The resource of woolly mammoths is overexploited by hunters. The excessive hunting of mammoths is an example of what Garrett Hardin dubbed the *tragedy of the commons*.[8] A **tragedy of the commons** is a situation in which two or more people are using a common resource and exploit it beyond the level that is best for the group as a whole. Overfishing Chilean sea bass, excessive deforestation of the Amazon jungle, and extracting oil too fast from a common reservoir are examples of the tragedy of the commons. Interdependence between players (and what economists call an "externality") is at the heart of this problem. When a hunter kills a woolly mammoth, he doesn't take into account the negative effect his action will have on the well-being of other hunters (i.e., they'll have fewer mammoths to kill). As a result, from the perspective of the human population as a whole, each hunter kills too many mammoths.

Surely the most important current example of the tragedy of the commons is global climate change. According to the U.S. Environmental Protection Agency, "Since the beginning of the industrial revolution, atmospheric concentrations of carbon dioxide have increased by nearly 30%, methane concentrations have more than doubled, and nitrous oxide concentrations have risen by about 15%."[9] During that same period, the average surface temperature of the planet has increased by $\frac{1}{2}$ to 1 degree Fahrenheit and sea level has risen 4–8 inches. Those are the facts about which there is little disagreement. Where controversy lies is whether the atmospheric changes have caused the rise in temperature. If, indeed, it has, then the only way to solve this tragedy of the commons is through coordinated action that limits behavior, such as was proposed with the Kyoto Accord.

▸ SITUATION: CHARITABLE GIVING AND THE POWER OF MATCHING GRANTS

In this final example, payoff functions are not hill shaped and, in fact, are not even continuous. This means that the method used in the previous two examples will not work here. So why do I present this example? First, it is a reminder that you should not willy-nilly use the calculus-based approach described at the start of Section 6.3. Such an approach requires that the payoff function be differentiable (continuous and with no kinks) and hill shaped. You must make sure that it satisfies those properties before applying that method. Second, even when the method cannot be used, calculus can still be useful in deriving a player's best reply function.

Suppose a philanthropist wants to raise \$3,000,000 for his favorite charity. Though he is quite wealthy, this sum is too much even for him. In order to spur others to contribute, he establishes a matching grant whereby he'll donate \$1,000,000 if \$2,000,000 is raised from other donors. Anything less than \$2,000,000, and he'll contribute nothing. This is hardly a novel scheme, as many charities and nonprofit organizations use it. Indeed, National Public Radio often uses similar schemes during its fund drives. Game theory shows how matching grants can generate more donations.

Suppose there are 10 prospective donors who are simultaneously deciding how much to contribute. Let s_i denote the donation of donor i and s_{-i} be the sum of all donations excluding that of donor i:

$$s_{-i} = s_1 + \cdots + s_{i-1} + s_{i+1} + \cdots + s_{10}.$$

Assume that a donor's strategy set is the interval from 0 to 500,000, measured in dollars. The donor i's payoff is specified as

$$\left(\frac{1}{5}\right)(s_{-i} + s_i) - s_i$$

and is made up of two parts: $(\frac{1}{5})(s_{-i} + s_i)$ is the benefit derived from money going to a worthy cause and depends only on the total contribution; $-s_i$ is the personal cost for making a contribution.

If there were no matching grant, would any contributions be made? Without a matching grant, donor 1's payoff function is

$$V_1(s_1, \ldots, s_n) = \left(\frac{1}{5}\right)(s_1 + \cdots + s_{10}) - s_1.$$

This payoff function is not hill shaped with respect to a donor's strategy. In fact, it is much simpler than that. Taking the first derivative of $V_1(s_1, \ldots, s_n)$ with respect to s_1, we have

$$\frac{\partial V_1(s_1, \ldots, s_n)}{\partial s_1} = -\frac{4}{5}.$$

A donor's payoff, then, always decreases with her contribution. For each dollar she contributes, the personal cost to her is \$1 and the benefit she attaches to it is only 20 cents. Thus, her payoff declines by 80 cents (or $\frac{4}{5}$ of a dollar) for every dollar she contributes; the more she gives, the worse she feels. Contributing nothing is then optimal. Since this is true regardless of the other donors' contribution, a zero contribution is the dominant strategy. Finally, because donor 1 is no different from the other nine donors, they have a zero contribution as a dominant strategy as well. There is then a unique Nash equilibrium in which all 10 donors contribute nothing. Our fund-raising campaign is off to a rather inauspicious start.

Now suppose there is a matching grant. Then donor 1's payoff function looks like this:

$$V_1(s_1, \ldots, s_n) = \begin{cases} (\frac{1}{5})(s_1 + \cdots + s_{10}) - s_1 & \text{if } s_1 + \cdots + s_{10} < 2{,}000{,}000 \\ (\frac{1}{5})(s_1 + \cdots + s_{10} + 1{,}000{,}000) - s_1 & \text{if } 2{,}000{,}000 \leq s_1 + \cdots + s_{10} \end{cases}.$$

If total contributions fall short of 2,000,000, then the payoff is the same as without a matching grant. However, if they reach that 2,000,000 threshold, then each donor's payoff jumps by the amount $(\frac{1}{5}) \times 1{,}000{,}000$, or 200,000. At this jump, the payoff function is not continuous and thus not differentiable, so we can't just start taking derivatives. A bit more care is required, but calculus will still come in handy.

Let's derive a donor's best-reply function. (Since the game is symmetric, donors have the same best-reply function.) Consider donor 1, and let $s_{-1}(= s_2 + \cdots + s_{10})$ denote the sum of the contributions of the other 9 donors. First note that if $s_{-1} \geq 2{,}000{,}000$, then the matching grant occurs regardless of donor 1's contribution, so her payoff is

$$\left(\frac{1}{5}\right)(s_1 + s_{-1} + 1{,}000{,}000) - s_1$$

for all values of s_1. The derivative of this expression with respect to s_1 is $-\frac{4}{5}$; thus, donor 1's payoff always decreases with her donation. Hence, the optimal contribution is the lowest feasible contribution, which is zero.

Now suppose $s_{-1} < 2{,}000{,}000$, so the other donors are not contributing enough to get the matching grant. As long as donor 1's contribution results in total contributions falling short of the 2,000,000 threshold—that is, if $s_1 + s_{-1} < 2{,}000{,}000$—then donor 1's payoff is

$$\left(\frac{1}{5}\right)(s_1 + s_{-1}) - s_1,$$

the derivative of which is $-\frac{4}{5}$, so her payoff strictly decreases with her contribution. Consequently, if donor 1 is not going to give enough to get the matching grant, then she ought to give zero. Next, suppose her contribution is sufficient to achieve the matching grant—that is, $s_1 + s_{-1} \geq 2{,}000{,}000$. Her payoff is then

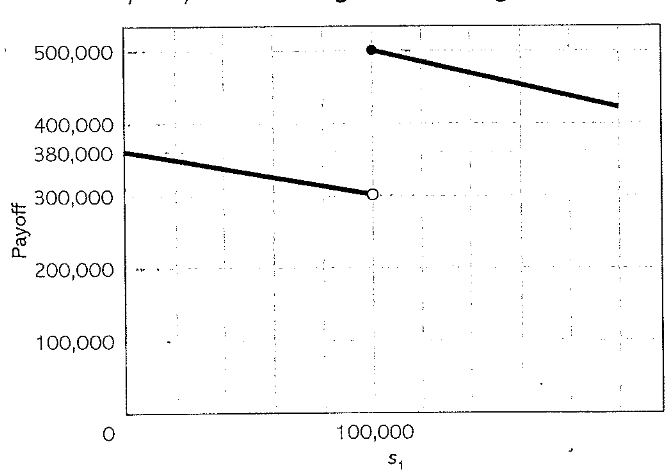
$$\left(\frac{1}{5}\right)(s_1 + s_{-1} + 1{,}000{,}000) - s_1.$$

Again, the derivative is $-\frac{4}{5}$, so her payoff is higher when she contributes less. Thus, conditional on giving enough to get the matching grant, donor 1 would find it best to give the smallest amount that does so, which is $2{,}000{,}000 - s_{-1}$.

To summarize, if $s_{-1} < 2{,}000{,}000$, then donor 1's best reply is either zero or the minimum amount required to get total contributions to 2,000,000. A couple of examples should solidify the logic behind this statement. Suppose $s_{-1} = 1{,}900{,}000$, so that donor 1's payoff function is as shown in FIGURE 6.18. If $s_1 < 100{,}000$, then total contributions fall short of the 2,000,000 goal, and in that range donor 1's payoff is

$$\left(\frac{1}{5}\right)(s_1 + 1{,}900{,}000) - s_1 = 380{,}000 - \left(\frac{4}{5}\right)s_1.$$

FIGURE 6.18 Donor 1's Payoff Function If $s_{-1} = 1{,}900{,}000$. Her Optimal Donation Is 100,000, Which Results in the 1,000,000 Matching Grant Kicking In



When s_1 hits 100,000, the payoff jumps to 500,000 as the 1,000,000 matching grant kicks in:

$$\left(\frac{1}{5}\right)(100{,}000 + 1{,}900{,}000 + 1{,}000{,}000) - 100{,}000 = 600{,}000 - 100{,}000 = 500{,}000.$$

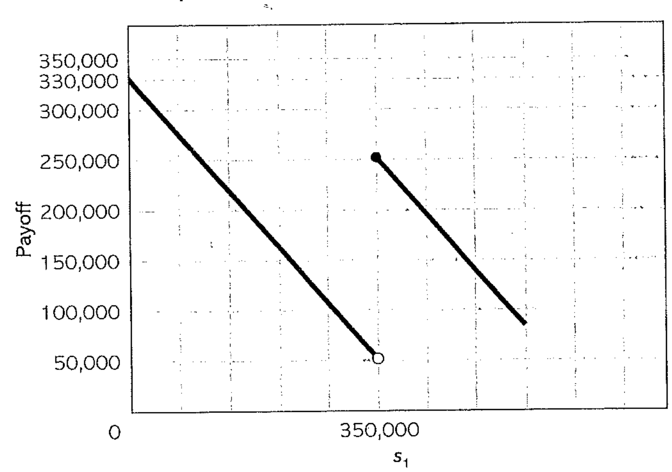
For $s_1 > 100{,}000$, donor 1's payoff is decreasing once more:

$$\left(\frac{1}{5}\right)(s_1 + 1{,}900{,}000 + 1{,}000{,}000) - s_1 = 580{,}000 - \left(\frac{4}{5}\right)s_1.$$

As Figure 6.18 indicates, donor 1's optimal donation is 100,000, which is the minimum amount required to get the matching grant.

Now suppose instead that $s_{-1} = 1{,}650{,}000$. In this case, it'll take a donation of 350,000 from donor 1 to get the matching grant. As depicted in **FIGURE 6.19**, donor 1 then prefers to contribute nothing. As before, her payoff declines until it reaches a level such that total contributions equal 2,000,000. At that point, it jumps from a payoff of 50,000 to 250,000 and again declines thereafter. Donor 1's payoff is maximized with a zero contribution. A donation of 350,000 to get the matching grant is just too much for any donor.

FIGURE 6.19 Donor 1's Payoff If s_{-1} = 1,650,000. Her Optimal Donation Is Zero



When the other donors have not given enough to get the matching grant, we have narrowed a donor's optimal contribution to being either zero or the minimum amount needed to get the grant. The next step is to compare these two options and determine when one is preferred over the other. Assuming that $s_{-1} < 2{,}000{,}000$, we calculate that donor 1 prefers to contribute so that total contributions just reach 2,000,000 rather than contribute zero when

$$\left(\frac{1}{5}\right)3{,}000{,}000 - (2{,}000{,}000 - s_{-1}) \geq \left(\frac{1}{5}\right)s_{-1}.$$

The left-hand side of this inequality is the payoff from contributing $2{,}000{,}000 - s_{-1}$, and the right-hand side is that from contributing zero. Solving the inequality for s_{-1}, we get

$$\left(\frac{4}{5}\right)s_{-1} \geq 1{,}400{,}000 \text{ or } s_{-1} \geq 1{,}750{,}000.$$

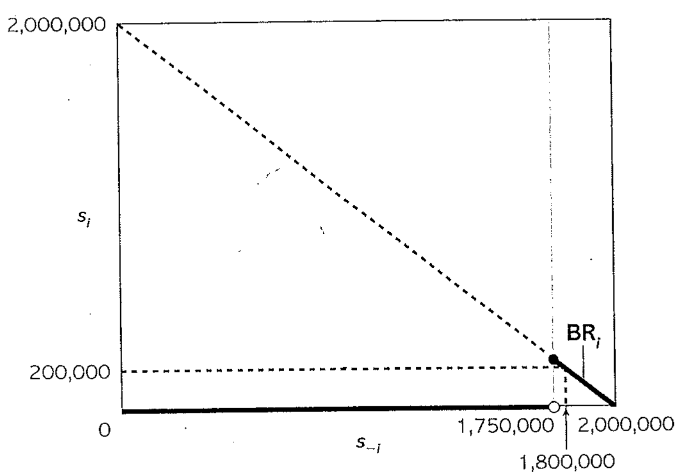
Thus, if $s_{-1} > 1{,}750{,}000$, then donor 1 optimally donates $2{,}000{,}000 - s_{-1}$ and secures the matching grant. If $s_{-1} < 1{,}750{,}000$, then donor 1 contributes zilch. She is indifferent between those two options when $s_{-1} = 1{,}750{,}000$ (and we will suppose she contributes 250,000).

By symmetry, this argument works for any donor. Thus, the best-reply function for donor i is

$$\mathrm{BR}_i = \begin{cases} 0 & \text{if } s_{-i} < 1{,}750{,}000 \\ 2{,}000{,}000 - s_{-i} & \text{if } 1{,}750{,}000 \leq s_{-i} < 2{,}000{,}000, \\ 0 & \text{if } 2{,}000{,}000 \geq s_{-i} \end{cases}$$

as depicted in **FIGURE 6.20**.

FIGURE 6.20 The Best-Reply Function for Donor i



Using the best-reply function, let's focus on finding symmetric Nash equilibria. We want to find a donation amount such that if the other 9 donors donate that amount, then it is optimal for an individual donor to do likewise. Figure 6.20 shows that one symmetric equilibrium is a zero donation. If each of the other 9 donors contribute zero, then $s_{-i} = 0$ and, according to donor i's best-reply function, his optimal donation is similarly zero.

Is there an equilibrium in which donors make a nonzero donation? Recall that if a donor contributes, her optimal contribution is the minimum amount necessary to achieve the 2,000,000 threshold in total donations. With 10 donors and given our focus on a symmetric strategy profile, this means that each donor contributes 200,000. In that case, $s_{-i} = 1{,}800{,}000$ (as the other 9 donors are each giving 200,000), and we can see in Figure 6.20 that an individual donor finds it optimal to respond with 200,000 as well. Hence, it is a Nash equilibrium for each donor to contribute 200,000 and thereby ensure that the matching grant kicks in.

Without the presence of a matching grant, the only equilibrium has no contributions being made. Thus, by offering to donate 1,000,000 if at least 2,000,000 in donations is raised, there is now an equilibrium in which donors contribute a total of 2,000,000 in order to get the matching grant. What the matching grant does is juice up the marginal impact of a donation. Given that the other donors contribute 1,800,000 in total, a contributor who gives 200,000 actually increases contributions by 1,200,000, so the matching grant induces her to contribute. Since this logic applies to all donors, each sees himself as making that incremental donation which brings forth the matching grant.

Summary

This chapter explored games with **continuous strategy sets**, as represented by an interval of real numbers. With an infinite number of strategy profiles, the exhaustive search is not a viable method for finding Nash equilibria. In Section 6.2, we showed how you can eliminate many strategy profiles as candidates for Nash equilibria by understanding players' incentives. In the example of price competition with identical products, each firm has an incentive to slightly undercut its rival's price when that price exceeds the firm's cost for the product. Because this undercutting incentive is present as long as shops price above cost, no strategy profile with price above cost is a Nash equilibrium. Using this idea allowed us to eliminate many possibilities and ultimately led us to the conclusion that shops pricing at cost is the unique Nash equilibrium.

In Section 6.3, we introduced a method for using calculus to solve for Nash equilibria. When a player's payoff function is differentiable (continuous and with no kinks) and hill shaped, his best reply is that strategy at which the first derivative of his payoff function (with respect to his strategy) is zero. If the derivative is positive (negative), then a player can increase his payoff by raising (lowering) his strategy. Only when the derivative is zero is that not possible and thus the payoff is maximized. This realization gave us an equation that could be easily solved for a player's best-reply function. All the players' best-reply functions could then be used to solve for a Nash equilibrium.

The calculus-based method just described was used to solve for Nash equilibrium in two games—first, when companies offer differentiated products and compete by choosing price, and second, when primitive humans exert an effort in hunting. An example exploring charitable donations reminded us that if we are to deploy the calculus-based method, it must first be determined that the payoff function is differentiable and hill shaped.

A common feature of Nash equilibria is that they are not payoff dominant among the set of all strategy profiles. That is, all players could be made better off relative to a Nash equilibrium if they all changed their strategies in a particular way. This is because, while each player individually maximizes her own payoff, she ignores the consequences of her strategy selection for the payoffs of other players. Players then do not act in their best collective