t} \leq 0$.

- The profit maximization problem for the firm can now be written as selecting its output level and pollution to solve

$$\max_{y,m} \; py - c(y, m; \mathbf{w}) - tm$$

- Suppose (y, m), (y', m') and (y'', m'') maximize profits for tax levels t, t' and t'', respectively, where $t'' = \alpha t + (1 - \alpha)t'$ for any $\alpha \in (0, 1)$. By profit maximization it follows that

$$\begin{aligned} \pi(p, \mathbf{w}, t) &= py - c(y, m; \mathbf{w}) - tm \geq py'' - c(y'', m''; \mathbf{w}) - tm'', \text{ and} \\ \pi(p, \mathbf{w}, t') &= py' - c(y', m'; \mathbf{w}) - t'm' \geq py'' - c(y'', m''; \mathbf{w}) - t'm'' \end{aligned}$$

Hence, the linear combination of profit functions $\pi(p, \mathbf{w}, t)$ and $\pi(p, \mathbf{w}, t')$ yields

$$\begin{aligned} & \alpha\pi(p, \mathbf{w}, t) + (1 - \alpha)\pi(p, \mathbf{w}, t') \\ \geq \; & py'' - c(y'', m''; \mathbf{w}) - [\alpha t + (1 - \alpha)t']m'' \\ = \; & py'' - c(y'', m''; \mathbf{w}) - t''m'' = \pi(p, \mathbf{w}, t'') \end{aligned}$$

Then the profit function $\pi(p, \mathbf{w}, t)$ is convex in the tax level t. Intuitively, since profits are strictly decreasing in the emission fee t, the maximal profit that the firm can obtain from a linear combination of fees t and t', i.e., $t'' = \alpha t + (1 - \alpha)t'$, is lower than the linear combination of profits when the firm faces either a fee t or t'; as figure 1 depicts.

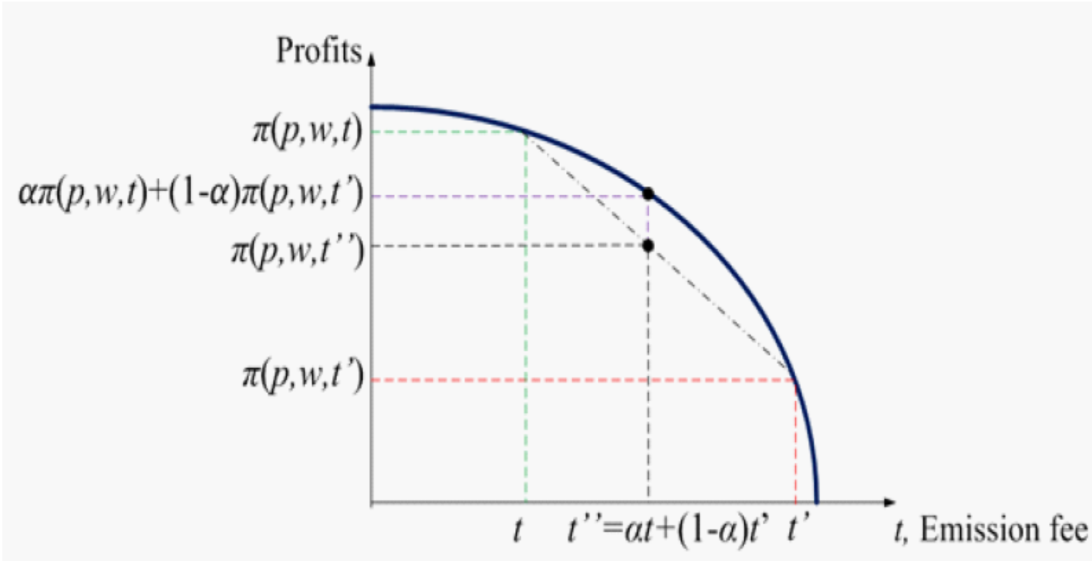

Figure 1. Convexity of profit function $\pi(p, w, t)$.

- Let $\mathbf{x}$ and $\mathbf{x}'$ be the input vectors for the profit-maximizing plans y and y' associated with taxes t and t', respectively. By profit maximization,

$$\pi(p, \mathbf{w}, t) = py - \mathbf{wx} - tm \geq py' - \mathbf{wx}' - tm'$$

and rearranging,

$$-p(y' - y) + \mathbf{w}(\mathbf{x}' - \mathbf{x}) + t(m' - m) \geq 0 \tag{1}$$

Similarly for tax level t',

$$\pi(p, \mathbf{w}, t') = py' - \mathbf{wx}' - t'm' \geq py - \mathbf{wx} - t'm$$

and rearranging,

$$p(y' - y) - \mathbf{w}(\mathbf{x}' - \mathbf{x}) - t'(m' - m) \geq 0 \tag{2}$$

Hence, by adding inequalities (1) and (2), we obtain

$$(t' - t)(m' - m) \leq 0$$

which means that the firm's choice of pollution level, m, decreases as the tax t increases, or in differential terms

$$\frac{\partial m^*}{\partial t} \leq 0.$$

2. **Regulating externalities under incomplete information.** Consider a setting where a regulator does not observes the marginal profits that a polluting firm obtains

from emitting additional pollution, but observes the damage that such additional pollution causes on consumers. In particular, suppose that the firm's marginal benefit from an additional unit of pollution, h, is

$$\frac{\partial \pi(h, \theta)}{\partial h} = \beta - bh + \theta,$$

and that the marginal utility from an additional unit of pollution for the consumer is

$$\frac{\partial \phi(h, \eta)}{\partial h} = \gamma - ch + \eta,$$

where θ is a random variable with expectation $E[\theta] = 0$, and strictly positive realizations, i.e., $\theta > 0$. Parameters b, c and γ are also strictly positive by definition, i.e., $b, c, \gamma > 0$. In this exercise, we will first determine which is the best quota and emission fee that the regulator can design given that he operates under incomplete information. Afterwards, we will evaluate the welfare that arises under each of these policy instruments, to determine which is better from a social point of view.

(a) *Setting a quota.* In this incomplete information setting, determine which is the best quota $\widehat{h}^*$ that a social planner can select in order to maximize the expected value of aggregate surplus.

- The firm must produce an output level exactly equal to the quota. The social planner determines the optimal quantity $\hat{h}^*$ by choosing the value of h that maximizes the expected value of aggregate surplus (since the social planner does not know the precise realization of parameter θ),

  $$\max_{\hat{h}} \phi(h, \eta) + E_\theta[\pi(h, \theta)]$$

  And taking first order condition with respect to h, we obtain

  $$\frac{\partial \phi(\hat{h}^*, \eta)}{\partial h} + E_\theta\left[\frac{\partial \pi(\hat{h}^*, \theta)}{\partial h}\right] \leq 0$$

  We can now substitute the functional forms for the marginal benefit for consumers, $\frac{\partial \phi(h,\eta)}{\partial h}$, and the marginal profits for the firm, $\frac{\partial \pi(h,\theta)}{\partial h}$, obtaining

  $$\gamma - c\hat{h}^* + \eta + \beta - b\hat{h}^* + E[\theta] \leq 0.$$

  Using $E[\theta] = 0$, we can solve form $\hat{h}^*$ to have

  $$\hat{h}^* \geq \frac{\gamma + \beta + \eta}{c + b}, \text{ with equality for } \hat{h}^* \geq 0$$

(b) *Setting an emission fee.* Find the best tax t^* that this social planner can set under the context of incomplete information described above.

- Given a tax t^*, the firm maximizes profits. That is, it chooses the level of h that maximizes its profits (net of tax payments), as follows

$$\max_h \pi(h, \theta) - th$$

The firm, hence, takes first order condition with respect to h, yielding

$$\frac{\partial \pi(h, \theta)}{\partial h} - t = 0$$

And since we know that $\frac{\partial \pi(h,\theta)}{\partial h} = \beta - bh + \theta$ by definition, the above first order condition becomes $\beta - bh + \theta - t = 0$. Solving for h, we obtain the firm's profit-maximizing externality $h(t, \theta)$, as a function of the tax rate t and its "type" θ, as follows

$$h(t, \theta) = \frac{\theta + \beta - t}{b}$$

Importantly, note that $h(t, \theta)$ describes the firm's "reaction function"(or "best response function") after observing that the regulator imposes a particular tax rate t. Provided this best response function, we can now find the optimal tax that the social planner imposes, anticipating the firm's best response function, as follows

$$\max_{t^*} \phi(h(t, \theta), \eta) + E[\pi(h(t, \theta), \theta)]$$

(where note that, rather than writing a general level of h, we wrote the level of h that the firm optimally chooses in the second stage, after observing the tax rate t imposed by the regulator in the first stage). Taking first order conditions with respect to h, yields

$$-\frac{\partial \phi(h(t, \theta), \eta)}{\partial h} \cdot \frac{\partial h(t, \theta)}{\partial t} = E\left[\frac{\partial \pi(h(t, \theta), \theta)}{\partial h} \cdot \frac{\partial h(t, \theta)}{\partial t}\right]$$

(note that we use the chain rule). Intuitively, the regulator equals the marginal disutility of additional pollution to consumers (which he can perfectly asses), as represented in the left-hand side of the equality; and the expected marginal profits from additional pollution for the firm (which he cannot perfectly observe), represented in the right-hand side of the above expression.

- Since $h(t, \theta) = \frac{\theta+\beta-t}{b}$ then the derivative $\frac{\partial h(t,\theta)}{\partial t} = -\frac{1}{b}$ is a constant, that can be taken out of the expectation operator. That is,

$$-\frac{\partial \phi(h(t, \theta), \eta)}{\partial h} \cdot \frac{\partial h(t, \theta)}{\partial t} = \frac{\partial h(t, \theta)}{\partial t} E\left[\frac{\partial \pi(h(t, \theta), \theta)}{\partial h}\right]$$

Therefore, we can cancel out the $\frac{\partial h(t,\theta)}{\partial t}$ term on both sides of the equality, which yields

$$-\frac{\partial \phi(h(t, \theta), \eta)}{\partial h} = E\left[\frac{\partial \pi(h(t, \theta), \theta)}{\partial h}\right].$$

Substituting the functional form of our marginal benefit and marginal profit functions, the above first-order condition becomes:

$$-\gamma + c \cdot h(t,\theta) - \eta = \beta - b \cdot h(t,\theta) + E[\theta]$$

Recalling that $E[\theta] = 0$, and that $h(t,\theta) = \frac{\theta+\beta-t}{b}$, the above expression can be simplified to

$$-\gamma + c\left(\frac{\theta+\beta-t}{b}\right) - \eta = \beta - b\left(\frac{\theta+\beta-t}{b}\right)$$

and solving for t, we find a fee

$$t^* = \theta + \frac{\beta c - b(\gamma+\eta)}{b+c}.$$

(c) *Policy comparison.* Compare the emission fee and the quota in terms of their associated deadweight loss. Under which conditions an uninformed regulator prefers to choose the emission fee?

- We need to compare the expected difference in losses in order to determine when a tax or a quota instrument is better. Figure 2 illustrates the welfare loss associated to tax t^*, which induces an externality level of $h(t^*, \theta)$.

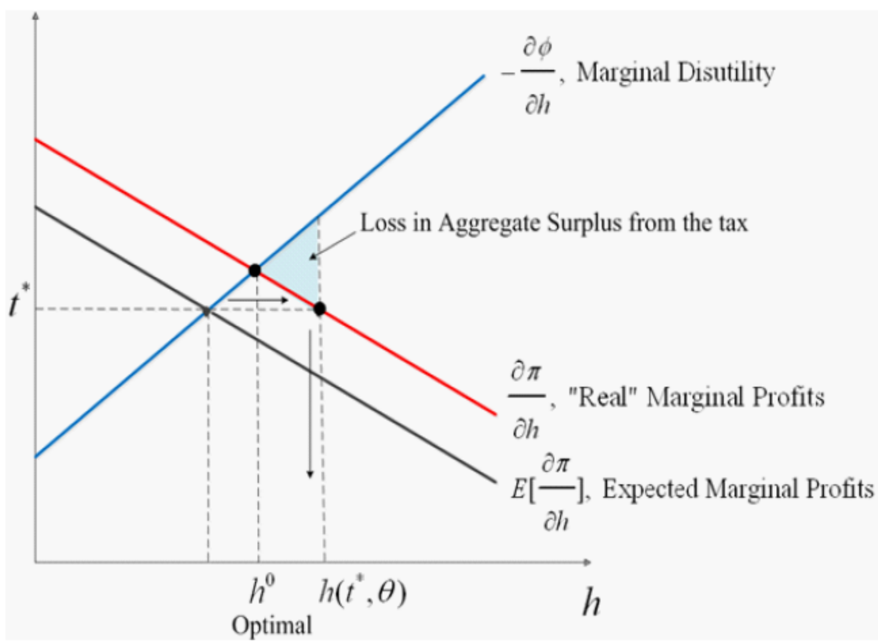

Figure 2. Setting an emission fee under incomplete information.

The figure considers that the regulator sets a tax based on the certain marginal disutility from the externality and the expected marginal profit. However, the realization of parameter θ implies that the real and expected marginal profits do not coincide, thus giving rise to a welfare loss associated to an imprecise tax, i.e., due to the regulator's imprecise information.

- While in the case of imposing a quota, $\widehat{h}^*$, figure 3 illustrates the associated welfare loss.

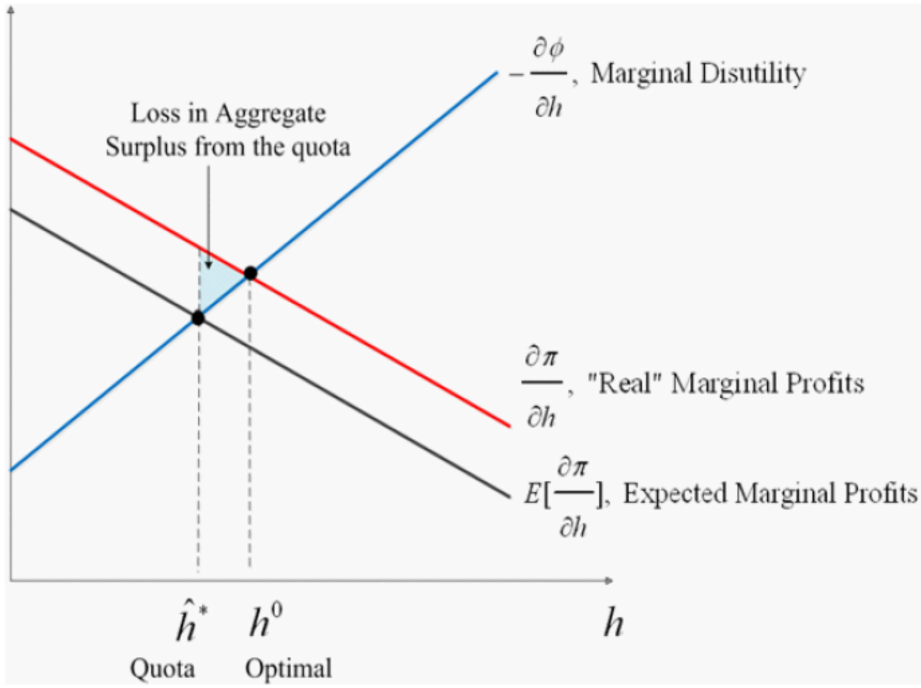

Figure 3. Setting a quota under incomplete information.

- *Welfare loss from the fee.* In order to compute the welfare loss from the tax, WL_t, we first need to find the socially optimal level of externality, h^0, given a realization θ. In particular, h^* solves

$$-\gamma + ch^o + \eta = \beta - bh^o + \theta$$

this yielding a socially optimal quota (if the regulator was perfectly informed) of $h^o = \frac{\beta+\gamma+\theta-\eta}{b+c}$. Moreover, the emission fee $t^* = \theta + \frac{\beta c - b(\gamma+\eta)}{b+c}$ induces an externality level of

$$\begin{aligned} h(t^*, \theta) &= \frac{\theta + \beta + t^*}{b} = \frac{\theta + \beta + \left(\theta + \frac{\beta c - b(\gamma+\eta)}{b+c}\right)}{b} \\ &= \frac{\beta + \gamma + \eta}{b + c}. \end{aligned}$$

Finally, we need to evaluate the marginal disutility function $-\frac{\partial \phi}{\partial h} = -\gamma + ch + \eta$ at $h(t^*, \theta) = \frac{\beta+\gamma+\eta}{b+c}$, which yields

$$-\gamma + ch(t^*, \theta) - \eta = -\gamma + c\frac{\beta + \gamma + \eta}{b + c} - \eta.$$

Hence, the WL_t is given by the area of the shaded triangle in figure 3,

$$\begin{aligned} WL_t &= \frac{1}{2}[h(t^*,\theta) - h^o] \cdot [(-\gamma + ch(t^*,\theta) - \eta) - t^*] \\ &= \frac{1}{2}\left[\frac{\beta+\gamma+\eta}{b+c} - \frac{\beta+\gamma+\theta-\eta}{b+c}\right] \cdot \\ &\quad \left[\left(-\gamma + c\frac{\beta+\gamma+\eta}{b+c} - \eta\right) - \left(\theta + \frac{\beta c - b(\gamma+\eta)}{b+c}\right)\right] \end{aligned}$$

which can be simplified to

$$WL_t = \frac{(\theta - 2\eta)\left[2c\left(\beta+\gamma+\eta\right) + \theta(b+c)\right]}{2(b+c)^2}$$

- *Welfare loss from the quota.* If, in constraint, the regulator uses a quota of $\widehat{h}^* = \frac{\gamma+\beta+\eta}{b+c}$, then we first need to evaluate the real marginal profits of the quota, that is
$$\beta - b\widehat{h}^* + \theta = \beta - \frac{b(\gamma+\beta+\eta)}{b+c} + \theta.$$
Second, we need to evaluate the expected marginal profit, $\beta - bh + E[\theta] = \beta - bh$, at the quota $\widehat{h}^*$, i.e., $\beta - b\frac{\gamma+\beta+\eta}{b+c}$. Therefore, the welfare loss from the quota is the area of the shaded triangle in figure 4. That is,
$$WL_q = \frac{1}{2}(h^o - \widehat{h}^*)\left[\left(\beta - \frac{b(\gamma+\beta+\eta)}{b+c} + \theta\right) - \left(\beta - \frac{b(\gamma+\beta+\eta)}{b+c}\right)\right]$$
which simplifies to
$$WL_q = \frac{1}{2}(h^o - \widehat{h}^*) \cdot \theta = \frac{\theta^2}{2(b+c)}$$

- *Comparing welfare losses.* Comparing WL_t and WL_q, we obtain that the difference $WL_t - WL_q$ is
$$WL_t - WL_q = \frac{(\beta+\gamma)c - b\eta\theta - 2c\eta(\beta+\gamma+\eta)}{(b+c)^2}$$
and solving for θ yields that $WL_t > WL_q$ if and only if $b < \bar{b}$, where
$$\bar{b} \equiv \frac{c(\beta+\gamma)}{\eta} - \frac{2c(\beta+\gamma+\eta)}{\theta}.$$
Hence, for all $b < \bar{b}$, the emission fee generates a larger welfare loss than the quota, thus implying that the quota is preferred. For illustrative purposes, figure 4 plots cutoff $\bar{b}$ evaluated at $\beta = \gamma = \frac{1}{2}$, $\eta = \frac{1}{4}$ and $\theta = 1$, thus becoming

$\bar{b} = \frac{3c}{2}$ (Other parameter values yield similar results.)

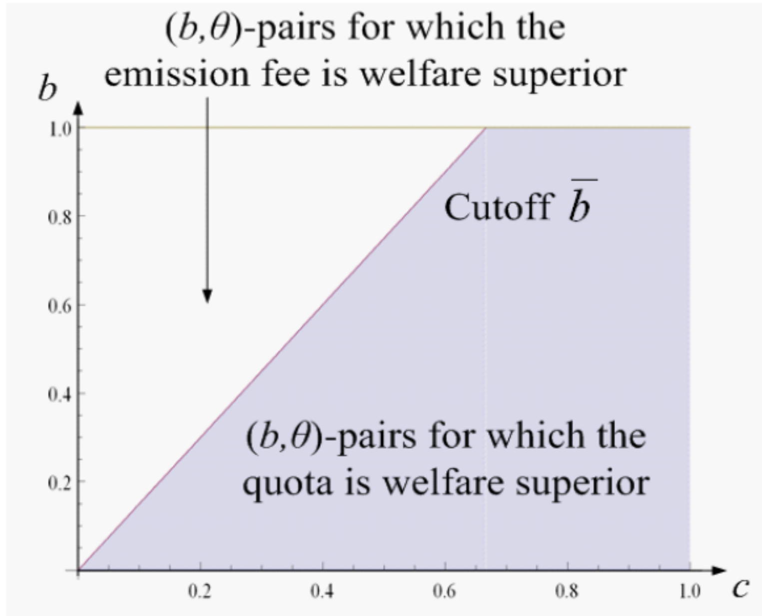

Figure 4. Policy comparison.

In particular, $(b, c)-$pairs below cutoff $\bar{b}$ (shaded area) illustrate settings in which the quota generates a lower welfare loss, thus becoming preferred by the uninformed regulator. In this setting, the marginal damage function (marginal profit function) is very sensitive (relatively insensitive, respectively) to additional amounts of the externality, i.e., further units of pollution are very damaging for consumers. The converse argument applies to $(b, c)-$pairs above cutoff $\bar{b}$ (unshaded area), where now marginal profits are relatively sensitive (i.e., rapidly increase) if the firm is allowed to increase pollution. Summarizing, for a given elasticity of the marginal profit function at the socially optimal level of the externality, the quota (emission fee) performs better when the marginal damage function is relatively inelastic (elastic, respectively).[2]

[2] For more details about the welfare properties of emission fees and quotas under contexts in which the regulator is imperfectly informed, see Weitzman (1974). For an application of Weitzman's results to a setting with marginal profit functions and marginal damage functions are linear (as in this exercise), see Mas-Colell, Whinston and Greene (1995).