Advanced Microeconomics Theory

Chapter 8: Game Theory and Imperfect Competition

Outline

- Game Theory Tools
- Bertrand Model of Price Competition
- Cournot Model of Quantity Competition
- Product Differentiation
- Dynamic Competition
- Capacity Constraints
- Endogenous Entry
- Repeated Interaction

Introduction

- Monopoly: a single firm
- Oligopoly: a limited number of firms
  - When allowing for N firms, the equilibrium predictions embody the results in perfectly competitive and monopoly markets as special cases.

Game Theory Tools

Game Theory Tools

- Consider a setting with I players (e.g., firms, individuals, or countries) each choosing a strategy s_i from a strategy set S_i, where $s_i \in S_i$ and $i \in I$.
  - An output level, a price, or an advertising expenditure
- Let (s_i, s_{-i}) denote a strategy profile where s_{-i} represents the strategies selected by all firms $i \neq j$, i.e., $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_I)$.
- **Dominated strategy**: Strategy s_i^* strictly dominates another strategy $s_i' \neq s_i^*$ for player i if

$$\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}) \text{ for all } s_{-i}$$

  - That is, s_i^* yields a strictly higher payoff than s_i' does, regardless of the strategy s_{-i} selected by all of player i's rivals.

Game Theory Tools

- Payoff matrix (Normal Form Game)

		Firm B	
		Low prices	High prices
Firm A	Low prices	5, 5	9, 1
	High prices	1, 9	7, 7

- "Low prices" yields a higher payoff than "high prices" both when a firm's rival chooses "low prices" and when it selects "high prices."
  - "Low prices" is a strictly ***dominant strategy*** for both firms (i.e., s_i^*).
  - "High prices" is referred to as a strictly ***dominated strategy*** (i.e., s_i').

Game Theory Tools

- A strictly dominated strategy can be deleted from the set of strategies a rational player would use.
- This helps to reduce the number of strategies to consider as optimal for each player.
- In the above payoff matrix, both firms will select “low prices” in the unique equilibrium of the game.
- However, games do not always have a strictly dominated strategy.

Game Theory Tools

		Firm B	
		Adopt	Not adopt
Firm A	Adopt	3, 1	0, 0
	Not adopt	0, 0	1, 3

- $Adopt$ is better than $Not\ adopt$ for firm A if its opponent selects $Adopt$, but becomes worse otherwise.
- Similar argument applies for firm B.
- Hence, no strictly dominated strategies for either player.
- What is the equilibrium of the game then?

Game Theory Tools

- A strategy profile (s_i^*, s_{-i}^*) is a ***Nash equilibrium (NE)*** if, for every player i,
  $$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_{-i}^*) \text{ for every } s_i \neq s_i^*$$
  - That is, s_i^* is player i's best response to his opponents choosing s_{-i}^* as s_i^* yields a better payoff than any of his available strategies $s_i \neq s_i^*$.

Game Theory Tools

- In the previous game:
  - Firm A's best response to firm B's playing $Adopt$ is $BR_A(Adopt) = Adopt$, while to firm B playing $Not\ adopt$ is $BR_A(Not\ adopt) = Not\ adopt$.
  - Similarly, firm B's best response to firm A choosing U is $BR_B(Adopt) = Adopt$, whereas best response to firm A selecting D is $BR_B(Not\ adopt) = Not\ adopt$.
  - Hence, strategy profiles $(Adopt, Adopt)$ and $(Not\ adopt, Not\ adopt)$ are mutual best responses (i.e., the two Nash equilibria).

Mixed-Strategy Nash Equilibrium

- Insofar we restricted players to use one of their available strategies 100% of the time (commonly known as "**pure strategies**")
- Generally, players could randomize (mix) their choices.
  - *Example*: Choose strategy A with probability p and strategy B with probability $1-p$.

Mixed-Strategy Nash Equilibrium

- **Mixed-strategy Nash equilibrium** (msNE): Consider a strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, where σ_i is a mixed strategy for player i. Strategy profile σ_i is a msNE if and only if

$$\pi_i(\sigma_i, \sigma_{-i}) \geq \pi_i(s_i', \sigma_{-i}) \text{ for all } s_i' \in S_i$$

  - That is, σ_i is a best response of player i, i.e., $\sigma_i \in BR_i(\sigma_i)$, to the strategy profile σ_{-i} of the other $N - 1$ players.

Mixed-Strategy Nash Equilibrium

- Three points about msNE:
  1. Players must be indifferent among all (or at least some) of their pure strategies.
  2. Since players never use strictly dominated strategies, a msNE assigns a zero probability to dominated strategies.
  3. In games with a finite set of players and a finite set of available actions, there is generally an odd number of equilibria.

Mixed-Strategy Nash Equilibrium

- ***Example*** (no NE in pure strategies):

		Firm B	
		Adopt	*Not adopt*
Firm A	*Adopt*	$\underline{3}$, -3	-4, $\underline{0}$
	Not adopt	-3, $\underline{1}$	$\underline{2}$, -2

  – There is no cell of the matrix in which players select mutual best responses.

  – Thus we cannot find a NE in pure strategies.

  – Firm A (B) seeks to coordinate (miscoordinate) its decision with that of firm B (A, respectively).

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Given their opposed incentives, firm A would like to make its choice difficult to anticipate.
  - If firm A chooses a specific action with certainty, firm B will be driven to select the opposite action.
  - An analogous argument applies to firm B.
  - As a consequence, players have incentives to randomize their actions.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Let p (q) denote the probability with which firm A (B, respectively) adopts the technology.
  - If firm A is indifferent between adopting and not adopting the technology, its expected utility must satisfy

$$EU_A(Adopt) = EU_A(Not\ adopt)$$
$$3q + (-4)(1-q) = -3q + 2(1-q)$$
$$6q = 6(1-q) \Rightarrow q = 1/2$$

  - Hence firm B adopts the technology with probability $q = 1/2$.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Similarly, firm B must be indifferent between adopting and not adopting the technology:

$$EU_B(Adopt) = EU_B(Not\ adopt)$$
$$-3p + 1(1-p) = 0p + (-2)(1-p)$$
$$1 - p = p \Rightarrow p = 1/2$$

  - Hence firm A adopts the technology with probability $p = 1/2$.
  - Combining our results, we obtain the msNE

$$\left(\frac{1}{2}Adopt\frac{1}{2}Not\ adopt, \frac{1}{2}Adopt\frac{1}{2}Not\ adopt\right)$$

Mixed-Strategy Nash Equilibrium

- ***Example*** (Technology adoption game):

		Firm B	
		Adopt	*Not adopt*
Firm A	*Adopt*	3, 1	0, 0
	Not adopt	0, 0	1, 3

  - The game has two Nash equilibria in pure strategies: $(Adopt, Adopt)$ and $(Not\ Adopt, Not\ Adopt)$.
  - There is, however, a third Nash equilibria in which both firms use a mixed strategy.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Let p (q) denote the probability with which firm A (B, respectively) adopts the technology.
  - If firm A is indifferent between adopting and not adopting the technology, its expected utility must satisfy

$$EU_A(Adopt) = EU_A(Not\ adopt)$$
$$3q + 0(1-q) = 0q + 1(1-q)$$
$$3q = 1 - q \Rightarrow q = 1/4$$

  - Hence firm B adopts the technology with probability $q = 1/4$.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Similarly, firm B must be indifferent between adopting and not adopting the technology:

$$EU_B(Adopt) = EU_B(Not\ adopt)$$

$$1p + 0(1-p) = 0p + 3(1-p)$$

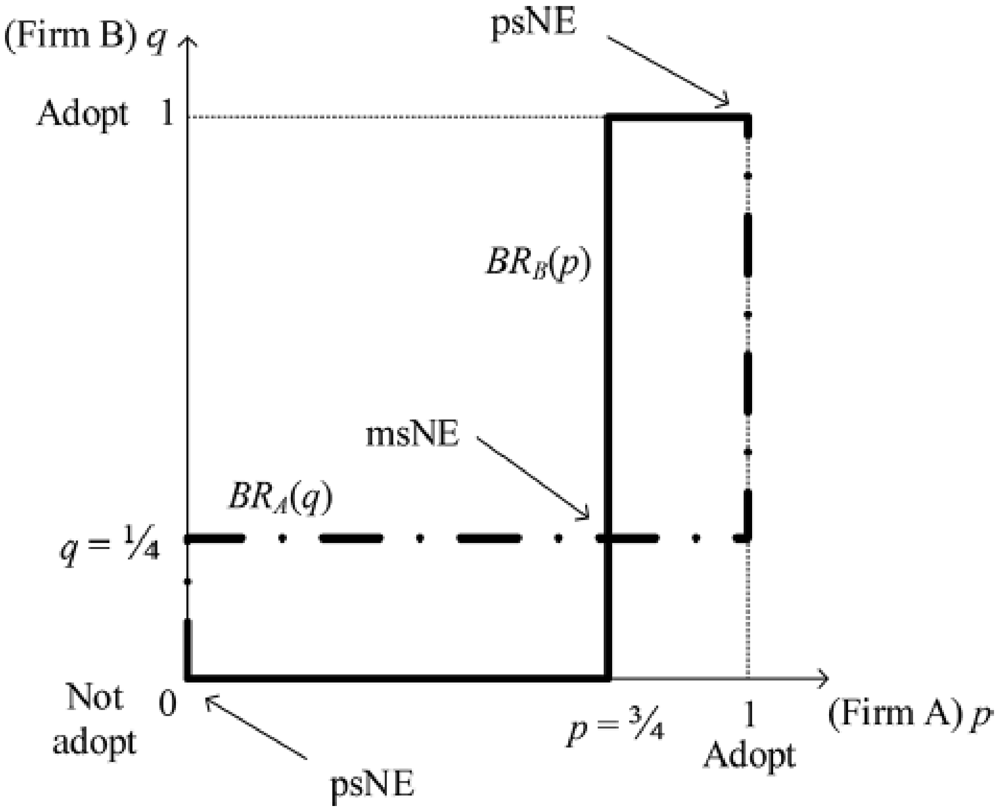
$$p = 3 - 3p \Rightarrow p = 3/4$$

  - Hence firm A adopts the technology with probability $p = 3/4$.
  - Combining our results, we obtain the msNE

$$\left(\frac{3}{4}Adopt\frac{1}{4}Not\ adopt, \frac{1}{4}Adopt\frac{3}{4}Not\ adopt\right)$$

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued): best-response



Sequential-Move Games

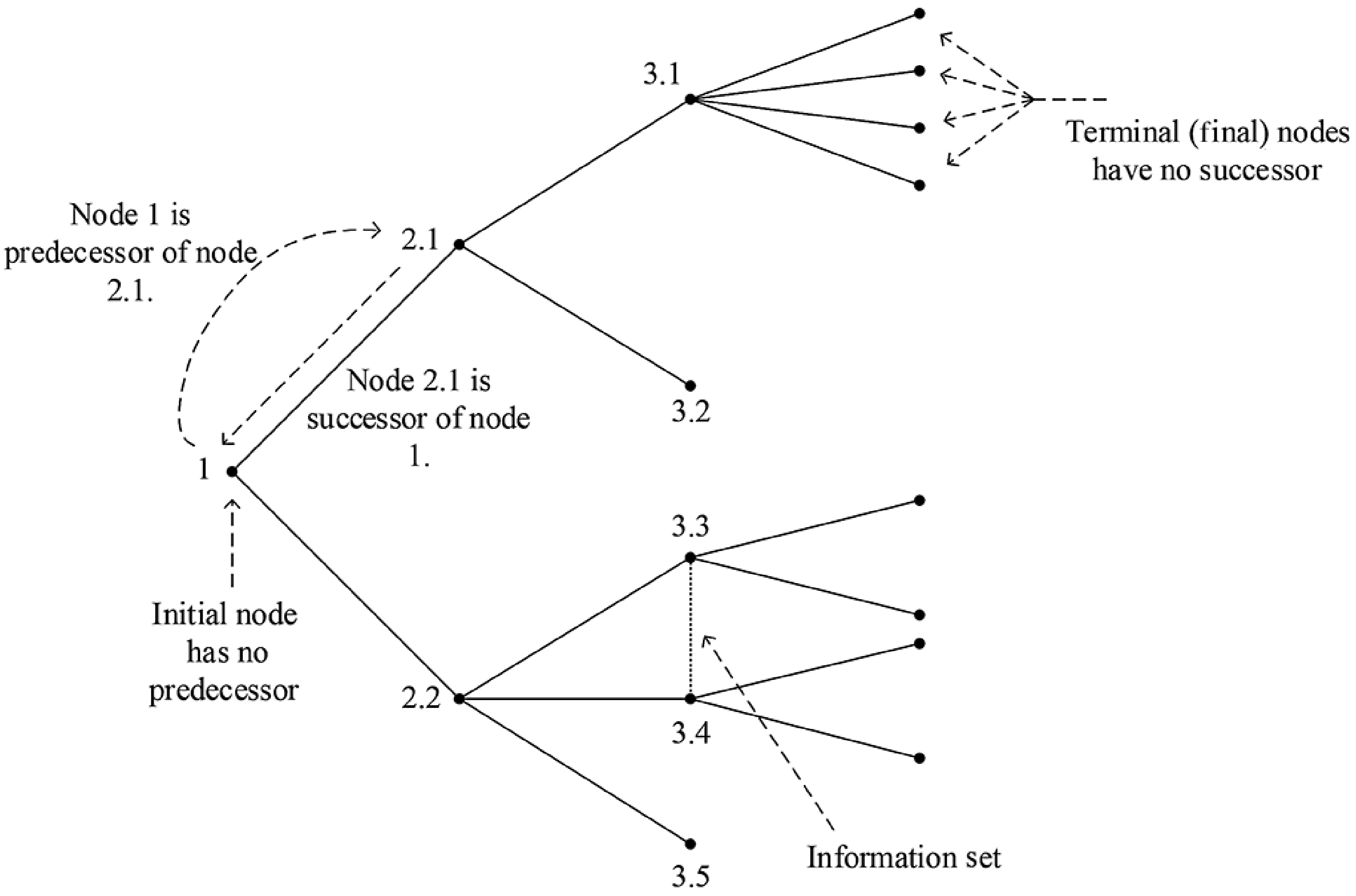
Sequential-Move Games

- When players choose their strategies *sequentially*, rather than *simultaneously*, the definition of strategy becomes more involved.
- Strategy is now a complete contingent plan describing what action player i chooses at each point at which he is called on to move, given the previous history of play.
- Such history can be observable or not observable by player i.

Sequential-Move Games

- Sequential-move games are represented using **game trees** rather than with matrices.
  - The “root” of the tree, where the game starts, is referred to as the **initial node**.
  - The last nodes of the tree, where no more branches originate, are the **terminal nodes**.

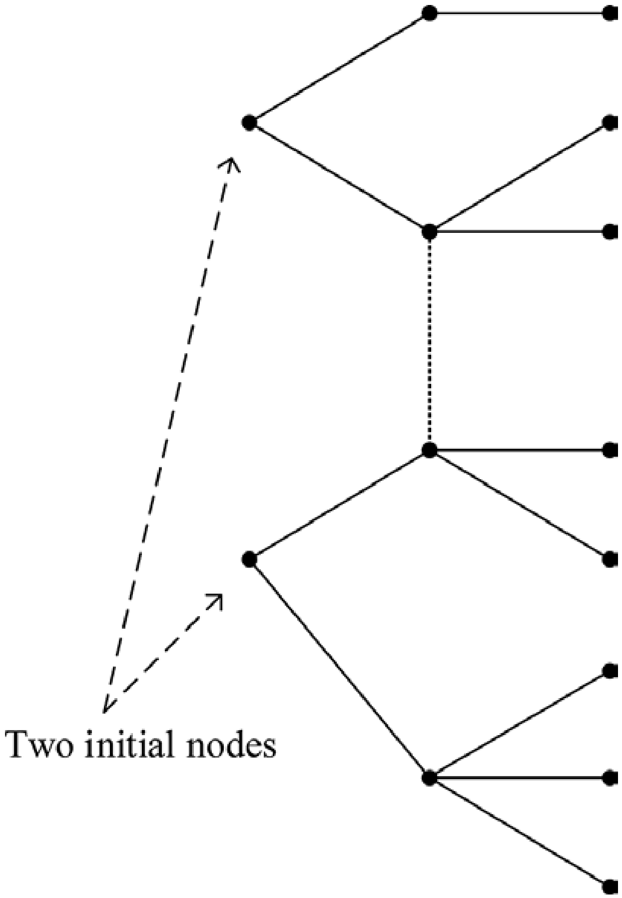
Sequential-Move Games



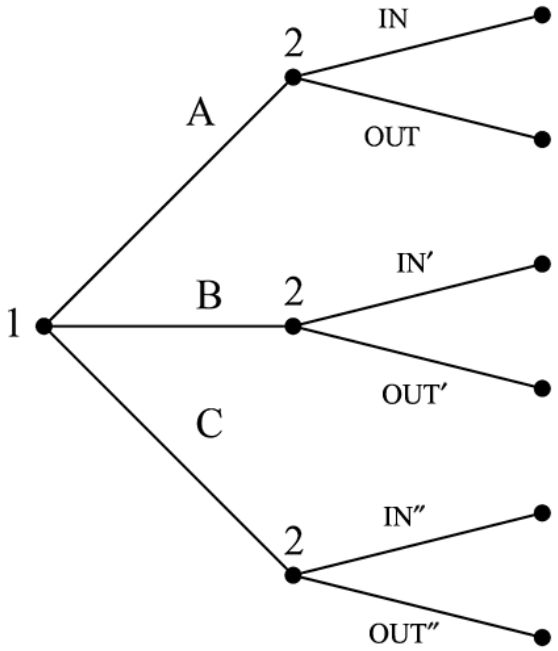
Sequential-Move Games

- Basic rules:
  1. A tree must have only one initial node.
  2. Every node of the tree has exactly one immediate predecessor except the initial node, which has no predecessor.
  3. Multiple branches extending from the same node must have different action labels.
  4. Every information set contains decision nodes for only one of the players in the game.
  5. All nodes in a given information set have the same immediate successors.

Sequential-Move Games



Sequential-Move Games



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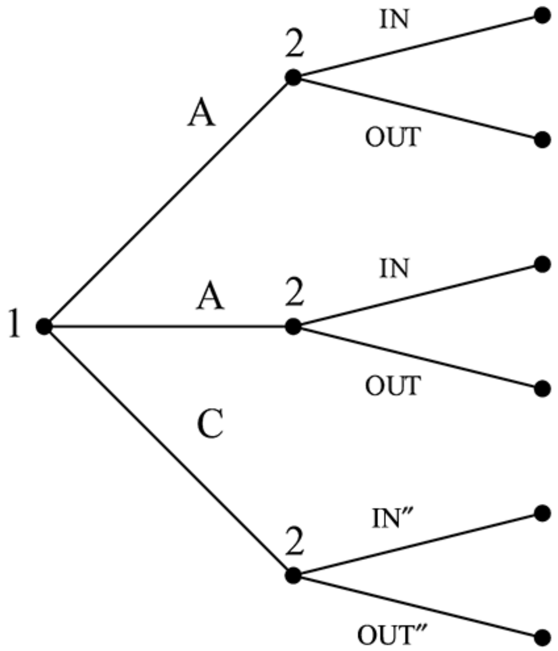
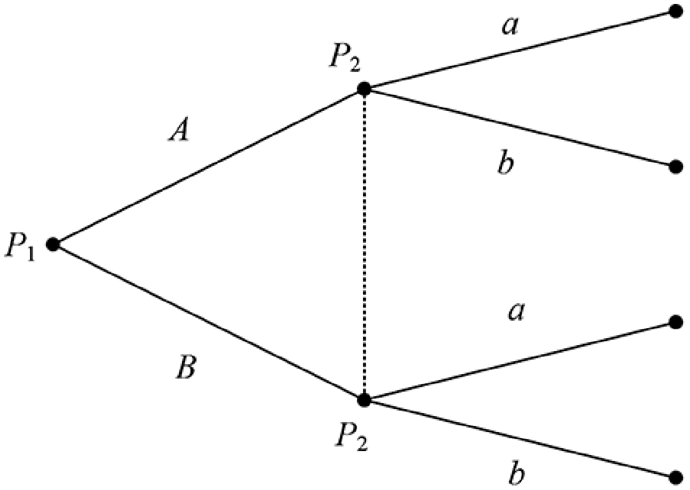


Incorrect

Sequential-Move Games

- **Information sets** are used to denote a group of nodes among which a player cannot distinguish.
- A common feature of trees representing games of **incomplete information**.
- Information sets arise when a player does not observe the action that his predecessor chose.

Sequential-Move Games



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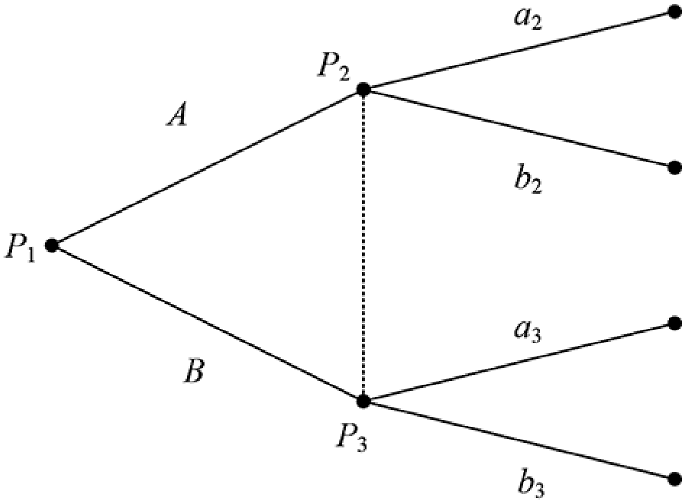

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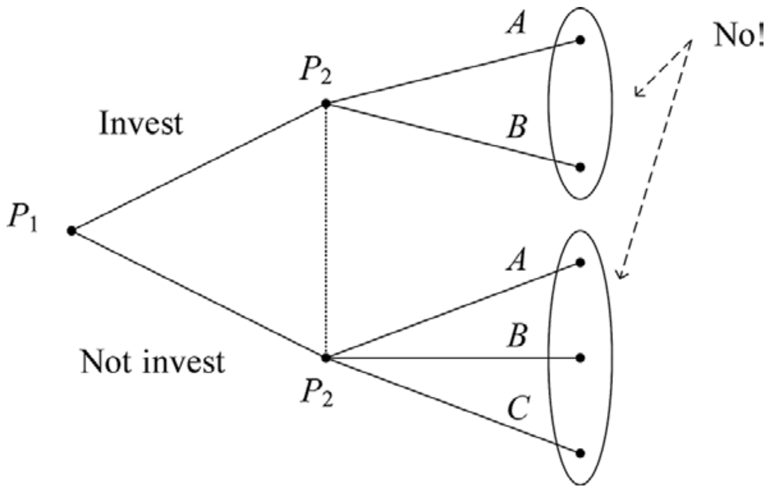
Sequential-Move Games

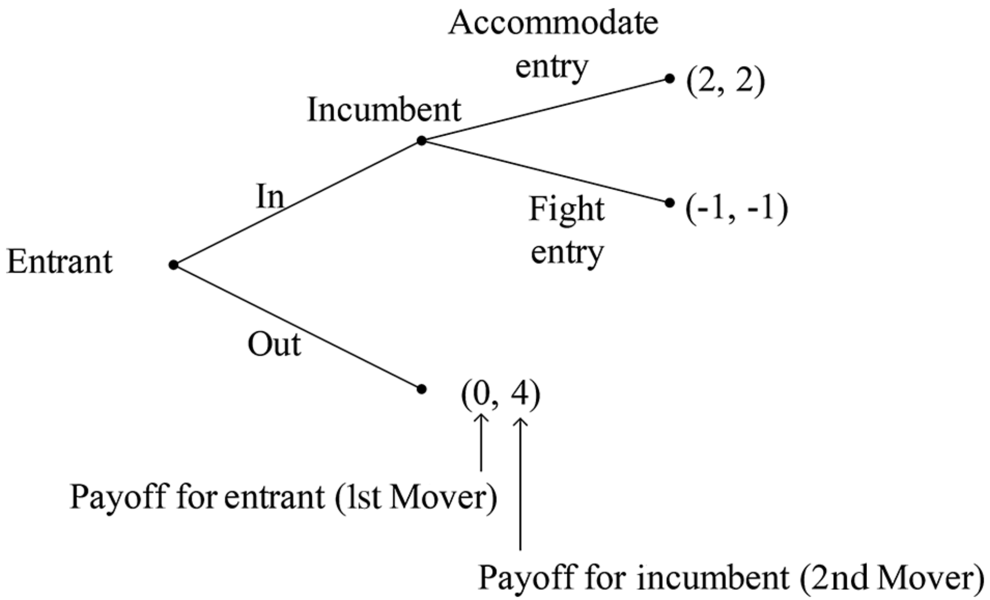


Sequential-Move Games

- Can we simply use the NE solution concept in order to find equilibrium predictions in sequential-move games?
- We can, but some of the NE predictions are not very sensible (credible).

Sequential-Move Games

- ***Example*** (Entry and predation game):
  - Consider an entrant's decision on whether to enter into an industry where an incumbent firm operates or to stay out.



Sequential-Move Games

- ***Example*** (continued):
  - In order to find the NE of this game, it is useful to represent the game in matrix form.

		Incumbent	
		Accommodate	*Fight*
Entrant	*In*	$\underline{2}, \underline{2}$	-1, -1
	Out	$0, \underline{4}$	$\underline{0}, \underline{4}$

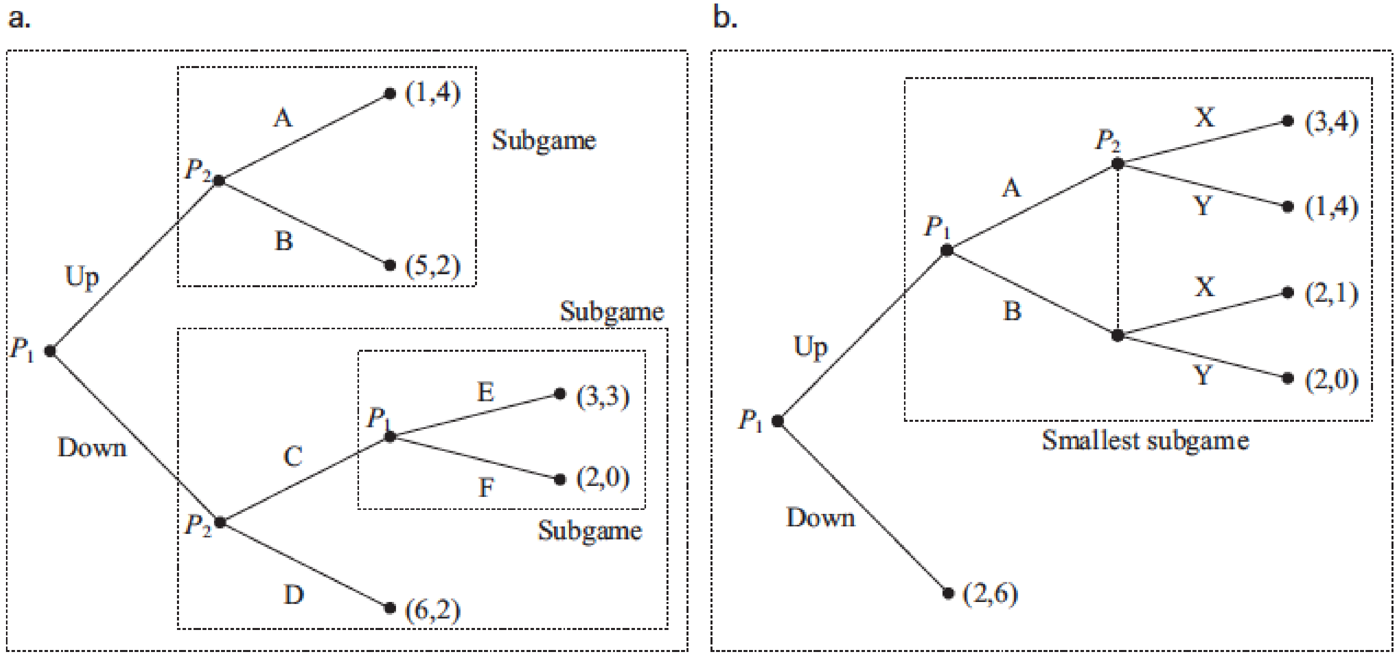
  - Two NEs: $(In, Accommodate)$ and $(Out, Fight)$
  - The first equilibrium seem credible, while the second equilibrium does not look credible at all.

Sequential-Move Games

- The preceding example indicates the need to require a notion of credibility in sequential-move games that did not exist in the NE solution concept
  - A requirement commonly known as "**sequential rationality**"
- Player i's strategy is sequentially rational if it specifies an optimal action for player i at any node (or information set) of the game, even those information sets that player i does not believe will be reached in the equilibrium of the game.
  - That is, player i behaves optimally at every node (or information set), both nodes that belong to the equilibrium path of the game tree and those that lie off-the-equilibrium path.
- How can we guarantee that it holds when finding equilibria in sequential-move games?
  - **Backward induction**: starting from every terminal node, each player uses optimal actions at every *subgame* of the game tree.

Sequential-Move Games

- A **subgame** can be identified by drawing a rectangle around a section of the game tree without "breaking" any information set



The game as a whole

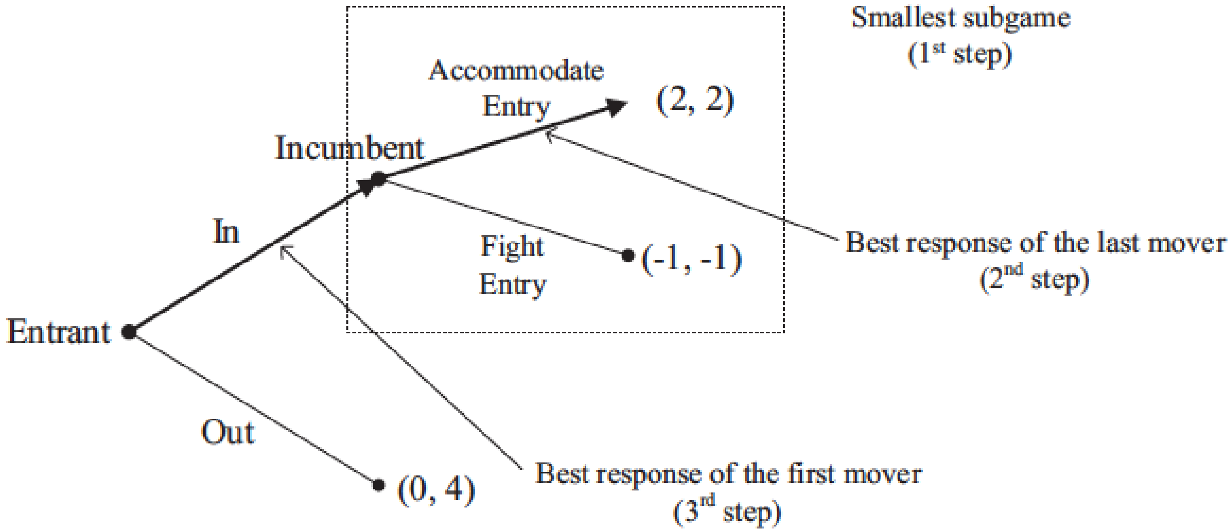
The game as a whole is the second smallest subgame

Sequential-Move Games

- The backward induction requires us to find the strategy that every player i finds optimal at every subgame along the game tree.
  - Start by identifying the optimal behavior of the player who acts last (in the last subgame of the tree).
  - Taking the optimal action of this player into account, move to the previous to the last player and identify his optimal behavior.
  - Repeat this process until the initial node.
- **Subgame perfect Nash equilibrium** (SPNE): A strategy profile $(s_1^*, s_2^*, \dots, s_N^*)$ is a SPNE if it specifies a NE for each subgame.

Sequential-Move Games

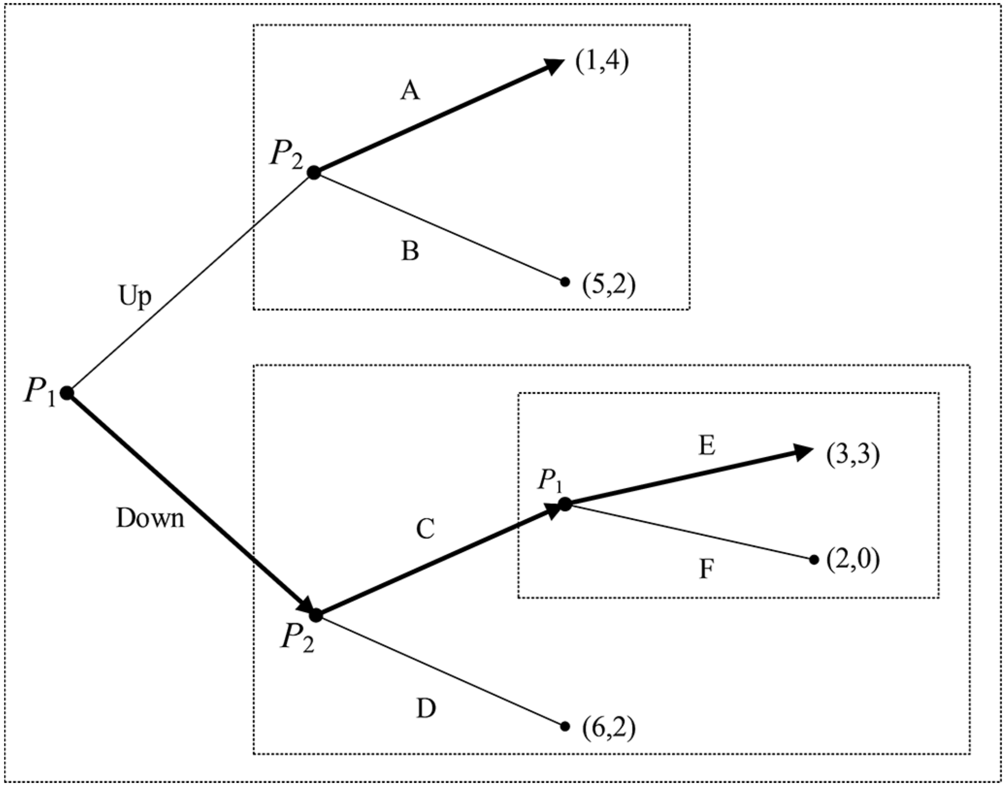
- ***Example*** (Entry and predation game):
  - Identify the subgames of the game tree



  - The SPNE is $(In, Accommodate)$, which coincides with one of the NE of this game.

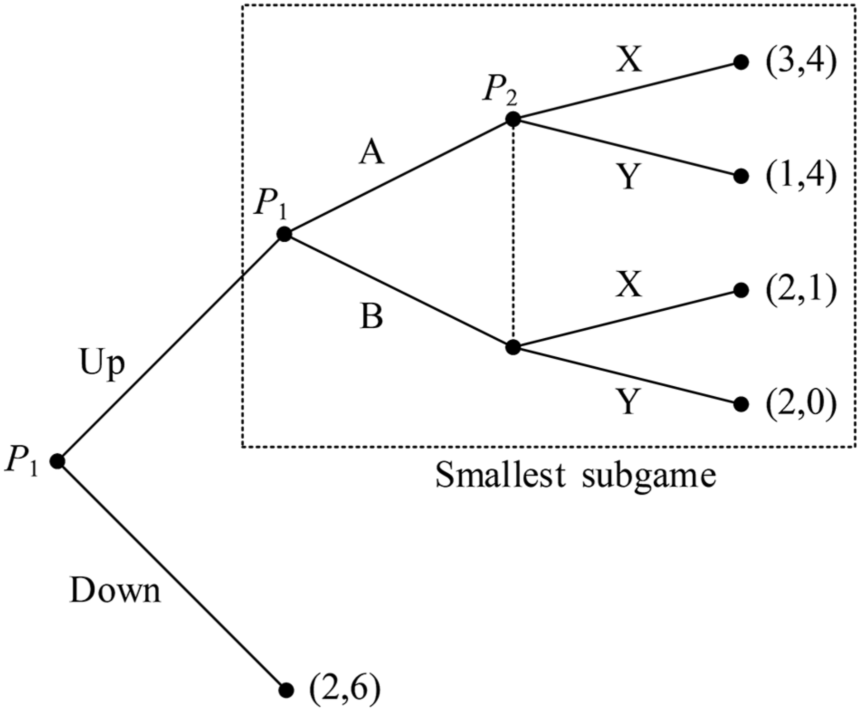
Sequential-Move Games

- ***Example*** (backward induction in three steps):



Sequential-Move Games

- ***Example*** (backward induction in information sets):



Sequential-Move Games

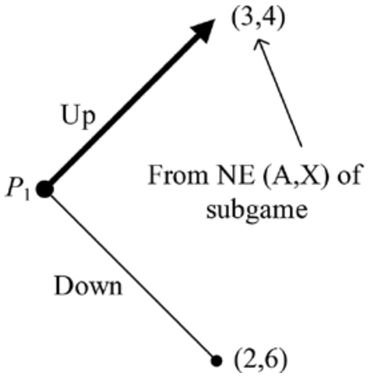
- ***Example*** (continued):
  - The smallest subgame is is strategically equivalent to one in which player 1 and 2 choose their actions simultaneously.

		Player 2	
		X	Y
Player 1	A	$\underline{3}, \underline{4}$	$1, \underline{4}$
	B	$2, \underline{1}$	$\underline{2}, 0$

  - The NE of the subgame is (A, X).

Sequential-Move Games

- ***Example*** (continued):
  - Once we have a reduced-form game tree, we can move one step backward (the initial node)



  - The SPNE of this game is $(Up|A, X)$.
  - Player 1's strategy: play Up in the first node and A afterwards
  - Player 2's strategy: play X

Sequential-Move Games

- ***Example*** (continued):
  - Normal-form representation of the sequential game

		Player 2	
		X	Y
Player 1	Up/A	$\underline{3}$, $\underline{4}$	1, $\underline{4}$
	Up/B	2, $\underline{1}$	$\underline{2}$, 0
	Down/A	2, $\underline{6}$	$\underline{2}$, $\underline{6}$
	Down/B	2, $\underline{6}$	$\underline{2}$, $\underline{6}$

  - Three NEs: $(Up|A, X)$, $(Down|A, Y)$, $(Down|B, Y)$.
  - Only the first equilibrium is sequentially rational.

Simultaneous-Move Games of Incomplete Information

Simultaneous-Move Games of Incomplete Information

- The strategic settings previously analyzed assume that all players are perfectly informed about all relevant details of the game.
- There are often real-life situations where players operate without such information.
- Players act under "**incomplete information**" if at least one player cannot observe a piece of information.
    - *Example*: marginal costs of rival firms

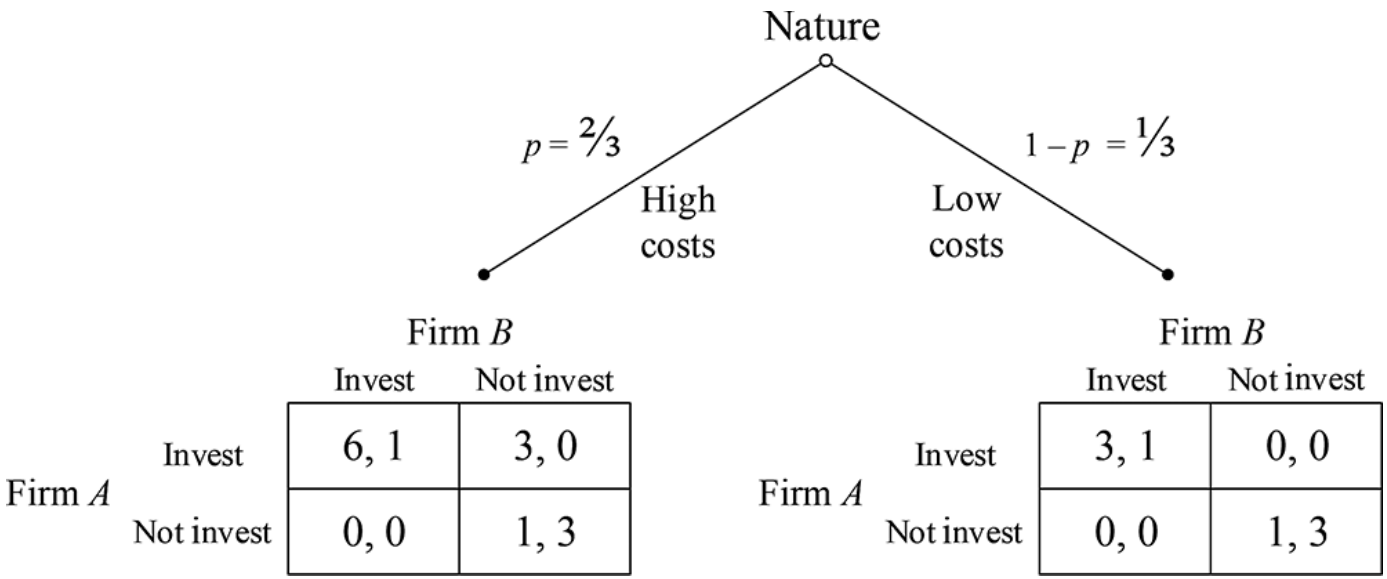
Simultaneous-Move Games of Incomplete Information

- For compactness, we refer to private information as player i's "type" and denote it as θ_i.
- While player j might not observe player i's type, he knows the probability distribution of each type.
- *Example*:
  - Marginal costs can be either high or low, whereby $\Theta_i = \{H, L\}$.
  - The probability of firm i's costs being high is $Prob(\theta_i = H) = p$ and the probability of its costs being low is $Prob(\theta_i = L) = 1 - p$, where $p \in (0,1)$.

Simultaneous-Move Games of Incomplete Information

- ***Example*** (technology adoption):
  - A first move of nature determines the precise type of θ_i.
  - Firm A has two possible types, either high or low costs, with associated probabilities 2/3 and 1/3.
  - Firm A observes its own type, but firm B cannot observe it.
  - Graphically, firm A knows which payoff matrix firms are playing, while firm B can only assign a probability 2/3 (1/3) to playing the left-hand (right-hand) matrix.

Simultaneous-Move Games of Incomplete Information



		Firm B	
		Invest	Not invest
Firm A	Invest	6, 1	3, 0
	Not invest	0, 0	1, 3

		Firm B	
		Invest	Not invest
Firm A	Invest	3, 1	0, 0
	Not invest	0, 0	1, 3

Simultaneous-Move Games of Incomplete Information

- Every player i's strategy in an incomplete information context needs to be a function of its privately observed type θ_i

$$s_i(\theta_i)$$

- Player i's strategy is not conditioned on other players' types

$$\theta_{-i} = (\theta_1, \theta_2, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$$

- That is, we do not write $s_i(\theta_i, \theta_{-i})$ because player i cannot observe the types of all other players.
  - If all players could observe the types of all of their rivals, we would be describing a complete information game.
- For simplicity, types are independently distributed, which entails that every player i cannot infer his rivals' types θ_{-i} after observing his own type θ_i.

Simultaneous-Move Games of Incomplete Information

- **Bayesian Nash equilibrium** (BNE): A strategy profile $(s_1^*(\theta_1), s_2^*(\theta_2), \dots, s_N^*(\theta_N))$ is a BNE of a game of incomplete information if

$$EU_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}) \geq EU_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i})$$

  for every strategy $s_i^*(\theta_i) \in S_i$, every type $\theta_i \in \Theta_i$, and every player i.

- When all other players select equilibrium strategies, the expected utility that player i obtains from selecting $s_i^*(\theta_i)$ when his type is θ_i is larger than that of deviating to any other strategy $s_i(\theta_i)$.

Simultaneous-Move Games of Incomplete Information

- **Approach 1**: Four steps to find all BNEs in simultaneous-move games of incomplete information.
- ***Example*** (technology adoption):
  1. *Strategy sets*: Identify the strategy set for each player, which can be a function of his privately observed type

$$S_1 = \{I_H I_L, I_H NI_L, NI_H I_L, NI_H NI_L\}$$
$$S_2 = \{I, NI\}$$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  2. *Bayesian normal-form representation*: Use the strategy sets identified in step 1 to construct the "Bayesian normal-form" representation of the incomplete information game.

		Firm B	
		I	NI
Firm A	$I_H I_L$		
	$I_H NI_L$		
	$NI_H I_L$		
	$NI_H NI_L$		

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  3. *Expected payoffs*: Find the expected payoffs that would go in every cell.

		Firm B	
		I	NI
Firm A	$I_H I_L$	5, 1	2, 0
	$I_H NI_L$	4, 2/3	2 1/3, 1
	$NI_H I_L$	1, 1/3	2/3, 2
	$NI_H NI_L$	0, 0	1, 3

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  4. *Find best responses for each player*: Follow an approach similar to that in simultaneous-move games of complete information to find best-response payoffs.

     – The BNEs are $(I_H I_L, I)$ and $(I_H NI_L, NI)$.

Simultaneous-Move Games of Incomplete Information

- **Approach 2**: Find the set of BNEs by first analyzing best responses for the privately informed player, and then use those in our identification of best responses for the uninformed player.

Simultaneous-Move Games of Incomplete Information

- ***Example*** (technology adoption):
  - The two possible games that firms could be playing.

			Firm B	
			Invest β	Not invest $1-\beta$
Firm A	Invest	α	6, 1	3, 0
	Not invest	$1-\alpha$	0, 0	1, 3

Firm A is high type with probability p

			Firm B	
			Invest β	Not invest $1-\beta$
Firm A	Invest	γ	3, 1	0, 0
	Not invest	$1-\gamma$	0, 0	1, 3

Firm A is low type with probability $1-p$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  - First, we look at the privately informed firm A.
  - If firm A is of the high type, $Invest$ strictly dominates $Not\ invest$.
  - If firm A is of the low type, neither strategy strictly dominates the other.
  - Need to compare the expected utilities
  $$EU_A(Invest|Low) = 3 \cdot \beta + 0 \cdot (1-\beta) = 3\beta$$
  $$EU_A(Not\ invest|Low) = 0 \cdot \beta + 1 \cdot (1-\beta) = 1-\beta$$
  - Firm A invests if $3\beta \geq 1-\beta$ or $\beta \geq 1/4$.

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  - Next, we look at the uninformed firm B.
  - Since firm B does not know firm A's type, we have to model in the probability (p) that firm A is of the high type.

$$EU_B(Invest)$$

$$= \overbrace{1 \cdot p}^{\text{If firm A is high type, it invests}} + (1-p) \cdot \overbrace{\left[\underbrace{1 \cdot \gamma}_{Firm\ A\ invests\ when\ low\ type} + \underbrace{0 \cdot (1-\gamma)}_{Firm\ A\ does\ not\ invest\ when\ low\ type} \right]}^{\text{If firm A is low type}}$$

$$= p + (1-p)\gamma$$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):

$EU_B(Invest)$

$$= \underbrace{1\cdot p}_{\text{If firm A is high type, it invests}} + (1-p)\cdot \overbrace{\left[\underbrace{1\cdot\gamma}_{Firm\ A\ invests\ when\ low\ type} + \underbrace{0\cdot(1-\gamma)}_{Firm\ A\ does\ not\ invest\ when\ low\ type}\right]}^{\text{If firm A is low type}}$$

$= p + (1-p)\gamma$

$EU_B(Not\ invest)$

$$= \underbrace{0\cdot p}_{\text{If firm A is high type, it invests}} + (1-p)\cdot \overbrace{\left[\underbrace{0\cdot\gamma}_{Firm\ A\ invests\ when\ low\ type} + \underbrace{3\cdot(1-\gamma)}_{Firm\ A\ does\ not\ investwhen\ low\ type}\right]}^{\text{If firm A is low type}}$$

$= 3(1-p)(1-\gamma)$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  - Therefore, firm B invests if

$$p + (1 - p)\gamma \geq 3(1 - p)(1 - \gamma)$$

  - Since $p = 2/3$, the above inequality reduces to

$$2 \geq 3 - 4\gamma$$

$$\gamma \geq 1/4$$

  - Two BNEs:
    1. If $\gamma \geq 1/4$, $(I_H I_L, I)$.
    2. If $\gamma < 1/4$, $(I_H NI_L, NI)$.

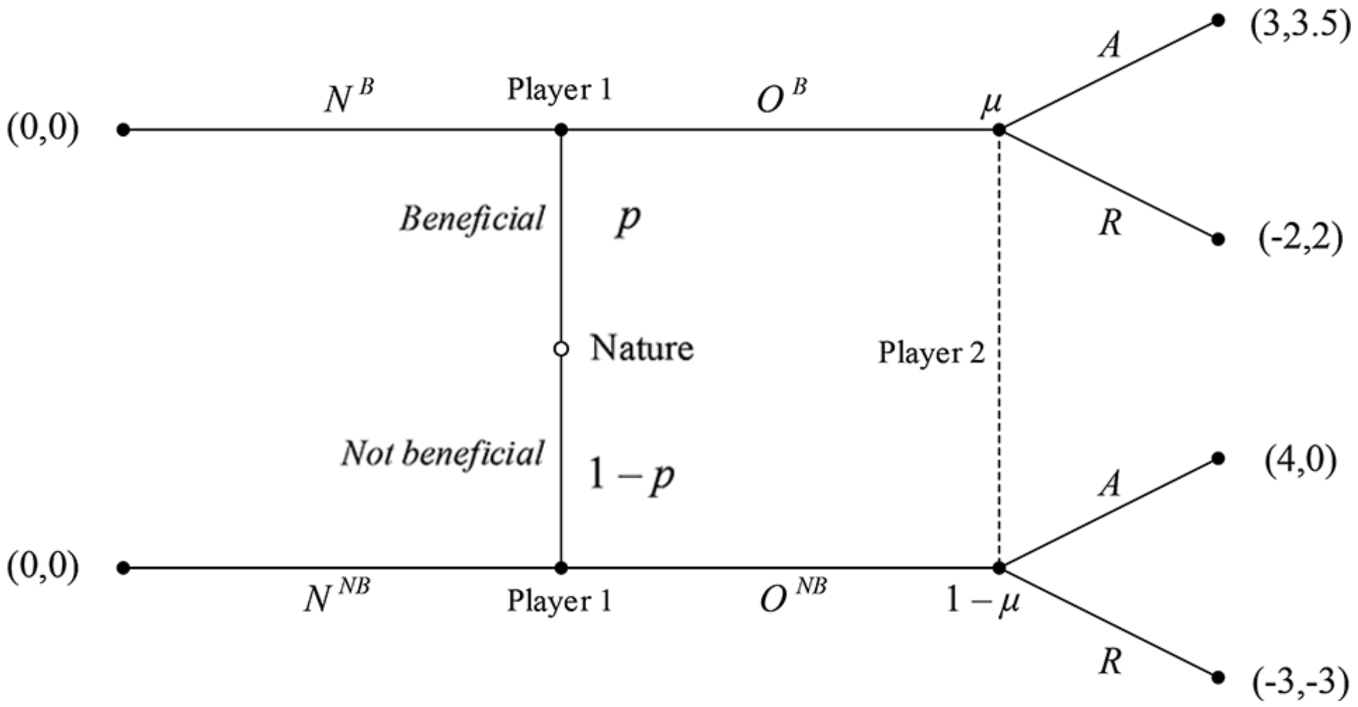
Sequential-Move Games under Incomplete Information

Sequential-Move Games under Incomplete Information

- The BNE solution concept helps us find equilibrium outcomes in settings where players interact under *incomplete information*.
- While the applications in the previous section considered that players act *simultaneously*, we can also find the BNEs of incomplete information games in which players act *sequentially*.

Sequential-Move Games under Incomplete Information

- Investment game:



Sequential-Move Games under Incomplete Information

- In order to find the set of BNEs, we first represent the Bayesian *normal-form* representation of the game tree.
- The matrix includes expected payoffs for each player.

		Player 2	
		A	R
Player 1	$O^B O^{NB}$	$\underline{4-p}$, $\underline{3.5p}$	-3+p, -3+5p
	$O^B N^{NB}$	3p, $\underline{3.5p}$	-2p, 2p
	$N^B O^{NB}$	4-4p, $\underline{0}$	-3+3p, -3+3p
	$N^B N^{NB}$	0, $\underline{0}$	$\underline{0}$, $\underline{0}$

Sequential-Move Games under Incomplete Information

- There are two BNEs in this game:

$$(O^B O^{NB}, A)$$
$$(N^B N^{NB}, R)$$

- The first BNE is rather sensible
  - Player 1 makes the offer regardless of his type, and thus the uninformed player 2 chooses to accept the offer if he receives one.

Sequential-Move Games under Incomplete Information

- The second BNE is difficult to rationalize
  - No type of sender makes an offer in equilibrium, and the responder rejects any offer presented to him.
  - If an offer was ever observed, the receiver should compare the expected utility of accepting and rejecting the offer, based on the off-the equilibrium belief μ.

$$EU_2(A) = 3.5 \cdot \mu + 0 \cdot (1 - \mu) = 3.5\mu$$

$$EU_2(R) = 2 \cdot \mu + (-3) \cdot (1 - \mu) = -3 + 5\mu$$

  - Player 2 accepts the offer, since $3.5\mu > -3 + 5\mu \Rightarrow 1.5\mu < 3$, which holds for all $\mu \in (0,1)$.
  - Therefore, the offer rejection that $(N^B N^{NB}, R)$ prescribes cannot be sequentially rational.

Sequential-Move Games under Incomplete Information

- In order to avoid identifying equilibrium predictions that are not sequentially rational, we apply the Perfect Bayesian Equilibrium (PBE) that can deal with sequential move games with incomplete information.
- **The Perfect Bayesian Equilibrium** (PBE): A strategy profit $(s_1, s_2, \dots, s_N)$ and beliefs μ over the nodes at all information sets are a PBE if:
  1. each player's strategies specify optimal actions, given the strategies of the other players, and given his beliefs, and
  2. beliefs are consistent with Bayes's rule, whenever possible.

Sequential-Move Games under Incomplete Information

- The first condition resembles the definition of BNE.
- The second condition was not present in the definition of BNE.
  - It states that beliefs must be consistent with Bayes's rule whenever possible
- Applying Bayes's rule in the investment game, player 2's probability that the investment is beneficial after receiving an offer is

$$p(B|\text{Offer}) = \frac{p(B) \cdot p(\text{Offer}|B)}{p(\text{Offer})}$$

$$= \frac{p(B) \cdot p(\text{Offer}|B)}{p(B) \cdot p(\text{Offer}|B) + p(NB) \cdot p(\text{Offer}|NB)}$$

Sequential-Move Games under Incomplete Information

- Denoting $\mu = p(B|\text{Offer})$, $\alpha_i = p(\text{Offer}|i)$, where $i = \{B, NB\}$, $p = p(B)$, and $1 - p = p(NB)$, player 2's belief can be expressed as

$$\mu = \frac{p \cdot \alpha_B}{p \cdot \alpha_B + (1-p) \cdot \alpha_{NB}}$$

- If player 2 assigns probabilities $\alpha_B = 1/8$ and $\alpha_{NB} = 1/16$, then

$$\mu = \frac{1/2 \cdot 1/8}{1/2 \cdot 1/8 + 1/2 \cdot 1/16} = \frac{2}{3}$$

- We refer to μ as off-the-equilibrium beliefs
  - The probability of being in a node of an information set that is actually not reached in equilibrium.

Sequential-Move Games under Incomplete Information

- **Procedure to Find PBEs:**
  1. Specify a strategy profile for the privately informed player.
     - In the investment example, there are four possible strategy profiles for the privately informed player 1.
     - Two separating strategy profiles: $O^B N^{NB}$, $N^B O^{NB}$.
     - Two pooling strategy profiles: $O^B O^{NB}$, $N^B N^{NB}$.

  2. Update the **uninformed** player's beliefs using Bayes's rule at all information sets, whenever possible.

Sequential-Move Games under Incomplete Information

- **Procedure to Find PBEs:** (continued)
  3. Given the **uninformed** player's updated beliefs, find his optimal response
     - In the investment example, we need to determine the optimal response of player 2 upon receiving an offer from player 1 given his updated belief.
  4. Given the optimal response of the uninformed player obtained in step 3, find the optimal action (message) for each type of **informed** player.
     - In the investment example, first check if player 1 makes an offer when the investment is beneficial.
     - Then check whether player 1 prefers to make an offer, when the investment is not beneficial.

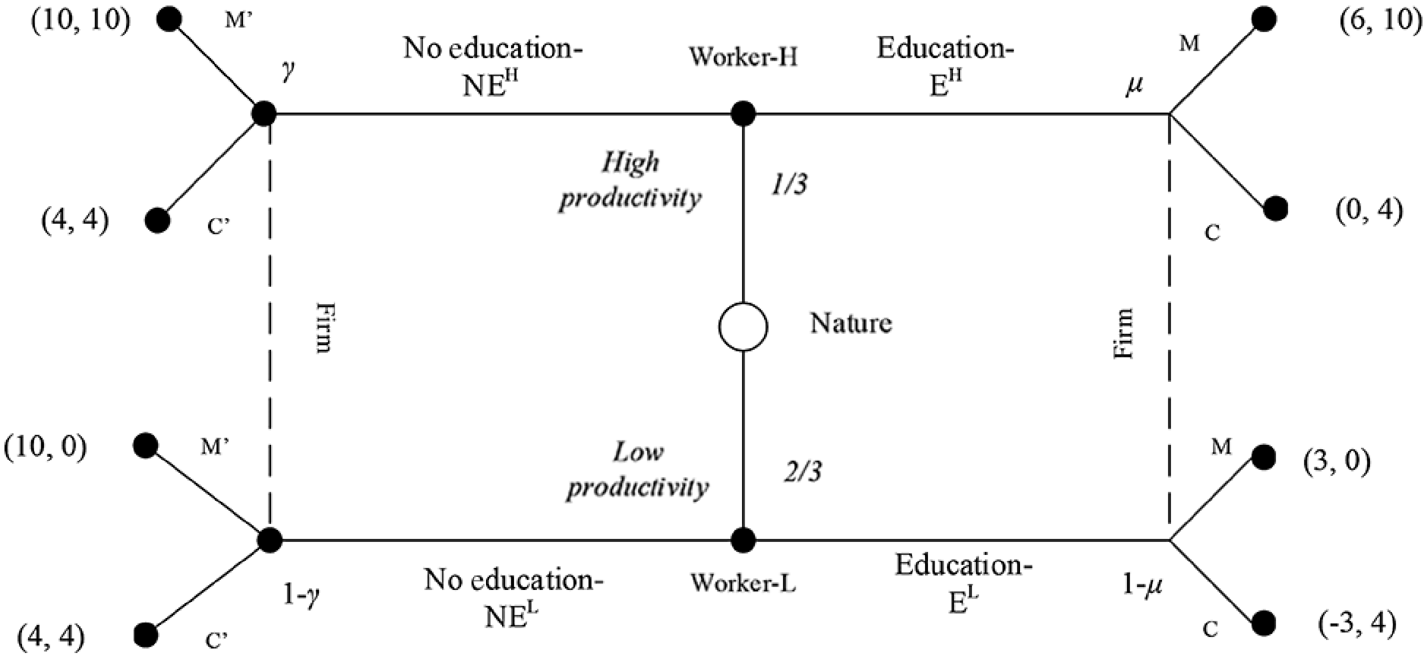
Sequential-Move Games under Incomplete Information

- **Procedure to Find PBEs:** (continued)
  5. Check if the strategy profile for the informed player found in step 4 coincides with the profile suggested in step 1.
     - If it coincides, then this strategy profile, updated beliefs, and optimal responses can be supported as a PBE of the incomplete information game.
     - Otherwise, we say that this strategy profile cannot be sustained as a PBE of the game.

Sequential-Move Games under Incomplete Information

- ***Example*** (Labor market signaling game):
  - The sequential game with incomplete information.
  - A worker privately observes whether he has a high productivity or a low productivity.
  - The worker then decides whether to pursue more education (e.g., an MBA) that he might use as a signal about his productivity.
  - The firm can either hire him as a manager (M) or as a cashier (C).

Sequential-Move Games under Incomplete Information



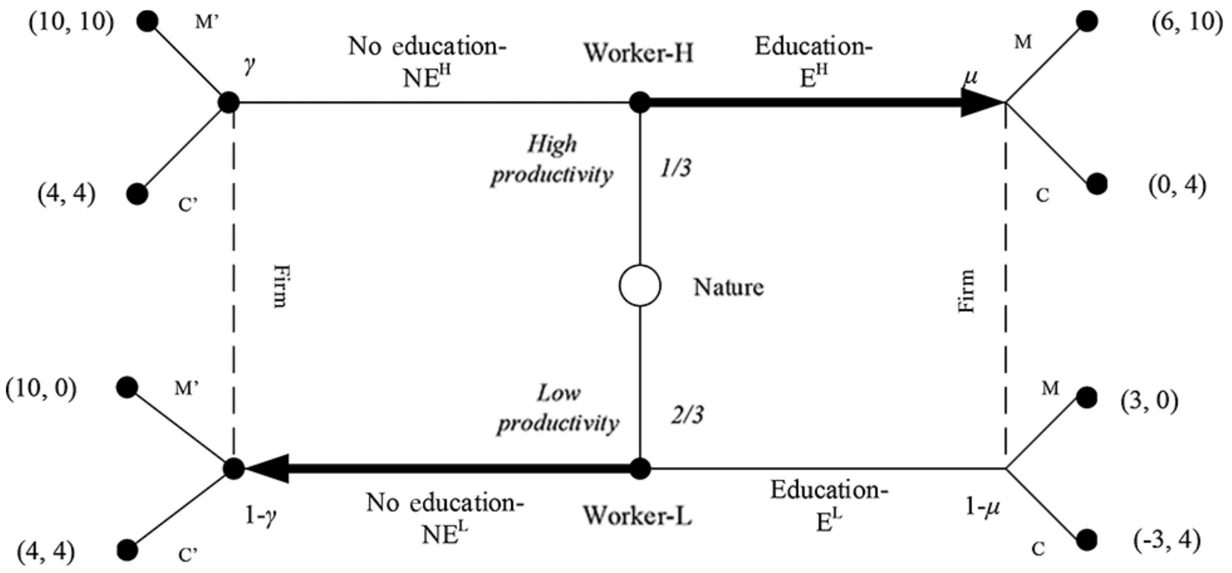
Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - We focus on:
    - Separating strategy profiles: (E^H, NE^L)
    - Pooling strategy profile: (NE^H, NE^L)
  - Exercise:
    - Separating strategy profiles: (NE^H, E^L)
    - Pooling strategy profile: (E^H, E^L)

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):

1. **Separating PBE $\left(E^H, NE^L\right)$:**
   – *Step 1*: Specify the separating strategy profile E^H, NE^L for the informed player.



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 2*: Use Bayes' rule to update the uninformed player's (firm) beliefs.
    - Taking into account that $\alpha_H = 1$ while $\alpha_L = 0$, the firm updates its beliefs for an educated applicant as

$$\mu = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 1$$

    - Intuitively, after observing that the applicant acquired education, the firm assigns full probability to the applicant being of high productivity.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Now, taking into account that $\alpha_E = 0$ while $\alpha_{NE} = 1$, the firm updates its beliefs for a less applicant as

$$\gamma = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 0$$

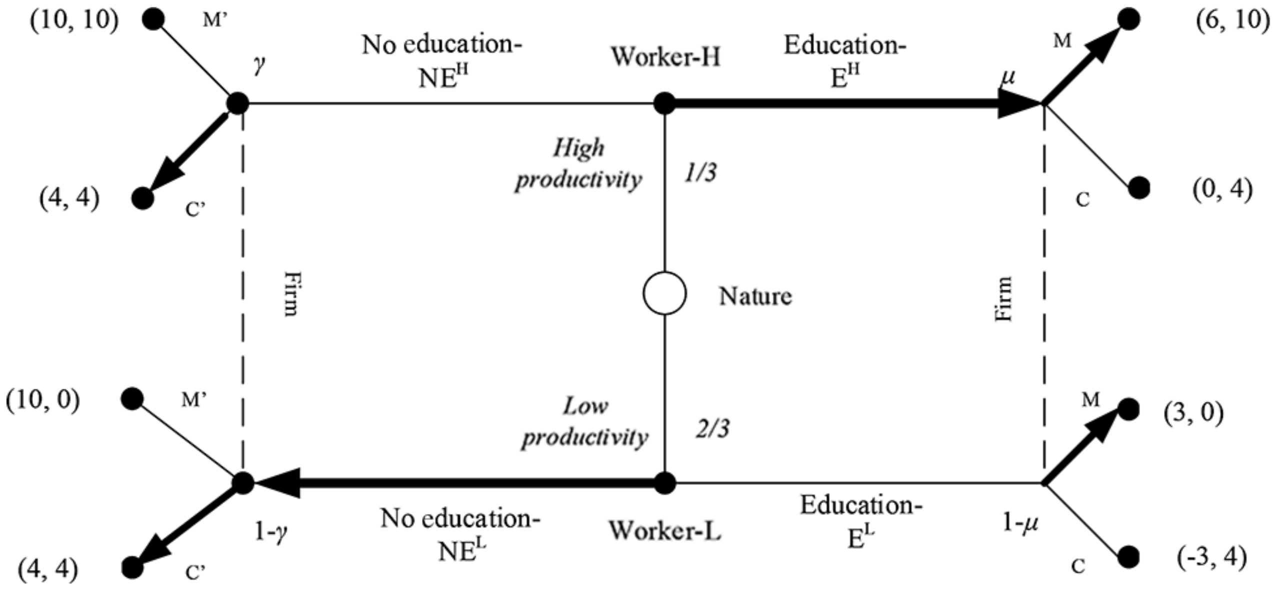
or that

$$1 - \gamma = 1$$

  - Intuitively, the firm that observes the less educated applicant believes that such an applicant must be of low productivity.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 3*: Given the firm's beliefs, determine the firm's optimal response, after observing the education level of the worker.



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 4*: Given these strategy profiles, examine the worker's optimal action.
    - *High-productivity type*: Does not have an incentive to deviate from the strategy profile (acquiring more education).
    - *Low-productivity type*: The cost of acquiring education is too high for the low-productivity worker; and thus that worker chooses not to pursue it.

Sequential-Move Games under Incomplete Information

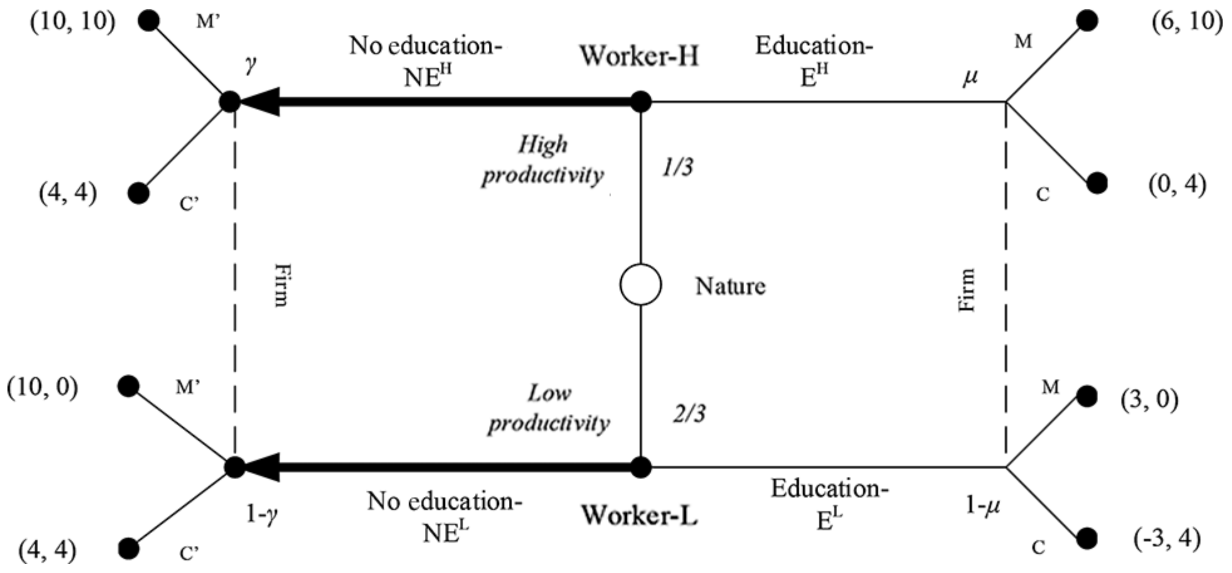
- ***Example*** (continued):
  - *Step 5*: The separating strategy profile (E^H, NE^L) can be sustained as the PBE of this incomplete information game.
    - Neither type of worker has the incentive to deviate from the prescribed separating strategy profile (E^H, NE^L).

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):

2. Pooling PBE (NE^H, NE^L):

  - *Step 1*: Specify the separating strategy profile NE^H, NE^L for the informed player.



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 2*: Use Bayes's rule to update the uninformed player's (firm) beliefs.
    - Taking into account that $\alpha_H = 1$ while $\alpha_L = 1$, the firm updates its beliefs for a less educated applicant as

$$\gamma = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 1/3$$

    - Intuitively, since neither type of applicant obtains education in this strategy profile, the firm's observation of an uneducated applicant does not allow the firm to further restrict its posterior beliefs about the applicant's type.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Taking into account that $\alpha_H = 0$ while $\alpha_L = 0$, the firm updates its beliefs for a more educated applicant as

$$\mu = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 0$$

  - This player's off-the-equilibrium beliefs are left unrestricted at $\mu \in [0,1]$.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 3*: Given the firm's beliefs, determine the firm's optimal response, after observing the education level of the worker.
    - Upon observing a less educated applicant:

$$EU_{\text{firm}}(M'|\text{No education}) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 = \frac{10}{3}$$

$$EU_{\text{firm}}(C'|\text{No education}) = \frac{1}{3}\cdot 4 + \frac{2}{3}\cdot 4 = 4$$

    - Hence, the firm optimally responds by offering the applicant a cashier position.

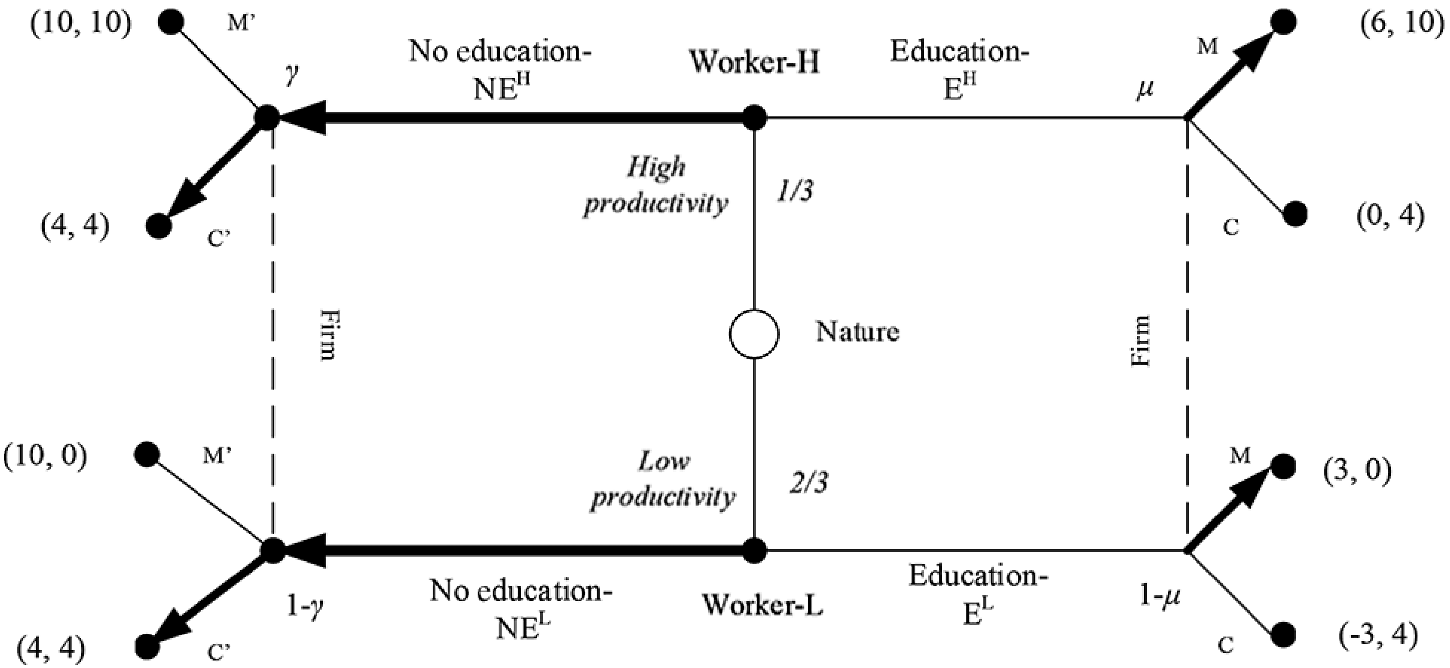
Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Upon observing a more educated applicant:
$$EU_{\text{firm}}(M|\text{Education}) = \mu \cdot 10 + (1-\mu) \cdot 0 = 10\mu$$
$$EU_{\text{firm}}(C|\text{Education}) = \mu \cdot 4 + (1-\mu) \cdot 4 = 4$$
  - The firm responds by offering the applicant a manager position if and only if
$$10\mu > 4 \Rightarrow \mu > 2/5$$
  - We thus need to divide the fifth step (the optimal actions of the worker) into two cases:
    1. $\mu > 2/5$, where the firm responds with M
    2. $\mu \leq 2/5$, where the firm responds with C

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 4*: Given these strategy profiles, examine the worker's optimal action.
  - Case 1: $\mu > 2/5$
    - *High-productivity type*: Has an incentive to deviate from the prescribed strategy profile. Thus it cannot be supported as a PBE.
    - *Low-productivity type*: Does not have incentives to deviate from the prescribed strategy profile.

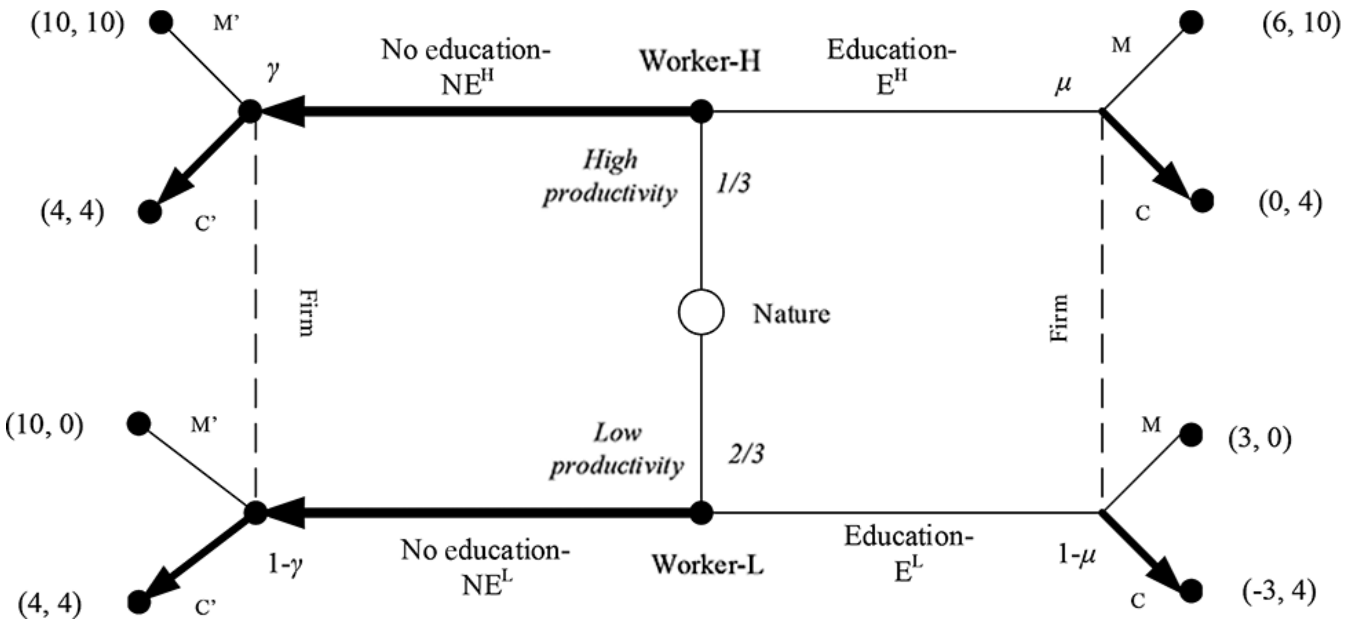
Sequential-Move Games under Incomplete Information



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Case 2: $\mu \leq 2/5$
    - *High-productivity type*: Does not have incentives to deviate from the prescribed strategy profile.
    - *Low-productivity type*: Does not have incentives to deviate from the prescribed strategy profile.

Sequential-Move Games under Incomplete Information



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 5*: The pooling strategy profile (NE^H, NE^L) can be supported as the PBE when off-the-equilibrium beliefs satisfy $\mu \leq 2/5$.

Bertrand Model of Price Competition

Bertrand Model of Price Competition

- Consider:
  - An industry with two firms, 1 and 2, selling a homogeneous product
  - Firms face market demand $x(p)$, where $x(p)$ is continuous and strictly decreasing in p
  - There exists a high enough price (choke price) $\bar{p} < \infty$ such that $x(p) = 0$ for all $p > \bar{p}$
  - Both firms are symmetric in their constant marginal cost $c > 0$, where $x(c) \in (0, \infty)$
  - Every firm *j simultaneously* sets a price p_j

Bertrand Model of Price Competition

- Firm *j*'s demand is

$$x_j(p_j, p_k) = \begin{cases} x(p_j) \text{ if } p_j < p_k \\ \frac{1}{2}x(p_j) \text{ if } p_j = p_k \\ 0 \text{ if } p_j > p_k \end{cases}$$

- *Intuition*: Firm *j* captures
  - all market if its price is the lowest, $p_j < p_k$
  - no market if its price is the highest, $p_j > p_k$
  - shares the market with firm k if the price of both firms coincides, $p_j = p_k$

Bertrand Model of Price Competition

- Given prices p_j and p_k, firm j's profits are therefore

$$(p_j - c) \cdot x_j\ (p_j, p_k)$$

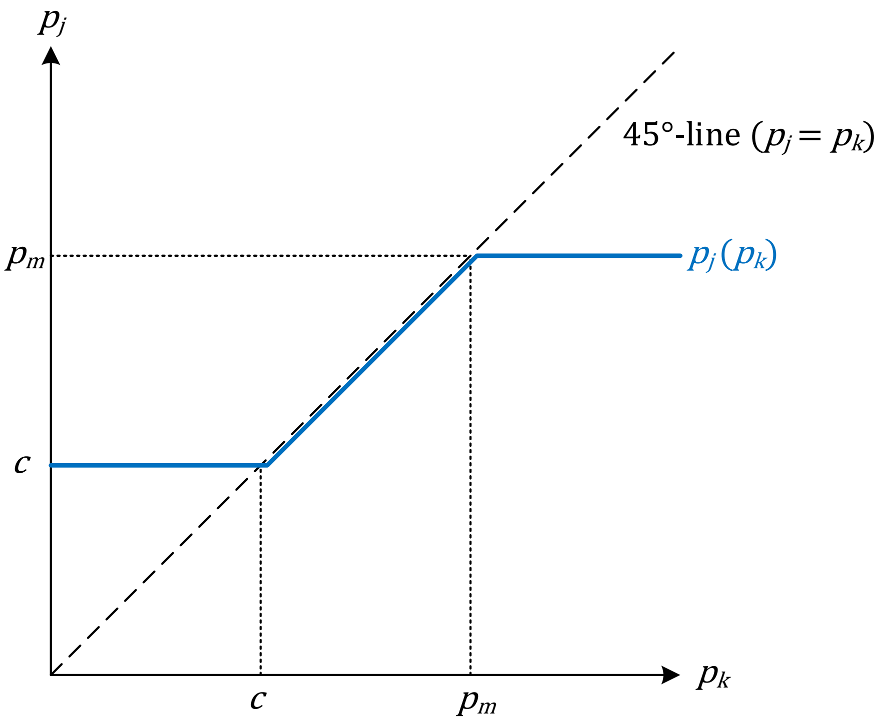
- We are now ready to find equilibrium prices in the Bertrand duopoly model.
  - There is a unique NE (p_j^*, p_k^*) in the Bertrand duopoly model. In this equilibrium, both firms set prices equal to marginal cost, $p_j^* = p_k^* = c$.

Bertrand Model of Price Competition

- Let us describe the best response function of firm j.
- If $p_k < c$, firm j sets its price at $p_j = c$.
  - Firm j does not undercut firm k since that would entail negative profits.
- If $c < p_k < p_m$, firm j slightly undercuts firm k, i.e., $p_j = p_k - \varepsilon$.
  - This allows firm j to capture all sales and still make a positive margin on each unit.
- If $p_k > p_m$, where p_m is a monopoly price, firm j does not need to charge more than p_m, i.e., $p_j = p_m$.
  - p_m allows firm j to capture all sales and maximize profits as the only firm selling a positive output.

Bertrand Model of Price Competition

- Firm *j*'s best response has:
  - a flat segment for all $p_k < c$, where $p_j(p_k) = c$
  - a positive slope for all $c < p_k < p_m$, where firm *j* charges a price slightly below firm *k*
  - a flat segment for all $p_k > p_m$, where $p_j(p_k) = p_m$

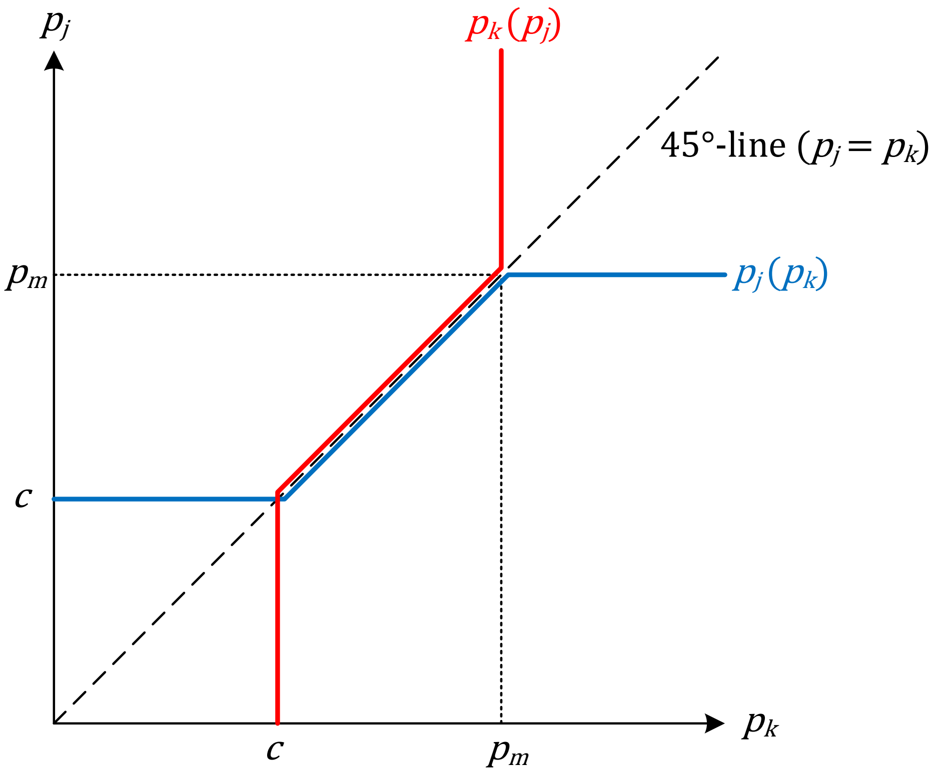


Bertrand Model of Price Competition

- A symmetric argument applies to the construction of the best response function of firm k.
- A mutual best response for both firms is

$$(p_1^*, p_2^*) = (c, c)$$

  where the two best response functions cross each other.
- This is the NE of the Bertrand model
  - Firms make no economic profits.



Bertrand Model of Price Competition

- With only two firms competing in prices we obtain the perfectly competitive outcome, where firms set prices equal to marginal cost.
- Price competition makes each firm j face an infinitely elastic demand curve at its rival's price, p_k.
  - Any increase (decrease) from p_k infinitely reduces (increases, respectively) firm j's demand.

Bertrand Model of Price Competition

- How much does Bertrand equilibrium hinge into our assumptions?
  - Quite a lot
- The competitive pressure in the Bertrand model with homogenous products is ameliorated if we instead consider:
  - Price competition (but allowing for heterogeneous products)
  - Quantity competition (still with homogenous products)
  - Capacity constraints

Bertrand Model of Price Competition

- ***Remark***:
  - How would our results be affected if firms face different production costs, i.e., $0 < c_1 < c_2$?
  - The most efficient firm sets a price equal to the marginal cost of the least efficient firm, $p_1 = c_2$.
  - Other firms will set a random price in the uniform interval

$$[c_1, c_1 + \eta]$$

where $\eta > 0$ is some small random increment with probability distribution $f(p, \eta) > 0$ for all p.

Cournot Model of Quantity Competition

Cournot Model of Quantity Competition

- Let us now consider that firms compete in quantities.
- Assume that:
  - Firms bring their output q_1 and q_2 to a market, the market clears, and the price is determined from the inverse demand function $p(q)$, where $q = q_1 + q_2$.
  - $p(q)$ satisfies $p'(q) < 0$ at all output levels $q \geq 0$,
  - Both firms face a common marginal cost $c > 0$
  - $p(0) > c$ in order to guarantee that the inverse demand curve crosses the constant marginal cost curve at an interior point.

Cournot Model of Quantity Competition

- Let us first identify every firm's best response function
- Firm 1's PMP, for a given output level of its rival, $\bar{q}_2$,

$$\max_{q_1 \geq 0} \; \underbrace{p(q_1 + \bar{q}_2)}_{\text{Price}} q_1 - cq_1$$

- When solving this PMP, firm 1 treats firm 2's production, $\bar{q}_2$, as a parameter, since firm 1 cannot vary its level.

Cournot Model of Quantity Competition

- FOCs:

$$p'(q_1 + \bar{q}_2)q_1 + p(q_1 + \bar{q}_2) - c \leq 0$$

with equality if $q_1 > 0$

- Solving this expression for q_1, we obtain firm 1's best response function (BRF), $q_1(\bar{q}_2)$.
- A similar argument applies to firm 2's PMP and its best response function $q_2(\bar{q}_1)$.
- Therefore, a pair of output levels (q_1^*, q_2^*) is NE of the Cournot model if and only if

$q_1^* \in q_1(\bar{q}_2)$ for firm 1's output
$q_2^* \in q_2(\bar{q}_1)$ for firm 2's output

Cournot Model of Quantity Competition

- To show that $q_1^*, q_2^* > 0$, let us work by contradiction, assuming $q_1^* = 0$.
  - Firm 2 becomes a monopolist since it is the only firm producing a positive output.
- Using the FOC for firm 1, we obtain

$$p'(0 + q_2^*)0 + p(0 + q_2^*) \leq c$$

$$\text{or } \; p(q_2^*) \leq c$$

- And using the FOC for firm 2, we have

$$p'(q_2^* + 0)q_2^* + p(q_2^* + 0) \leq c$$

$$\text{or } \; p'(q_2^*)q_2^* + p(q_2^*) \leq c$$

- This implies firm 2's MR under monopoly is lower than its MC. Thus, $q_2^* = 0$.

Cournot Model of Quantity Competition

- Hence, if $q_1^* = 0$, firm 2's output would also be zero, $q_2^* = 0$.
- But this implies that $p(0) < c$ since no firm produces a positive output, thus violating our initial assumption $p(0) > c$.
  - Contradiction!
- As a result, we must have that both $q_1^* > 0$ and $q_2^* > 0$.
- *Note*: This result does not necessarily hold when both firms are asymmetric in their production costs.

Cournot Model of Quantity Competition

- ***Example*** (symmetric costs):
  - Consider an inverse demand curve $p(q) = a - bq$, and two firms competing à la Cournot both facing a marginal cost $c > 0$.
  - Firm 1's PMP is
$$[a - b(q_1 + \bar{q}_2)]q_1 - cq_1$$
  - FOC wrt q_1:
$$a - 2bq_1 - b\bar{q}_2 - c \leq 0$$
with equality if $q_1 > 0$

Cournot Model of Quantity Competition

- ***Example*** (continue):
  - Solving for q_1, we obtain firm 1's BRF

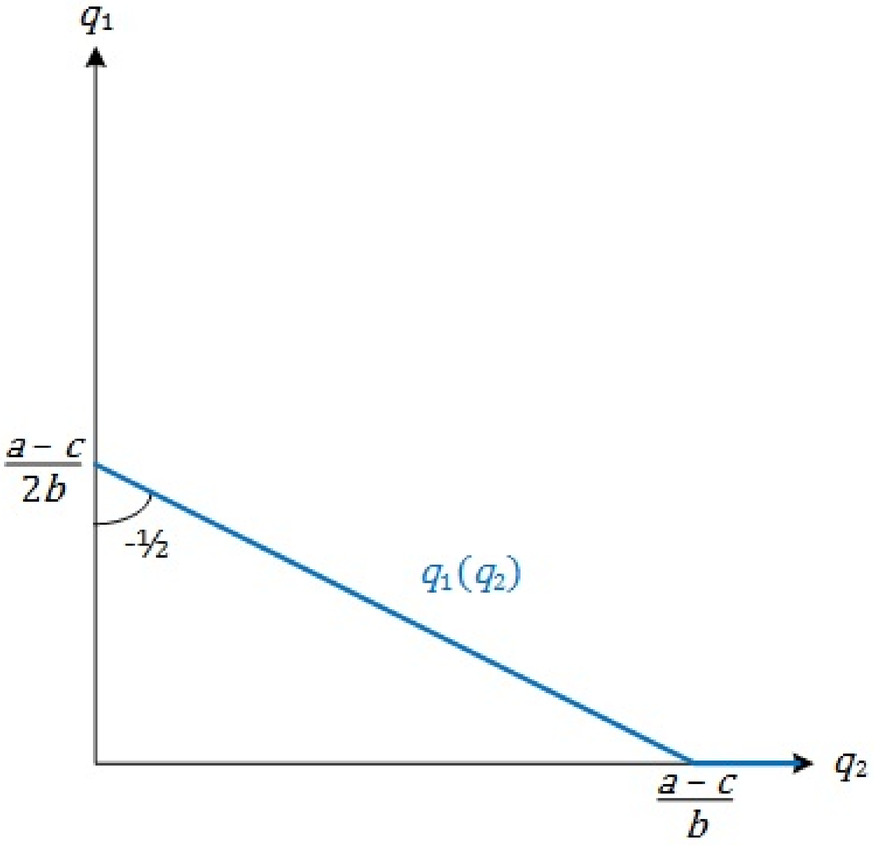
$$q_1(\bar{q}_2) = \frac{a-c}{2b} - \frac{\bar{q}_2}{2}$$

  - Analogously, firm 2's BRF

$$q_2(\bar{q}_1) = \frac{a-c}{2b} - \frac{\bar{q}_1}{2}$$

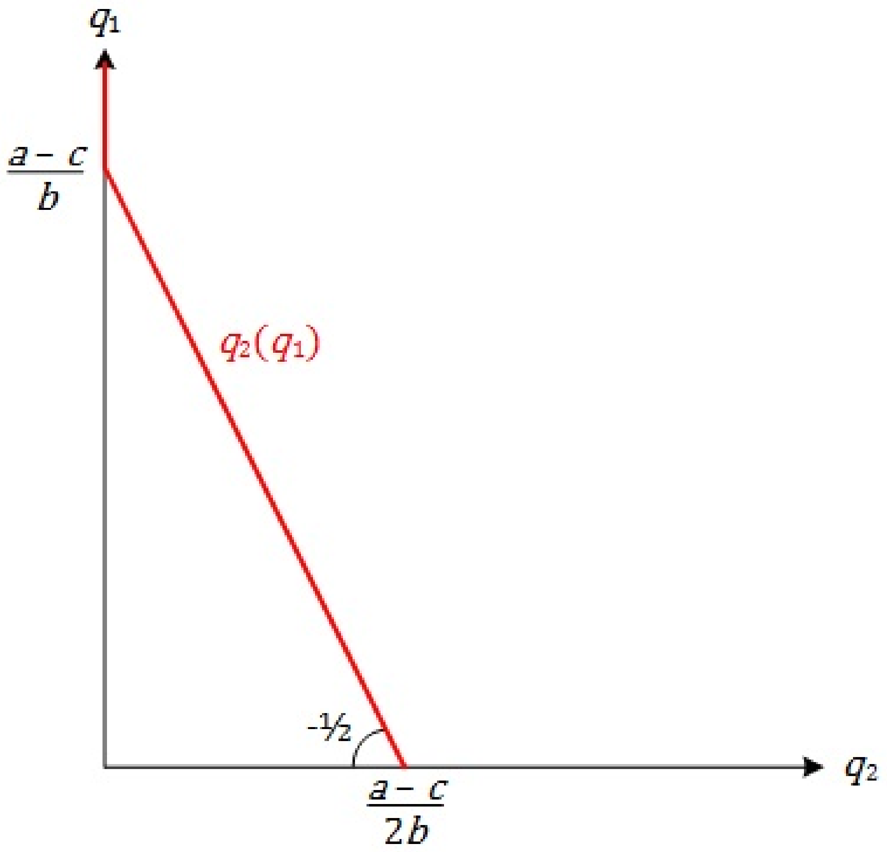
Cournot Model of Quantity Competition

- Firm 1's BRF:
  - When $q_2 = 0$, then $q_1 = \frac{a-c}{2b}$, which coincides with its output under monopoly.
  - As q_2 increases, q_1 decreases (i.e., firm 1's and 2's output are strategic substitutes)
  - When $q_2 = \frac{a-c}{b}$, then $q_1 = 0$.

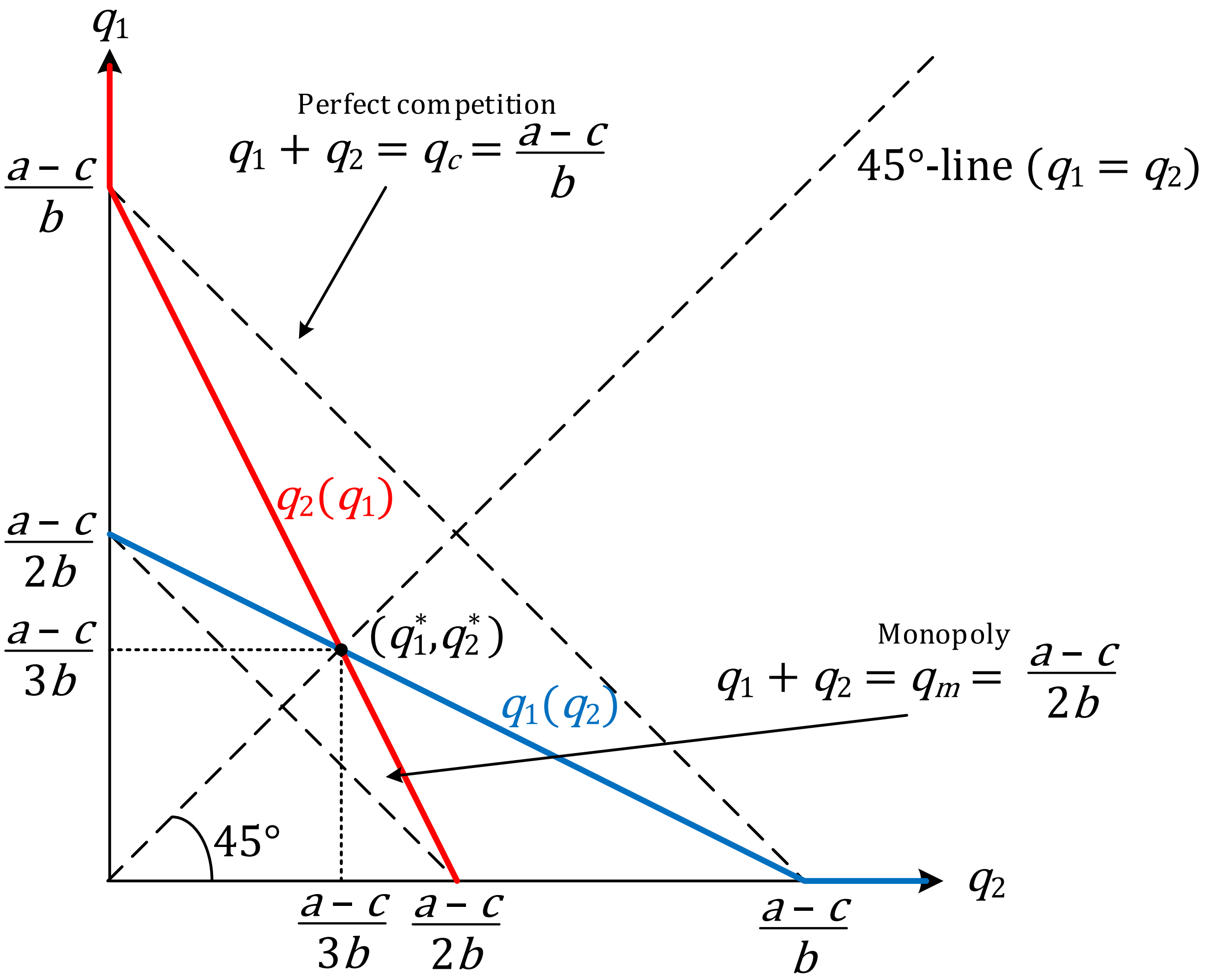


Cournot Model of Quantity Competition

- A similar argument applies to firm 2’s BRF.
- Superimposing both firms’ BRFs, we obtain the Cournot equilibrium output pair (q_1^*, q_2^*).



Cournot Model of Quantity Competition



Cournot Model of Quantity Competition

- Cournot equilibrium output pair (q_1^*, q_2^*) occurs at the intersection of the two BRFs, i.e.,

$$(q_1^*, q_2^*) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$$

- Aggregate output becomes

$$q^* = q_1^* + q_2^* = \frac{a-c}{3b} + \frac{a-c}{3b} = \frac{2(a-c)}{3b}$$

which is larger than under monopoly, $q_m = \frac{a-c}{2b}$, but smaller than under perfect competition, $q_c = \frac{a-c}{b}$.

Cournot Model of Quantity Competition

- The equilibrium price becomes

$$p(q^*) = a - bq^* = a - b\left(\frac{2(a-c)}{3b}\right) = \frac{a+2c}{3}$$

  which is lower than under monopoly, $p_m = \frac{a+c}{2}$, but higher than under perfect competition, $p_c = c$.

- Finally, the equilibrium profits of every firm j

$$\pi_j^* = p(q^*)q_j^* - cq_j^* = \left(\frac{a+2c}{3}\right)\left(\frac{a-c}{3b}\right) - c\left(\frac{a-c}{3b}\right) = \frac{(a-c)^2}{9b}$$

  which are lower than under monopoly, $\pi_m = \frac{(a-c)^2}{4b}$, but higher than under perfect competition, $\pi_c = 0$.

Cournot Model of Quantity Competition

- Quantity competition (Cournot model) yields less competitive outcomes than price competition (Bertrand model), whereby firms' behavior mimics that in perfectly competitive markets
  - That's because, the demand that every firm faces in the Cournot game is not infinitely elastic.
  - A reduction in output does not produce an infinite increase in market price, but instead an increase of $-p'(q_1 + q_2)$.
  - Hence, if firms produce the same output as under marginal cost pricing, i.e., half of $\frac{a-c}{b}$, each firm would have incentives to deviate from such a high output level by marginally reducing its output.

Cournot Model of Quantity Competition

- Equilibrium output under Cournot does not coincide with the monopoly output either.
  - That’s because, every firm i, individually increasing its output level q_i, takes into account how the reduction in market price affects its own profits, but ignores the profit loss (i.e., a negative external effect) that its rival suffers from such a lower price.
  - Since every firm does not take into account this external effect, aggregate output is too large, relative to the output that would maximize firms’ joint profits.

Cournot Model of Quantity Competition

- ***Example*** (Cournot vs. Cartel):
  - Let us demonstrate that firms’ Cournot output is larger than that under the cartel.
  - PMP of the cartel is

$$\max_{q_1,q_2} [(a - b(q_1{+}q_2))q_1 - cq_1] \\ + [(a - b(q_1{+}q_2))q_2 - cq_2]$$

  - Since $Q = q_1 + q_2$, the PMP can be written as

$$\max_{q_1,q_2} (a - b(q_1{+}q_2))(q_1{+}q_2) - c(q_1{+}q_2) \\ = \max_{Q} (a - bQ)Q - cQ = aQ - bQ^2 - cQ$$

Cournot Model of Quantity Competition

- ***Example*** (continued):
  - FOC with respect to Q

$$a - 2bQ - c \leq 0$$

  - Solving for Q, we obtain the aggregate output

$$Q^* = \frac{a-c}{2b}$$

which is positive since $a > c$, i.e., $p(0) = a > c$.

  - Since firms are symmetric in costs, each produces

$$q_i = \frac{Q}{2} = \frac{a-c}{4b}$$

Cournot Model of Quantity Competition

- ***Example*** (continued):
  - The equilibrium price is

$$p = a - bQ = a - b\frac{a-c}{2b} = \frac{a+c}{2}$$

  - Finally, the equilibrium profits are

$$\pi_i = p \cdot q_i - cq_i$$

$$= \frac{a+c}{2} \cdot \frac{a-c}{4b} - c\frac{a-c}{4b} = \frac{(a-c)^2}{8b}$$

which is larger than firms would obtain under Cournot competition, $\frac{(a-c)^2}{9b}$.

Cournot Model of Quantity Competition: Cournot Pricing Rule

- Firms' market power can be expressed using a variation of the Lerner index.
  - Consider firm *j*'s profit maximization problem

    $$\pi_j = p(q)q_j - c_j(q_j)$$
  - FOC for every firm *j*

    $$p'(q)q_j + p(q) - c_j = 0$$

    $$\text{or } \; p(q) - c_j = -p'(q)q_j$$
  - Multiplying both sides by q and dividing them by $p(q)$ yield

    $$q\frac{p(q) - c_j}{p(q)} = \frac{-p'(q)q_j}{p(q)}q$$

Cournot Model of Quantity Competition: Cournot Pricing Rule

– Recalling $\frac{1}{\varepsilon} = -p'(q) \cdot \frac{q}{p(q)}$, we have

$$q\frac{p(q)-c_j}{p(q)} = \frac{1}{\varepsilon}q_j$$

or $$\frac{p(q)-c_j}{p(q)} = \frac{1}{\varepsilon}\frac{q_j}{q}$$

– Defining $\alpha_j \equiv \frac{q_j}{q}$ as firm *j*'s market share, we obtain

$$\frac{p(q)-c_j}{p(q)} = \frac{\alpha_j}{\varepsilon}$$

which is referred to as the ***Cournot pricing rule***.

Cournot Model of Quantity Competition: Cournot Pricing Rule

- *Note*:
  - When $\alpha_j = 1$, implying that firm j is a monopoly, the IEPR becomes a special case of the Cournot pricing rule.
  - The larger the market share α_j of a given firm, the larger the price markup of firm j.
  - The more inelastic is the demand, the smaller is the value of ε, and the larger the price markup of firm j.

Cournot Model of Quantity Competition: Cournot Pricing Rule

- ***Example*** (Merger effects on Cournot Prices):
  - Consider an industry with n firms and a constant-elasticity demand function $q(p) = ap^{-1}$, where $a > 0$ and $\varepsilon = -1$.
  - Before merger, we have

$$\frac{p^B - c}{p^B} = \frac{1}{n} \implies p^B = \frac{nc}{n-1}$$

  - After the merger of $k < n$ firms $n - k + 1$ firms remain in the industry, and thus

$$\frac{p^A - c}{p^A} = \frac{1}{n-k+1} \implies p^A = \frac{(n-k+1)c}{n-k}$$

Cournot Model of Quantity Competition: Cournot Pricing Rule

- ***Example*** (continued):
  - The percentage change in prices is

$$\%\Delta p = \frac{p^A - p^B}{p^B} = \frac{\frac{(n-k+1)c}{n-k} - \frac{nc}{n-1}}{\frac{nc}{n-1}}$$

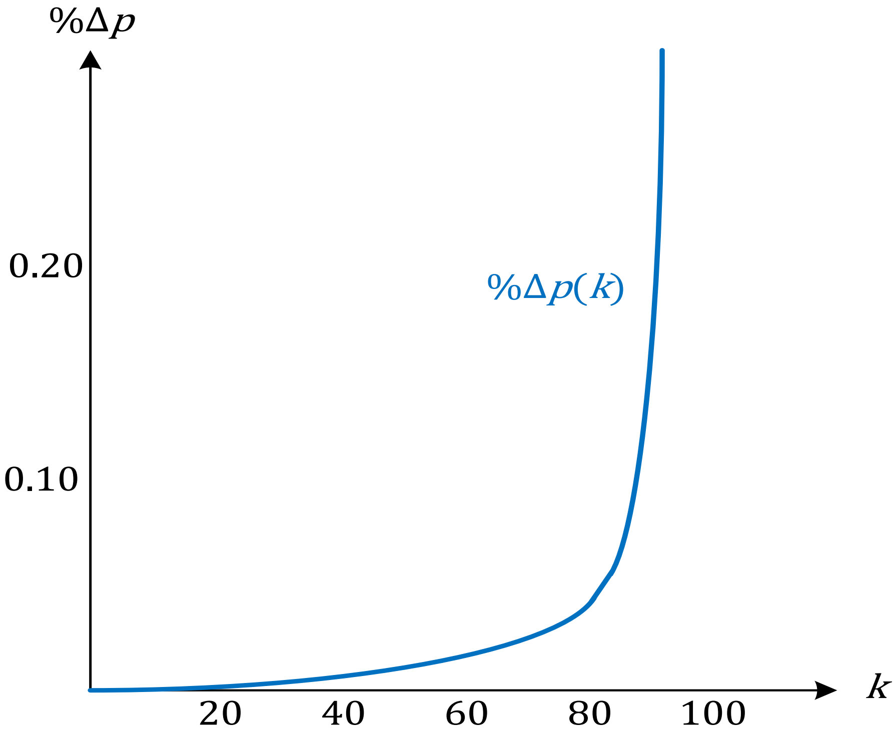
$$= \frac{k-1}{n(n-k)} > 0$$

  - Hence, prices increase after the merger.
  - Also, $\%\Delta p$ increases as the number of merging firms k increases

$$\frac{\partial \%\Delta p}{\partial k} = \frac{n-1}{n(n-k)^2} > 0$$

Cournot Model of Quantity Competition: Cournot Pricing Rule

- ***Example*** (continued):
  - The percentage increase in price after the merger, $\%\Delta p$, as a function of the number of merging firms, k.
  - For simplicity, $n = 100$.



Cournot Model of Quantity Competition: SOC

- Let us check if the first order (necessary) conditions are also sufficient.
- Recall that FOCs are

$$p'(q)q_j + p(q) - c_j'(q_j) \leq 0$$

- Differentiating FOCs wrt q_j yields

$$p''(q)q_j + 2p'(q) - c_j''(q_j) \leq 0$$

  - $p'(q) < 0$: by definition (negatively sloped inverse demand curve)
  - $c_j''(q_j) \geq 0$: by assumption (constant or increasing marginal costs)
  - $p''(q)q_j \leq 0$: as long as the demand curve decreases at a constant or decreasing rate

Cournot Model of Quantity Competition: SOC

- ***Example*** (linear demand):
  - The linear inverse demand curve is $p(q) = a - bq$ and constant marginal cost is $c > 0$.
  - Since $p'(q) = -b < 0, p''(q) = 0, c'(q) = c$ and $c''(q) = 0$, the SOC reduces to

$$0 - 2b - 0 = -2b < 0$$

    where $b > 0$ by definition.
  - Hence the equilibrium output is indeed profit maximizing.

Cournot Model of Quantity Competition: SOC

- Note that SOCs coincide with the cross-derivative

  $$\frac{\partial^2 \pi_j}{\partial q_j \partial q_k} = \frac{\partial}{\partial q_k}\left[p'(q)q_j + p(q) - c'(q_j)\right]$$

  $= p''(q)q_j + p'(q)$ for all $k \neq j$.

- Hence, the firm j's BRF decreases in q_k as long as $p''(q)q_j + p'(q) < 0$
  - That is, firm j's BRF is negatively sloped.

Cournot Model of Quantity Competition: Asymmetric Costs

- Assume that firm 1 and 2's constant marginal costs of production differ, i.e., $c_1 > c_2$, so firm 2 is more efficient than firm 1. Assume also that the inverse demand function is $p(Q) = a - bQ$, and $Q = q_1 + q_2$.
- Firm i's PMP is

$$\max_{q_i} \left(a - b(q_i + q_j)\right)q_i - c_i q_i$$

- FOC:

$$a - 2bq_i - bq_j - c_i = 0$$

Cournot Model of Quantity Competition: Asymmetric Costs

- Solving for q_i (assuming an interior solution) yields firm i's BRF

$$q_i(q_j) = \frac{a - c_i}{2b} - \frac{q_j}{2}$$

- Firm 1's optimal output level can be found by plugging firm 2's BRF into firm 1's

$$q_1^* = \frac{a - c_1}{2b} - \frac{1}{2}\left(\frac{a - c_2}{2b} - \frac{q_1^*}{2}\right) \Leftrightarrow q_1^* = \frac{a - 2c_1 + c_2}{3b}$$

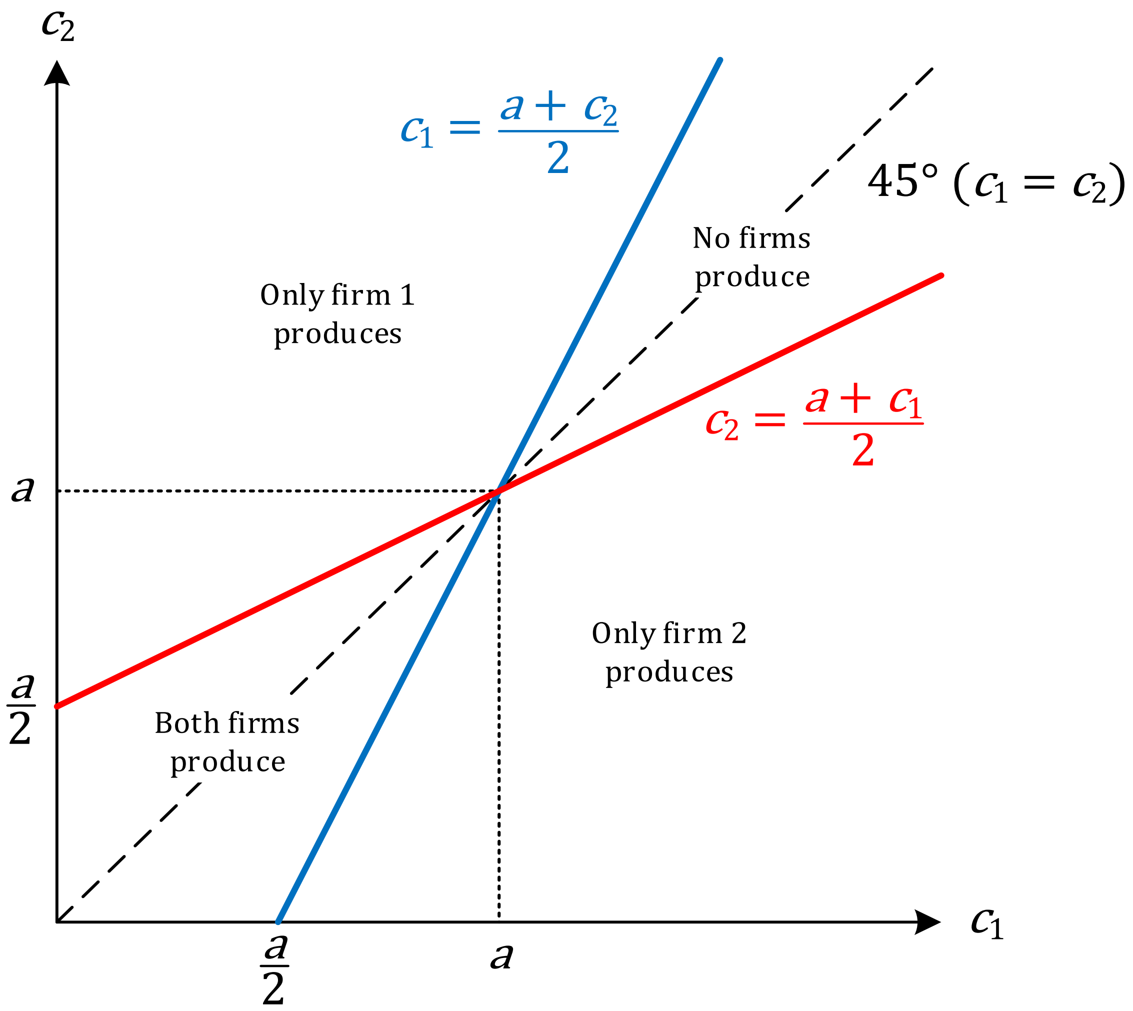
- Similarly, firm 2's optimal output level is

$$q_2^* = \frac{a - c_2}{2b} - \frac{q_1^*}{2} = \frac{a + c_1 - 2c_2}{3b}$$

Cournot Model of Quantity Competition: Asymmetric Costs

- If firm i's costs are sufficiently high it will not produce at all.
  - Firm 1: $q_1^* \leq 0$ if $\frac{a+c_2}{2} \leq c_1$
  - Firm 2: $q_2^* \leq 0$ if $\frac{a+c_1}{2} \leq c_2$
- Thus, we can identify three different cases:
  - If $c_i \geq \frac{a+c_j}{2}$ for all firms $i = \{1,2\}$, no firm produces a positive output
  - If $c_i \geq \frac{a+c_j}{2}$ but $c_j < \frac{a+c_i}{2}$, then only firm j produces a positive output
  - If $c_i < \frac{a+c_j}{2}$ for all firms $i = \{1,2\}$, both firms produce a positive output

Cournot Model of Quantity Competition: Asymmetric Costs



Cournot Model of Quantity Competition: Asymmetric Costs

- The output levels (q_1^*, q_2^*) also vary when (c_1, c_2) changes

$$\frac{\partial q_1^*}{\partial c_1} = -\frac{2}{3b} < 0 \text{ and } \frac{\partial q_1^*}{\partial c_2} = \frac{1}{3b} > 0$$

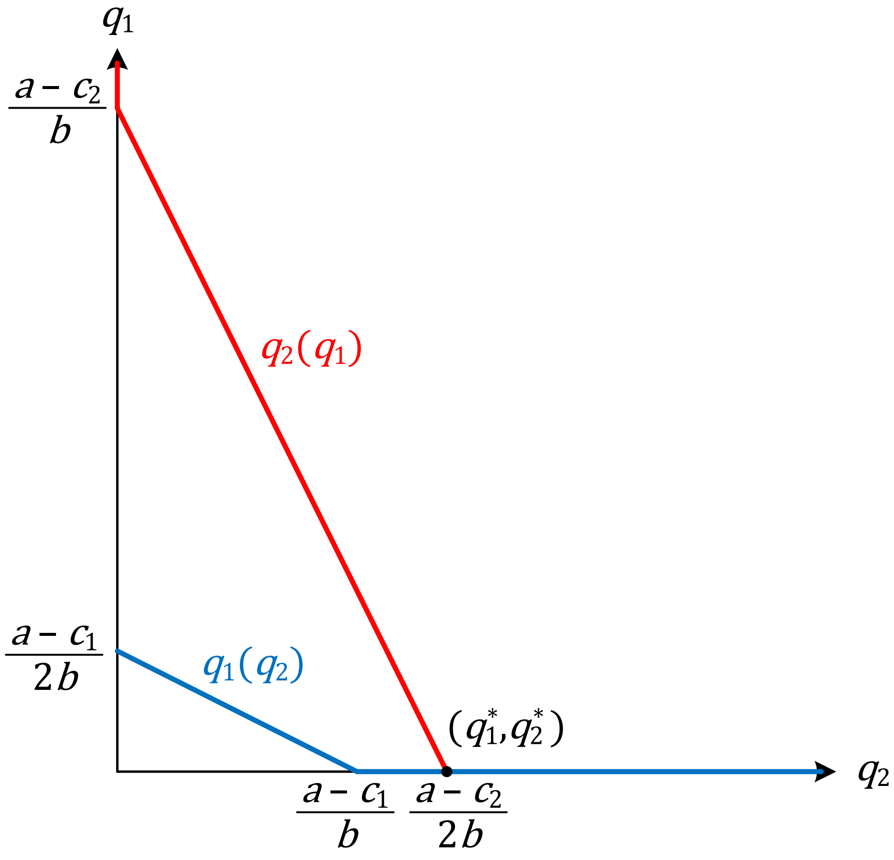
$$\frac{\partial q_2^*}{\partial c_1} = \frac{1}{3b} > 0 \text{ and } \frac{\partial q_2^*}{\partial c_2} = -\frac{2}{3b} < 0$$

- *Intuition*: Each firm's output decreases in its own costs, but increases in its rival's costs.

Cournot Model of Quantity Competition: Asymmetric Costs

- BRFs for firms 1 and 2 when $c_1 > \frac{a+c_2}{2}$ (i.e., only firm 2 produces).

- BRFs cross at the horizontal axis where $q_1^* = 0$ and $q_2^* > 0$ (i.e., a corner solution)



Cournot Model of Quantity Competition: $J > 2$ firms

- Consider $J > 2$ firms, all facing the same constant marginal cost $c > 0$.
- The linear inverse demand curve is $p(Q) = a - bQ$, where $Q = \sum_k q_k$.
- Alternatively, we can write $p(Q) = a - b(q_i + Q_{-i})$, where $Q_{-i} = \sum_{k \neq i} q_k$ denotes the output from firm i's rivals.

- Firm i's PMP is

$$\max_{q_i} \; [a - b(q_i + Q_{-i})]q_i - cq_i$$

- FOC:

$$a - 2bq_i^* - bQ_{-i}^* - c \leq 0$$

Cournot Model of Quantity Competition: $J > 2$ firms

- Solving for q_i^*, we obtain firm i's BRF

$$q_i^* = \frac{a-c}{2b} - \frac{1}{2} Q_{-i}^*$$

- Since all firms are symmetric, their BRFs are also symmetric, implying $q_1^* = q_2^* = \cdots = q_J^*$. This implies that $Q_{-i}^* = Jq_i^* - q_i^* = (J-1)q_i^*$.
- Hence, the BRF becomes

$$q_i^* = \frac{a-c}{2b} - \frac{1}{2}(J-1)q_i^*$$

Cournot Model of Quantity Competition: $J > 2$ firms

- Solving for q_i^*

$$q_i^* = \frac{a - c}{(J + 1)b}$$

which is also the equilibrium output for other $J - 1$ firms.

- Therefore, aggregate output is

$$Q^* = Jq_i^* = \frac{J}{J + 1} \frac{a - c}{b}$$

and the corresponding equilibrium price is

$$p^* = a - bQ^* = \frac{a + Jc}{J + 1}$$

Cournot Model of Quantity Competition: $J > 2$ firms

- Firm i's equilibrium profits are

$$\pi_i^* = (a - bQ^*)q_i^* - cq_i^*$$

$$= \left(a - b\left(\frac{J}{J+1}\frac{a-c}{b}\right)\right)\left(\frac{a-c}{(J+1)b}\right) - c\left(\frac{a-c}{(J+1)b}\right)$$

$$= \frac{(a-c)^2}{b(J+1)^2}$$

Cournot Model of Quantity Competition: $J > 2$ firms

- We can show that

$$q_i^*(2) = \frac{a-c}{(2+1)b} = \frac{a-c}{3b}$$

$$Q^*(2) = \frac{2(a-c)}{(2+1)b} = \frac{2(a-c)}{3b}$$

$$p^*(2) = \frac{a+2c}{2+1} = \frac{a+2c}{3}$$

which exactly coincide with our results in the Cournot duopoly model.

Cournot Model of Quantity Competition: $J > 2$ firms

- We can show that

$$q_i^*(1) = \frac{a-c}{(1+1)b} = \frac{a-c}{2b}$$

$$Q^*(1) = \frac{1(a-c)}{(1+1)b} = \frac{a-c}{2b}$$

$$p^*(1) = \frac{a+1c}{(1+1)} = \frac{a+c}{2}$$

which exactly coincide with our findings in the monopolist's model.

Cournot Model of Quantity Competition: $J > 2$ firms

- We can show that when there are extremely large number of firms, that is, $J \to \infty$,

$$\lim_{J\to\infty} q_i^* = 0$$

$$\lim_{J\to\infty} Q^* = \frac{a-c}{b}$$

$$\lim_{J\to\infty} p^* = c$$

which coincides with the solution in a perfectly competitive market.

Product Differentiation

Product Differentiation

- So far we assumed that firms sell homogenous (undifferentiated) products.
- What if the goods firms sell are differentiated?
  - For simplicity, we will assume that product attributes are exogenous (not chosen by the firm), and production costs are zero.

Product Differentiation: Bertrand Model

- Consider the case where every firm i, for $i = \{1,2\}$, faces demand curve
$$q_i(p_i, p_j) = a - bp_i + cp_j$$
where $a, b, c > 0$ and $j \neq i$.
- Hence, an increase in p_j increases firm i's sales.
- Firm i's PMP:
$$\max_{p_i \geq 0} \; (a - bp_i + cp_j)p_i$$
- FOC:
$$a - 2bp_i + cp_j = 0$$

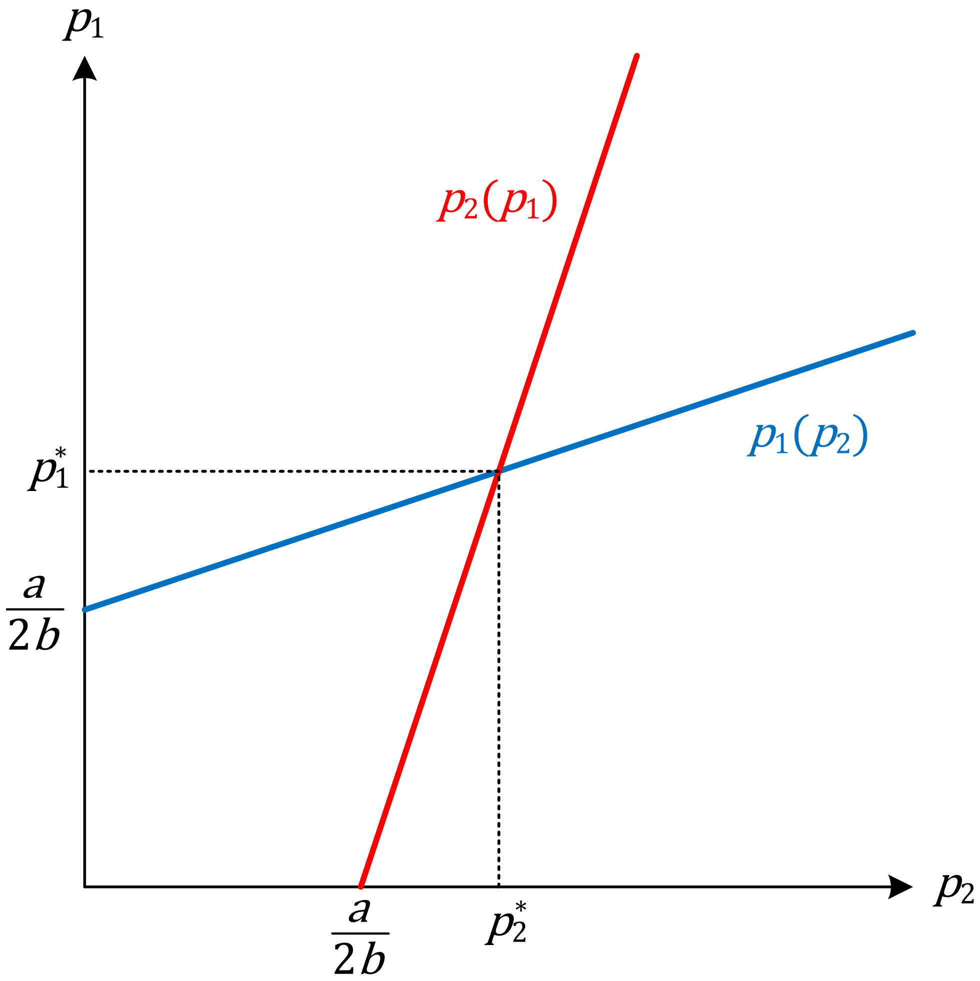
Product Differentiation: Bertrand Model

- Solving for p_i, we find firm i's BRF

$$p_i(p_j) = \frac{a + cp_j}{2b}$$

- Firm j also has a symmetric BRF.
- *Note*:
  - BRFs are now positively sloped
  - An increase in firm j's price leads firm i to increase his, and vice versa
  - In this case, firms' choices (i.e., prices) are strategic complements

Product Differentiation: Bertrand Model



Product Differentiation: Bertrand Model

- Simultaneously solving the two BRFs yields

$$p_i^* = \frac{a}{2b - c}$$

with corresponding equilibrium sales of

$$q_i^*(p_i^*, p_j^*) = a - bp_i^* + cp_j^* = \frac{ab}{2b - c}$$

and equilibrium profits of

$$\pi_i^* = p_i^* \cdot q_i^*(p_i^*, p_j^*) = \left(\frac{a}{2b - c}\right)\left(\frac{ab}{2b - c}\right)$$

$$= \frac{a^2 b}{(2b - c)^2}$$

Product Differentiation: Cournot Model

- Consider two firms with the following linear inverse demand curves
  $$p_1(q_1, q_2) = \alpha - \beta q_1 - \gamma q_2 \text{ for firm 1}$$
  $$p_2(q_1, q_2) = \alpha - \gamma q_1 - \beta q_2 \text{ for firm 2}$$
- We assume that $\beta > 0$ and $\beta > \gamma$
  - That is, the effect of increasing q_1 on p_1 is larger than the effect of increasing q_1 on p_2
  - Intuitively, the price of a particular brand is more sensitive to changes in its own output than to changes in its rival's output
  - In other words, ***own-price effects*** dominate the ***cross-price effects***.

Product Differentiation: Cournot Model

- Firm i's PMP is (assuming no costs)

$$\max_{q_i \geq 0} \ (\alpha - \beta q_i - \gamma q_j) q_i$$

- FOC:

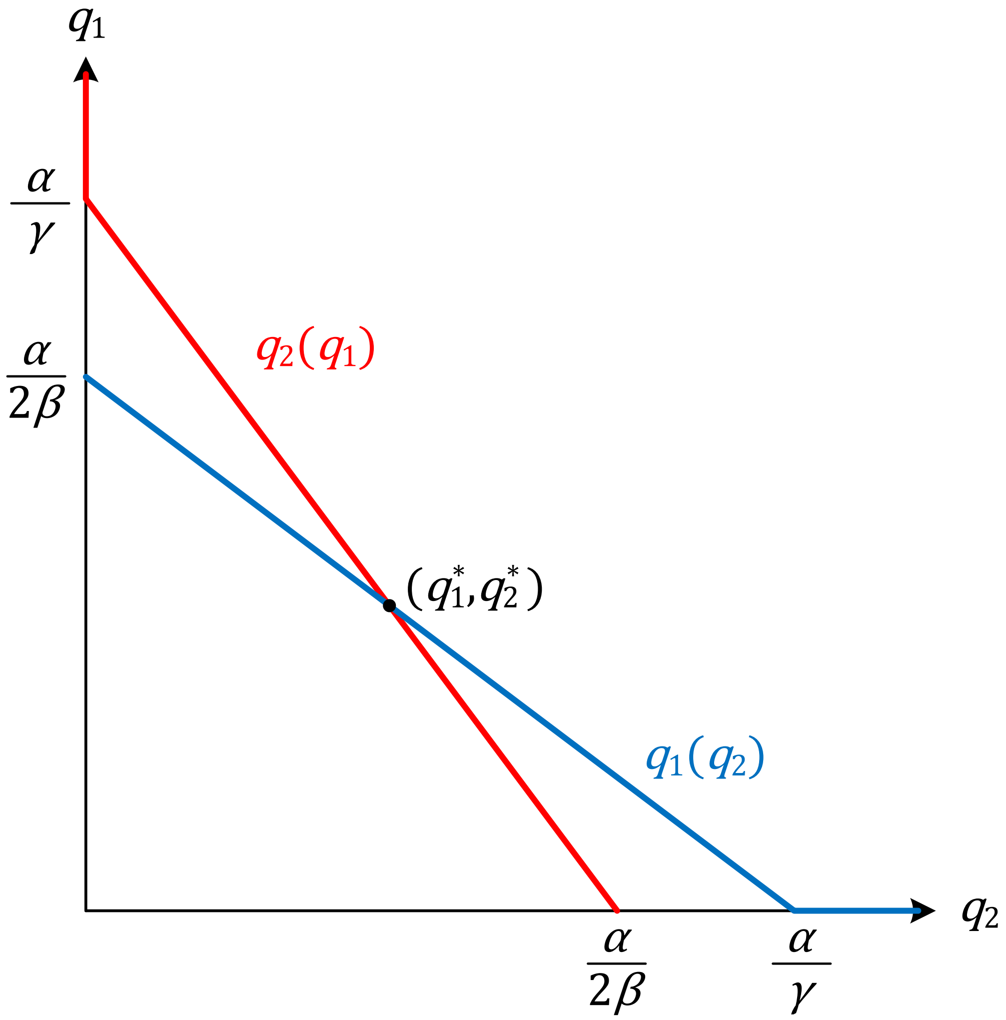
$$\alpha - 2\beta q_i - \gamma q_j = 0$$

- Solving for q_i we find firm i's BRF

$$q_i(q_j) = \frac{\alpha}{2\beta} - \frac{\gamma}{2\beta} q_j$$

- Firm j also has a symmetric BRF.

Product Differentiation: Cournot Model



Product Differentiation: Cournot Model

- Comparative statics of firm i's BRF
  - As $\beta \to \gamma$ (products become more homogeneous), BRF becomes steeper. That is, the profit-maximizing choice of q_i is more sensitive to changes in q_j (tougher competition)
  - As $\gamma \to 0$ (products become very differentiated), firm i's BRF no longer depends on q_j and becomes flat (milder competition)

Product Differentiation: Cournot Model

- Simultaneously solving the two BRF yields

$$q_i^* = \frac{\alpha}{2\beta + \gamma} \text{ for all } i = \{1,2\}$$

with a corresponding equilibrium price of

$$p_i^* = \alpha - \beta q_i^* - \gamma q_j^* = \frac{\alpha\beta}{2\beta + \gamma}$$

and equilibrium profits of

$$\pi_i^* = p_i^* q_i^* = \left(\frac{\alpha\beta}{2\beta + \gamma}\right)\left(\frac{\alpha}{2\beta + \gamma}\right) = \frac{\alpha^2\beta}{(2\beta + \gamma)^2}$$

Product Differentiation: Cournot Model

- *Note*:
  - As γ increases (products become more homogeneous), individual and aggregate output decrease, and individual profits decrease as well.
  - If $\gamma \rightarrow \beta$ (indicating undifferentiated products), then $q_i^* = \frac{\alpha}{2\beta+\beta} = \frac{\alpha}{3\beta}$ as in standard Cournot models of homogeneous products.
  - If $\gamma \rightarrow 0$ (extremely differentiated products), then $q_i^* = \frac{\alpha}{2\beta+0} = \frac{\alpha}{2\beta}$ as in monopoly.

Dynamic Competition

Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

- Assume that firm 1 chooses its price p_1 first, whereas firm 2 observes that price and responds with its own price p_2.

- Since the game is a sequential-move game (rather than a simultaneous-move game), we should use ***backward induction***.

Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

- Firm 2 (the follower) has a BRF given by

$$p_2(p_1) = \begin{cases} p^m \text{ if } p_1 > p^m \\ p_1 - \varepsilon \text{ if } p^m \geq p_1 > c \\ c \qquad \text{if } p_1 \leq c \end{cases}$$

- *Intuition*:
  - The follower charges monopoly price p^m if the leader charges prices above p^m;
  - undercuts the leader's price p_1 by a small $\varepsilon > 0$ for intermediate prices, $p^m \geq p_1 > c$; or
  - keeps its price at $p_2 = c$ if the leader sets $p_1 = c$.

Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

- The leader expects that its price will be:
  - undercut by the follower when $p_1 > c$ (thus yielding no sales)
  - mimicked by the follower when $p_1 = c$ (thus entailing half of the market share)
- Hence, the leader has (weak) incentives to set a price $p_1 = c$.
- As a consequence, the equilibrium price pair remains at $(p_1^*, p_2^*) = (c, c)$, as in the simultaneous-move version of the Bertrand model.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- Assume that firms sell differentiated products, where firm *j*'s demand is

$$q_j = D_j(p_j, p_k)$$

  - *Example*: $q_j(p_j, p_k) = a - bp_j + cp_k$, where $a, b, c > 0$ and $b > c$

- In the second stage, firm 2 (the follower) solves following PMP

$$\max_{p_2 \geq 0} \pi_2 = p_2 q_2 - TC(q_2)$$

$$= p_2 D_2(p_2, p_1) - TC(\underbrace{D_2(p_2, p_1)}_{q_2})$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- FOCs wrt p_2 yield

$$D_2(p_2,p_1) + p_2\frac{\partial D_2(p_2,p_1)}{\partial p_2} - \underbrace{\frac{\partial TC(D_2(p_2,p_1))}{\partial D_2(p_2,p_1)}\,\frac{\partial D_2(p_2,p_1)}{\partial p_2}}_{\text{Using the chain rule}} = 0$$

- Solving for p_2 produces the follower's BRF for every price set by the leader, p_1, i.e., $p_2(p_1)$.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- In the first stage, firm 1 (leader) anticipates that the follower will use BRF $p_2(p_1)$ to respond to each possible price p_1, hence solves the following PMP

$$\max_{p_1 \geq 0} \quad \pi_1 = p_1 q_1 - TC(q_1)$$

$$= p_1 D_1\left(p_1, \underbrace{p_2(p_1)}_{BRF_2}\right) - TC\left(\overbrace{D_1(p_1, p_2(p_1))}^{q_1}\right)$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- FOCs wrt p_1 yield

$$D_1(p_1,p_2) + p_1\left[\frac{\partial D_1(p_1,p_2)}{\partial p_1} + \underbrace{\frac{\partial D_1(p_1,p_2)}{\partial p_2(p_1)}\frac{\partial p_2(p_1)}{\partial p_1}}_{\text{New strategic effect}}\right]$$

$$-\frac{\partial TC(D_1(p_1,p_2))}{\partial D_1(p_1,p_2)}\left[\frac{\partial D_1(p_1,p_2)}{\partial p_1} + \underbrace{\frac{\partial D_1(p_1,p_2)}{\partial p_2(p_1)}\frac{\partial p_2(p_1)}{\partial p_1}}_{\text{New strategic effect}}\right] = 0$$

- Or more compactly as

$$D_1(p_1,p_2) + \left(p_1 - \frac{\partial TC(D_1)}{\partial D_1}\right)\left[\frac{\partial D_1(p_1,p_2)}{\partial p_1} + \underbrace{\frac{\partial D_1(p_1,p_2)}{\partial p_2(p_1)}\frac{\partial p_2(p_1)}{\partial p_1}}_{\text{New}}\right] = 0$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- In contrast to the Bertrand model with simultaneous price competition, an increase in firm 1’s price now produces an increase in firm 2’s price in the second stage.

- Hence, the leader has more incentives to raise its price, ultimately softening price competition.

- While a softened competition benefits both the leader and the follower, the real beneficiary is the follower, as its profits increase more than the leader’s.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- ***Example***:
  - Consider a linear demand $q_i = 1 - 2p_i + p_j$, with no marginal costs, i.e., $c = 0$.
  - *Simultaneous Bertrand model*: the PMP is

    $$\max_{p_j \geq 0} \quad \pi_j = p_j \cdot (1 - 2p_j + p_k) \text{ for any } k \neq j$$

    where FOC wrt p_j produces firm *j*'s BRF

    $$p_j(p_k) = \frac{1}{4} + \frac{1}{4} p_k$$
  - Simultaneously solving the two BRFs yields $p_j^* = \frac{1}{3} \simeq 0.33$, entailing equilibrium profits of $\pi_j^* = \frac{2}{9} \simeq 0.222$.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- ***Example*** (continued):
  - *Sequential Bertrand model*: in the second stage, firm 2's (the follower's) PMP is

$$\max_{p_2 \geq 0} \ \pi_2 = p_2 \cdot (1 - 2p_2 + p_1)$$

where FOC wrt p_2 produces firm 2's BRF

$$p_2(p_1) = \frac{1}{4} + \frac{1}{4}p_1$$

  - In the first stage, firm 1's (the leader's) PMP is

$$\max_{p_1 \geq 0} \ \pi_1 = p_1 \cdot \left[1 - 2p_1 + \underbrace{\left(\frac{1}{4} + \frac{1}{4}p_1 \right)}_{BRF_2} \right] = p_1 \cdot \left[\frac{1}{4}(5 - 7p_1) \right]$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

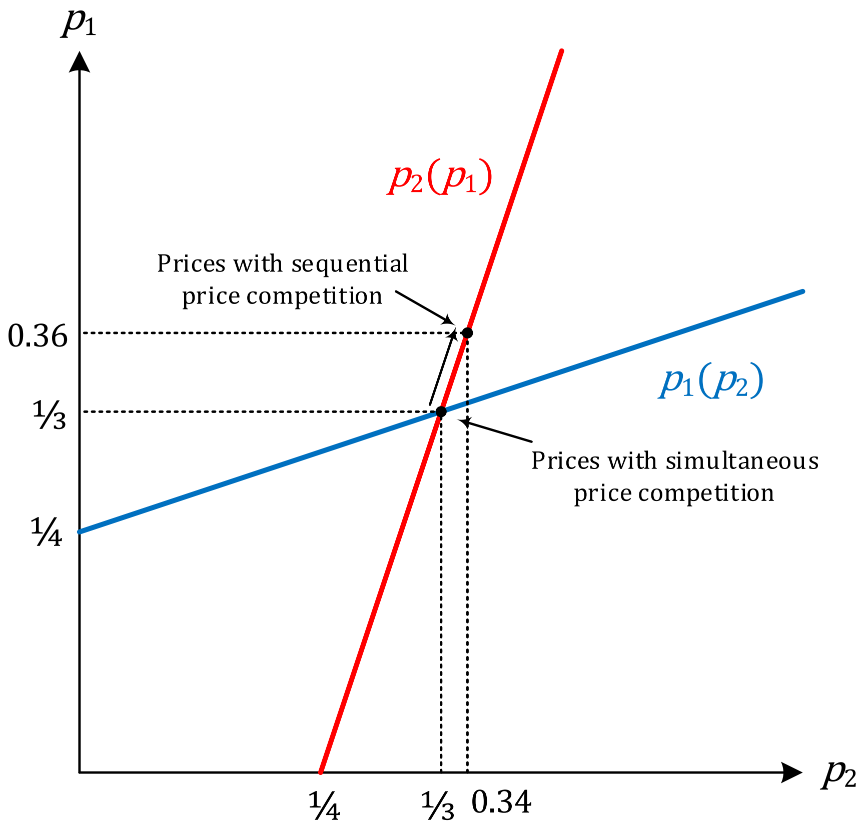
- ***Example*** (continued):
  - FOC wrt p_1, and solving for p_1, produces firm 1's equilibrium price $p_1^* = \frac{5}{14} = 0.36.$
  - Substituting p_1^* into the BRF of firm 2 yields $p_2^*(0.36) = \frac{1}{4} + \frac{1}{4}\left(\frac{5}{14}\right) = \frac{19}{56} = 0.34.$
  - Equilibrium profits are hence

$$\pi_1^* = 0.36\left[\frac{1}{4}\left(5 - 7(0.36)\right)\right] = 0.223 \text{ for firm 1}$$

$$\pi_2^* = 0.34\left(1 - 2(0.34) + (0.36)\right) = 0.23 \text{ for firm 2}$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- ***Example*** (continued):
  - Both firms' prices and profits are higher in the sequential than in the simultaneous game.
  - However, the follower earns more than the leader in the sequential game! (***second mover's advantage***)



Dynamic Competition: Sequential Cournot Model with Homogenous Products

- ***Stackelberg model***: firm 1 (the leader) chooses output level q_1, and firm 2 (the follower), observing the output decision of the leader, responds with its own output $q_2(q_1)$.

- By backward induction, the follower's BRF is $q_2(q_1)$ for any q_1.
- Since the leader anticipates $q_2(q_1)$ from the follower, the leader's PMP is

$$\max_{q_1 \geq 0} \; p\left(q_1 + \underbrace{q_2(q_1)}_{BRF_2}\right) q_1 - TC_1(q_1)$$

Dynamic Competition: Sequential Cournot Model with Homogenous Products

- FOCs wrt q_1 yields

$$p(q_1 + q_2(q_1)) + p'(q_1 + q_2(q_1))\left[1 + \frac{\partial q_2(q_1)}{\partial q_1}\right]q_1 - \frac{\partial TC_1(q_1)}{\partial q_1} = 0$$

or more compactly

$$p(Q) + p'(Q)q_1 + \underbrace{p'(Q)\frac{\partial q_2(q_1)}{\partial q_1}q_1}_{\text{Strategic Effect}} - \frac{\partial TC_1(q_1)}{\partial q_1} = 0$$

- This FOC coincides with that for standard Cournot model with simultaneous output decisions, except for the ***strategic effect***.

Dynamic Competition: Sequential Cournot Model with Homogenous Products

- The strategic effect is positive since $p'(Q) < 0$ and $\frac{\partial q_2(q_1)}{\partial q_1} < 0$.
- Firm 1 (the leader) has more incentive to raise q_1 relative to the Cournot model with simultaneous output decision.
- *Intuition* (***first-mover advantage***):
  - By overproducing, the leader forces the follower to reduce its output q_2 by the amount $\frac{\partial q_2(q_1)}{\partial q_1}$.
  - This helps the leader sell its production at a higher price, as reflected by $p'(Q)$; ultimately earning a larger profit than in the standard Cournot model.

Dynamic Competition: Sequential Cournot Model with Homogenous Products

- ***Example***:
  - Consider linear inverse demand $p = a - Q$, where $Q = q_1 + q_2$, and a constant marginal cost of c.
  - Firm 2's (the follower's) PMP is

$$\max_{q_2} \ (a - q_1 - q_2)q_2 - cq_2$$

  - FOC:

$$a - q_1 - 2q_2 - c = 0$$

  - Solving for q_2 yields the follower's BRF

$$q_2(q_1) = \frac{a-q_1-c}{2}$$

Dynamic Competition: Sequential Cournot Model with Homogenous Products

- ***Example*** (continued):
  - Plugging $q_2(q_1)$ into the leader's PMP, we get

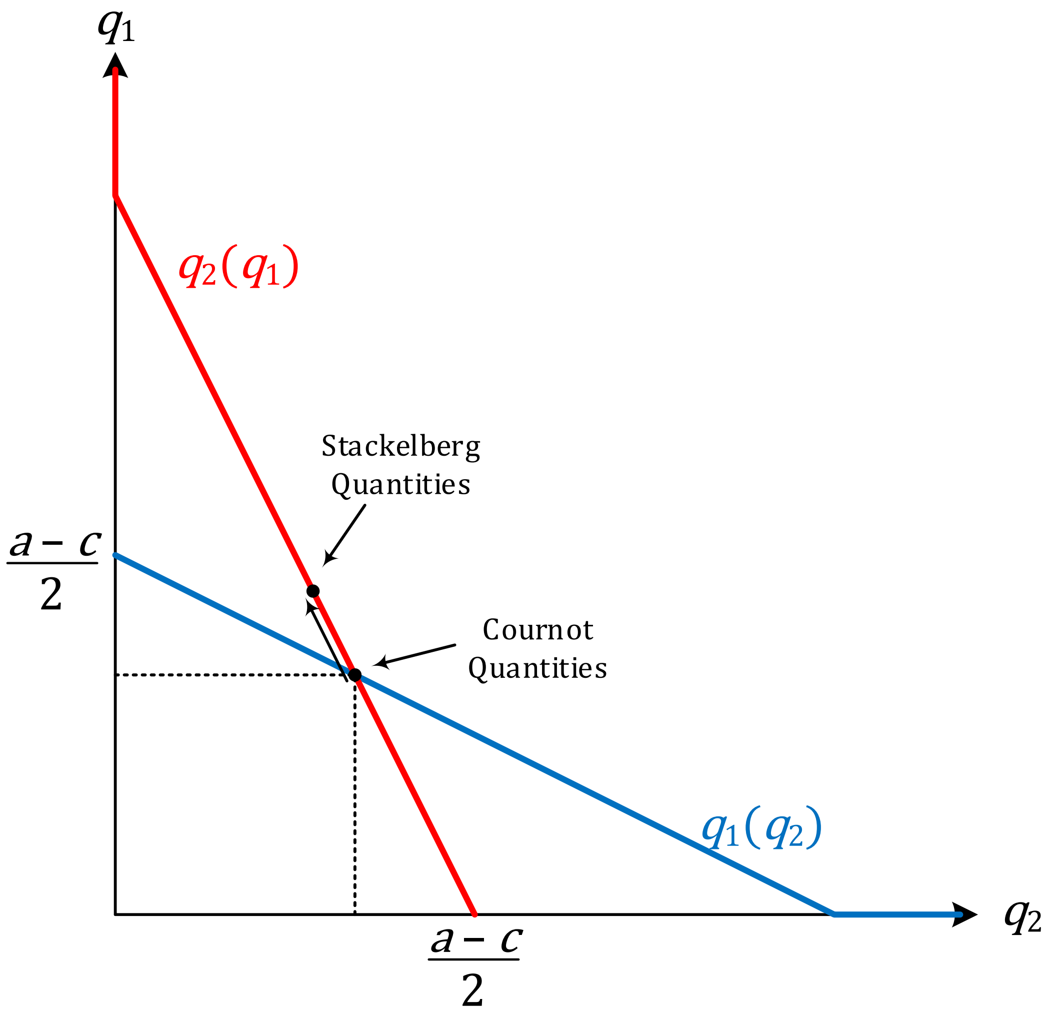
$$\max_{q_1} \left(a - q_1 - \frac{a-q_1-c}{2}\right) q_1 - cq_1 = \frac{1}{2}(a - q_1 - c)q_1$$

  - FOC:

$$\frac{1}{2}(a - 2q_1 - c) = 0$$

  - Solving for q_1, we obtain the leader's equilibrium output level $q_1^* = \frac{a-c}{2}$.
  - Substituting q_1^* into the follower's BRF yields the follower's equilibrium output $q_2^* = \frac{a-c}{4}$.

Dynamic Competition: Sequential Cournot Model with Homogenous Products



Dynamic Competition: Sequential Cournot Model with Homogenous Products

- ***Example*** (continued):
  - The equilibrium price is

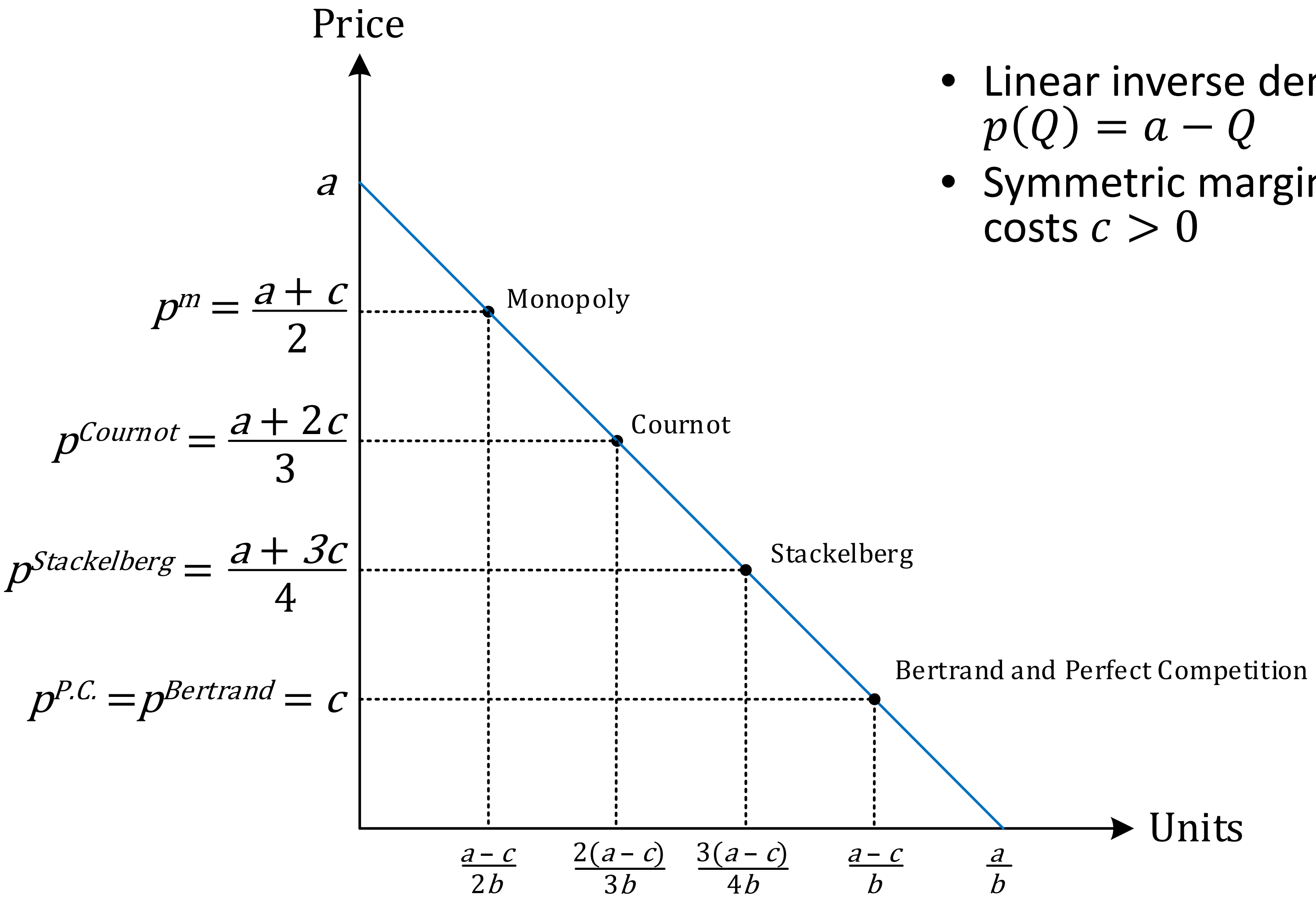
$$p = a - q_1^* - q_2^* = \frac{a + 3c}{4}$$

  - And the resulting equilibrium profits are

$$\pi_1^* = \left(\frac{a+3c}{4}\right)\left(\frac{a-c}{2}\right) - c\left(\frac{a-c}{2}\right) = \frac{(a-c)^2}{8}$$

$$\pi_2^* = \left(\frac{a+3c}{4}\right)\left(\frac{a-c}{4}\right) - c\left(\frac{a-c}{4}\right) = \frac{(a-c)^2}{16}$$

Dynamic Competition: Sequential Cournot Model with Homogenous Products



- Linear inverse demand $p(Q) = a - Q$
- Symmetric marginal costs $c > 0$

Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

- Assume that firms sell differentiated products, with inverse demand curves for firms 1 and 2

$$p_1(q_1, q_2) = \alpha - \beta q_1 - \gamma q_2 \text{ for firm 1}$$

$$p_2(q_1, q_2) = \alpha - \gamma q_1 - \beta q_2 \text{ for firm 2}$$

- Firm 2's (the follower's) PMP is

$$\max_{q_2} \ (\alpha - \gamma q_1 - \beta q_2) \cdot q_2$$

where, for simplicity, we assume no marginal costs.

- FOC:

$$\alpha - \gamma q_1 - 2\beta q_2 = 0$$

Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

- Solving for q_2 yields firm 2's BRF

$$q_2(q_1) = \frac{\alpha - \gamma q_1}{2\beta}$$

- Plugging $q_2(q_1)$ into the leader's firm 1's (the leader's) PMP, we get

$$\max_{q_1} \left(\alpha - \beta q_1 - \gamma\left(\frac{\alpha - \gamma q_1}{2\beta}\right)\right) q_1 =$$

$$\max_{q_1} \left(\alpha\left(\frac{2\beta - \gamma}{2\beta}\right) - \left(\frac{2\beta^2 - \gamma^2}{2\beta}\right) q_1\right) q_1$$

- FOC:

$$\alpha\left(\frac{2\beta - \gamma}{2\beta}\right) - \left(\frac{2\beta^2 - \gamma^2}{\beta}\right) q_1 = 0$$

Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

- Solving for q_1, we obtain the leader's equilibrium output level $q_1^* = \frac{\alpha(2\beta-\gamma)}{2(2\beta^2-\gamma^2)}$
- Substituting q_1^* into the follower's BRF yields the follower's equilibrium output

$$q_2^* = \frac{\alpha-\gamma q_1^*}{2\beta} = \frac{\alpha(4\beta^2-2\beta\gamma-\gamma^2)}{4\beta(2\beta^2-\gamma^2)}$$

- *Note*:
  - $q_1^* > q_2^*$
  - If $\gamma \to \beta$ (i.e., the products become more homogeneous), (q_1^*, q_2^*) convege to standard Stackelberg output $(\frac{\alpha}{2\beta}, \frac{\alpha}{4\beta})$.
  - If $\gamma \to 0$ (i.e., the products become very differentiated), (q_1^*, q_2^*) converge to the monopoly output $q^m = \frac{\alpha}{2\beta}$.

Capacity Constraints

Capacity Constraints

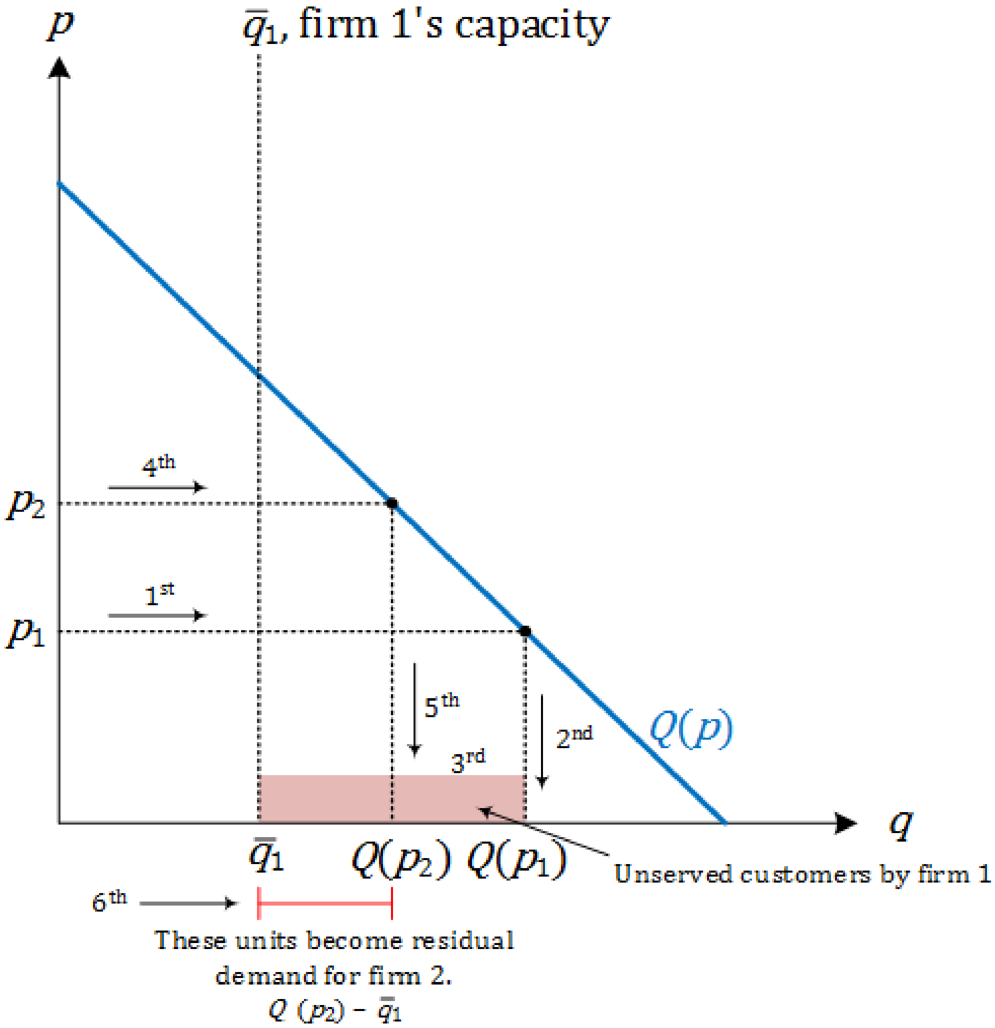
- How come are the equilibrium outcomes in standard Bertrand and Cournot models so different?
- Do firms really compete in prices without facing capacity constraints?
  - Bertrand model assumes a firm can supply infinitely large amount if its price is lower than its rivals.
- Extension of the Bertrand model:
  - **First stage**: firms set capacities, $\bar{q}_1$ and $\bar{q}_2$, with a cost of capacity $c > 0$
  - **Second stage**: firms observe each other's capacities and compete in prices, simultaneously setting p_1 and p_2

Capacity Constraints

- What is the role of capacity constraint?
  - When a firm's price is lower than its capacity, not all consumers can be served.
  - Hence, sales must be rationed through efficient rationing: the customers with the highest willingness to pay get the product first.
- Intuitively, if $p_1 < p_2$ and the quantity demanded at p_1 is so large that $Q(p_1) > \bar{q}_1$, then the first $\bar{q}_1$ units are served to the customers with the highest willingness to pay (i.e., the upper segment of the demand curve), while some customers are left in the form of residual demand to firm 2.

Capacity Constraints

- At p_1 the quantity demanded is $Q(p_1)$, but only $\bar{q}_1$ units can be served.
- Hence, the residual demand is $Q(p_1) - \bar{q}_1$.
- Since firm 2 sets a price of p_2, its demand will be $Q(p_2)$.
- Thus, a portion of the residual demand , i.e., $Q(p_2) - \bar{q}_1$, is captured.



Capacity Constraints

- Hence, firm 2's residual demand can be expressed as

$$\begin{cases} Q(p_2) - \bar{q}_1 & \text{if } Q(p_2) - \bar{q}_1 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- Should we restrict $\bar{q}_1$ and $\bar{q}_2$ somewhat?
  - Yes. A firm will never set a huge capacity if such capacity entails negative profits, independently of the decision of its competitor.

Capacity Constraints

- How to express this rather obvious statement with a simple mathematical condition?
  - The maximal revenue of a firm under monopoly is $\max_{q} (a-q)q$, which is maximized at $q=\frac{a}{2}$, yielding profits of $\frac{a^2}{4}$.
  - Maximal revenues are larger than costs if $\frac{a^2}{4} \geq c\bar{q}_j$, or solving for $\bar{q}_j$, $\frac{a^2}{4c} \geq \bar{q}_j$.
  - Intuitively, the capacity cannot be too high, as otherwise the firm would not obtain positive profits regardless of the opponent's decision.

Capacity Constraints: Second Stage

- By backward induction, we start with the second stage (pricing game), where firms simultaneously choose prices p_1 and p_2 as a function of the capacity choices $\bar{q}_1$ and $\bar{q}_2$.

- We want to show that in this second stage, both firms set a common price

$$p_1 = p_2 = p^* = a - \bar{q}_1 - \bar{q}_2$$

where demand equals supply, i.e., total capacity,

$$p^* = a - \bar{Q}, \text{ where } \bar{Q} \equiv \bar{q}_1 + \bar{q}_2$$

Capacity Constraints: Second Stage

- In order to prove this result, we start by assuming that firm 1 sets $p_1 = p^*$. We now need to show that firm 2 also sets $p_2 = p^*$, i.e., it does not have incentives to deviate from p^*.

- If firm 2 does not deviate, $p_1 = p_2 = p^*$, then it sells up to its capacity $\bar{q}_2$.

- If firm 2 reduces its price below p^*, demand would exceed its capacity $\bar{q}_2$. As a result, firm 2 would sell the same units as before, $\bar{q}_2$, but at a lower price.

Capacity Constraints: Second Stage

- If, instead, firm 2 charges a price above p^*, then $p_1 = p^* < p_2$ and its revenues become

$$p_2\hat{Q}(p_2) = \begin{cases} p_2(a - p_2 - \bar{q}_1) & \text{if } a - p_2 - \bar{q}_1 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- *Note*:
  - This is fundamentally different from the standard Bertrand model without capacity constraints, where an increase in price by a firm reduces its sales to zero.
  - When capacity constraints are present, the firm can still capture a residual demand, ultimately raising its revenues after increasing its price.

Capacity Constraints: Second Stage

- We now find the maximum of this revenue function. FOC wrt p_2 yields:

$$a - 2p_2 - \bar{q}_1 = 0 \iff p_2 = \frac{a - \bar{q}_1}{2}$$

- The non-deviating price $p^* = a - \bar{q}_1 - \bar{q}_2$ lies above the maximum-revenue price $p_2 = \frac{a-\bar{q}_1}{2}$ when

$$a - \bar{q}_1 - \bar{q}_2 > \frac{a - \bar{q}_1}{2} \iff a > \bar{q}_1 + 2\bar{q}_2$$

- Since $\frac{a^2}{4c} \geq \bar{q}_j$ (capacity constraint), we can obtain

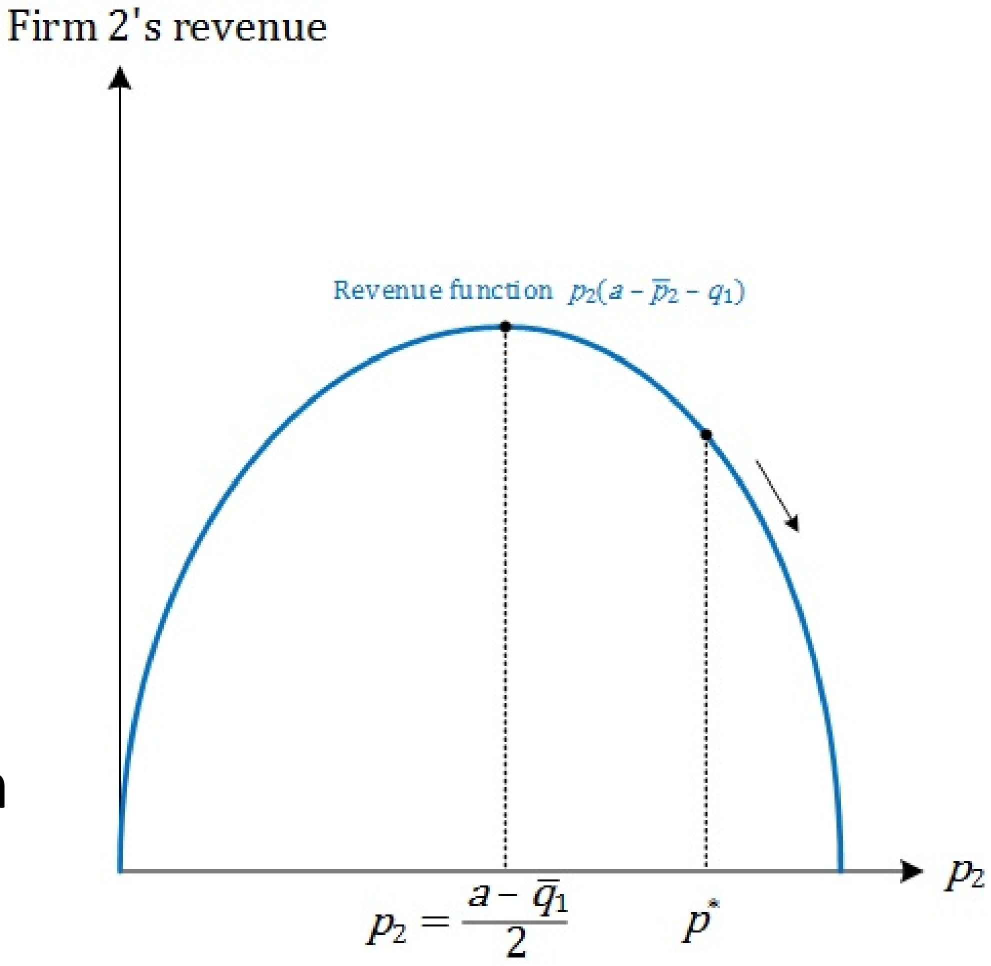
$$\frac{a^2}{4c} + 2\frac{a^2}{4c} > \bar{q}_1 + 2\bar{q}_2 \iff \frac{3a^2}{4c} > \bar{q}_1 + 2\bar{q}_2$$

Capacity Constraints: Second Stage

- Therefore, $a > \bar{q}_1 + 2\bar{q}_2$ holds if $a > \frac{3a^2}{4c}$ which, solving for a, is equivalent to $\frac{4c}{3} > a$.

Capacity Constraints: Second Stage

- When $\frac{4c}{3} > a$ holds, capacity constraint $\frac{a^2}{4c} \geq \bar{q}_j$ transforms into $\frac{3a^2}{4c} > \bar{q}_1 + 2\bar{q}_2$, implying $p^* > p_2 = a - \frac{\bar{q}_1}{2}$.
- Thus, firm 2 does not have incentives to increase its price p_2 from p^*, since that would lower its revenues.



Capacity Constraints: Second Stage

- In short, firm 2 does not have incentives to deviate from the common price
$$p^* = a - \bar{q}_1 - \bar{q}_2$$
- A similar argument applies to firm 1 (by symmetry).
- Hence, we have found an equilibrium in the pricing stage.

Capacity Constraints: First Stage

- In the first stage (capacity setting), firms simultaneously select their capacities $\bar{q}_1$ and $\bar{q}_2$.
- Inserting stage 2 equilibrium prices, i.e.,

$$p_1 = p_2 = p^* = a - \bar{q}_1 - \bar{q}_2,$$

into firm j's profit function yields

$$\pi_j(\bar{q}_1, \bar{q}_2) = \underbrace{(a - \bar{q}_1 - \bar{q}_2)}_{p^*}\bar{q}_j - c\bar{q}_j$$

- FOC wrt capacity $\bar{q}_j$ yields firm j's BRF

$$\bar{q}_j(\bar{q}_k) = \frac{a-c}{2} - \frac{1}{2}\bar{q}_k$$

Capacity Constraints: First Stage

- Solving the two BRFs simultaneously, we obtain a symmetric solution

$$\bar{q}_j = \bar{q}_k = \frac{a-c}{3}$$

- These are the same equilibrium predictions as those in the standard Cournot model.
- Hence, capacities in this two-stage game coincide with output decisions in the standard Cournot model, while prices are set equal to total capacity.

Endogenous Entry

Endogenous Entry

- So far the number of firms was exogenous
- What if the number of firms operating in a market is endogenously determined?
- That is, how many firms would enter an industry where
  - They know that competition will be à la Cournot
  - They must incur a fixed entry cost $F > 0$.

Endogenous Entry

- Consider inverse demand function $p(Q)$, where Q denotes aggregate output
- Every firm j faces the same total cost function, $c(q_j)$, of producing q_j units
- Hence, the Cournot equilibrium must be symmetric
  - Every firm produces the same output level $q(n)$, which is a function of the number of entrants.
- Entry profits for firm j are

$$\pi_j(n) = \underbrace{p\left(\underbrace{n \cdot q(n)}_{Q}\right)}_{p(Q)} q(n) - \underbrace{c\big(q(n)\big)}_{\text{Production Costs}} - \underbrace{F}_{\text{Fixed Entry Cost}}$$

Endogenous Entry

- Three assumptions (valid under most demand and cost functions):
  - individual equilibrium output $q(n)$ is decreasing in n;
  - aggregate output $q \equiv n \cdot q(n)$ increases in n;
  - equilibrium price $p(n \cdot q(n))$ remains above marginal costs regardless of the number of entrants n.

Endogenous Entry

- ***Equilibrium number of firms***:
  - The equilibrium occurs when no more firms have incentives to enter or exit the market, i.e., $\pi_j(n^e) = 0$.
  - Note that individual profits decrease in n, i.e.,

$$\pi_j'(n) = \overbrace{\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right]}^{+}\overbrace{\frac{\partial q(n)}{\partial n}}^{-} + \underbrace{q(n)p'\left(nq(n)\right)}_{-}\underbrace{\frac{\partial [nq(n)]}{\partial n}}_{+} < 0$$

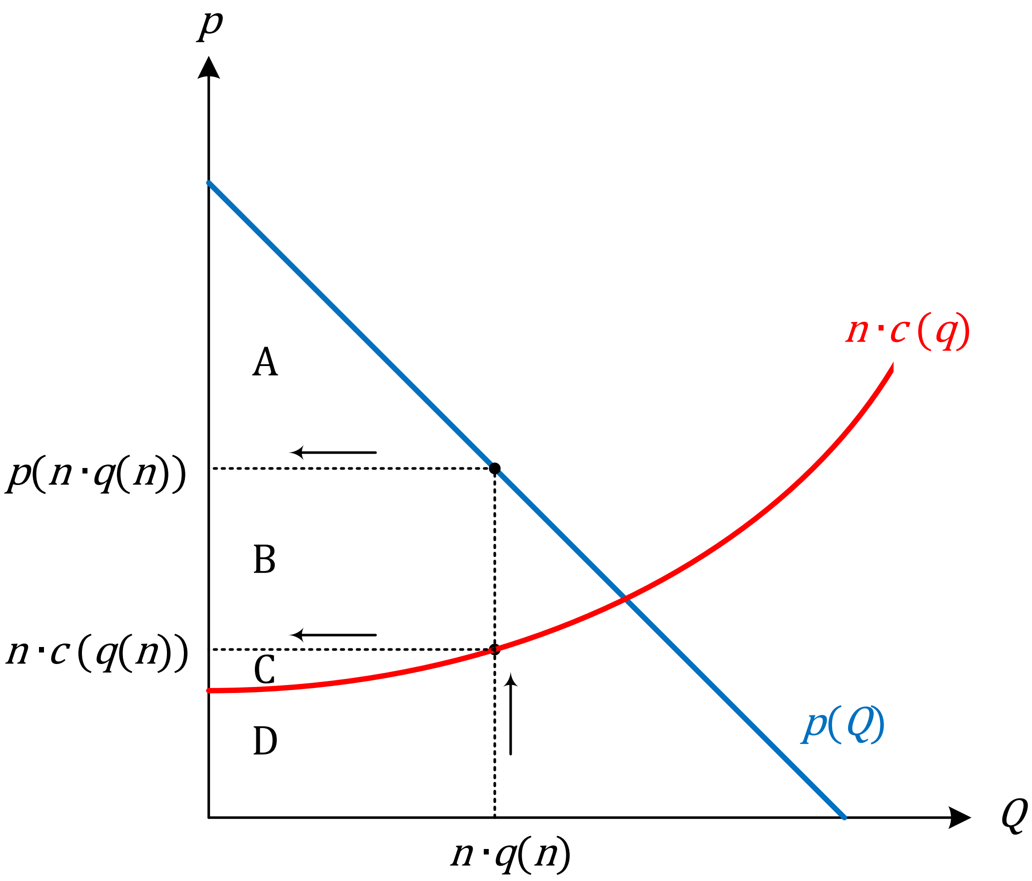
Endogenous Entry

- ***Social optimum***:
  - The social planner chooses the number of entrants n^o that maximizes social welfare

$$\max_{n} \; W(n) \equiv \int_0^{nq(n)} p(s)ds - n \cdot c\big(q(n)\big) - n \cdot F$$

Endogenous Entry

- $\int_0^{nq(n)} p(s)ds = A + B + C + D$
- $n \cdot c\big(q(n)\big) = C + D$
- Social welfare is thus $A + B$ minus total entry costs $n \cdot F$



Endogenous Entry

- Using Leibniz's rule, FOC wrt n yields

$$p(nq(n))\left[n\frac{\partial q(n)}{\partial n}+q(n)\right]-c(q(n))-nc'(q(n))\frac{\partial q(n)}{\partial n}-F=0$$

  or, rearranging,

$$\pi(n)+n\left[p(nq(n))-c'(q(n))\right]\frac{\partial q(n)}{\partial n}=0$$

- Hence, marginal increase in n entails two opposite effects on social welfare:
  a) the profits of the new entrant increase social welfare (+, ***appropriability effect***)
  b) the entrant reduces the profits of all previous incumbents in the industry as the individual sales of each firm decreases upon entry (–, ***business stealing effect***)

Endogenous Entry

- The "business stealing" effect is represented by:

$$n\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right]\frac{\partial q(n)}{\partial n} < 0$$

which is negative since $\frac{\partial q(n)}{\partial n} < 0$ and $n\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right] > 0$ by definition.

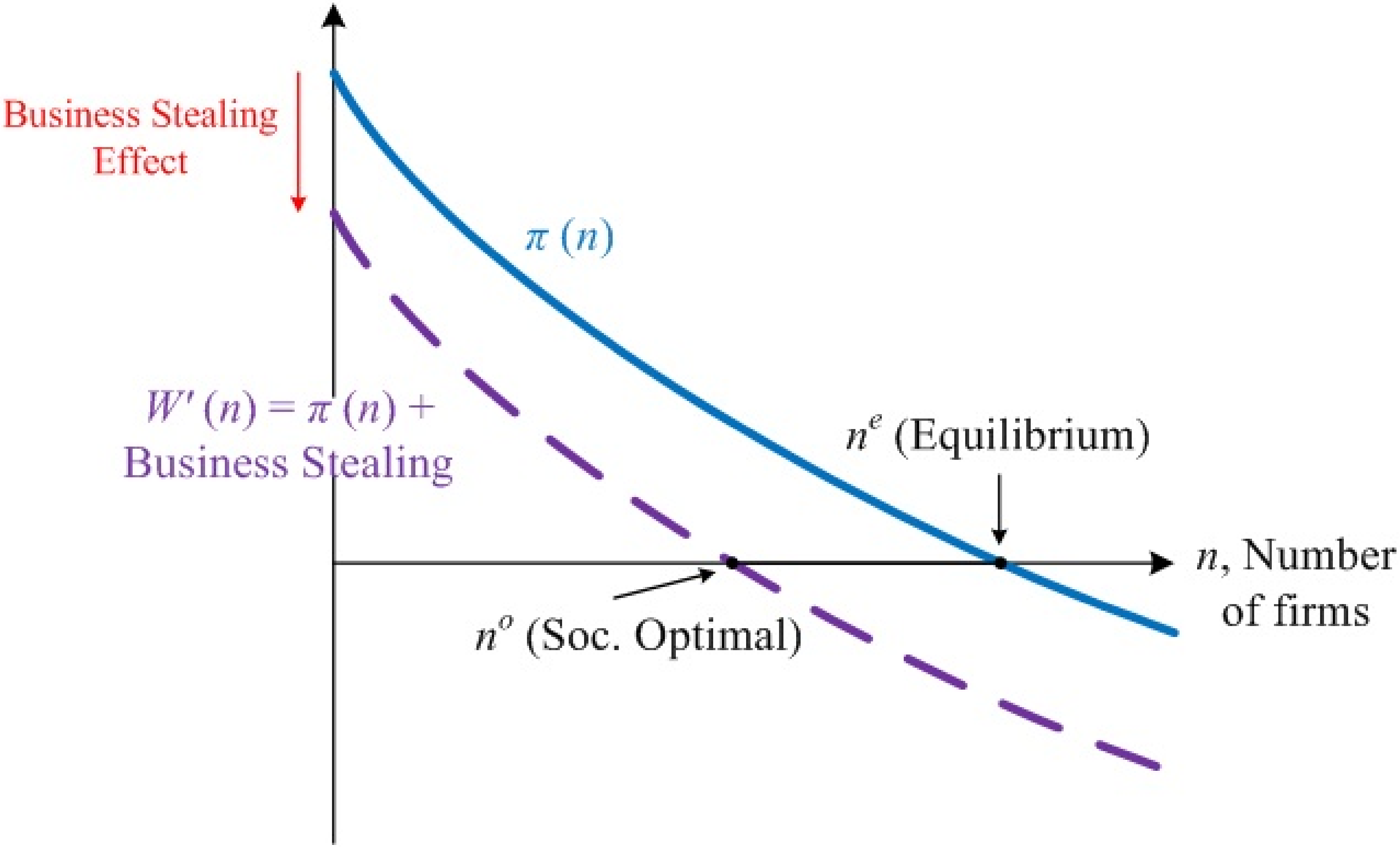
Endogenous Entry

- Given the negative sign of the business stealing effect, we can conclude that

$$W'(n) = \pi(n) + \underbrace{n\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right]\frac{\partial q(n)}{\partial n}}_{-} < \pi(n)$$

and therefore more firms enter in equilibrium than in the social optimum, i.e., $n^e > n^o$.

Endogenous Entry



Endogenous Entry

- ***Example***:
  - Consider a linear inverse demand $p(Q) = 1 - Q$ and no marginal costs.
  - The equilibrium quantity in a market with n firms that compete à la Cournot is

$$q(n) = \frac{1}{n+1}$$

  - Let's check if the three assumptions from above hold.

Endogenous Entry

- ***Example*** (continued):
  - First, individual output decreases with entry

$$\frac{\partial q(n)}{\partial n} = -\frac{1}{(n+1)^2} < 0$$

  - Second, aggregate output $nq(n)$ increases with entry

$$\frac{\partial [nq(n)]}{\partial n} = \frac{1}{(n+1)^2} > 0$$

  - Third, price lies above marginal cost for any number of firms

$$p(n) = 1 - n \cdot \frac{1}{n+1} = \frac{1}{n+1} > 0 \text{ for all } n$$

Endogenous Entry

- ***Example*** (continued):
  - Every firm earns equilibrium profits of

$$\pi(n) = \underbrace{\left(\frac{1}{n+1}\right)}_{p(n)} \underbrace{\frac{1}{n+1}}_{q(n)} - F = \frac{1}{(n+1)^2} - F$$

  - Since equilibrium profits after entry, $\frac{1}{(n+1)^2}$, is smaller than 1 even if only one firm enters the industry, $n = 1$, we assume that entry costs are in between 0 and 1, i.e., $0 \leq F \leq 1$.

Endogenous Entry

- ***Example*** (continued):
  - Social welfare is

$$W(n) = \int_0^{\frac{n}{n+1}} (1 - s)ds - n \cdot F$$

$$= \left(s - \frac{s^2}{2}\right)\Bigg|_0^{\frac{n}{n+1}} - n \cdot F$$

$$= \frac{n(n+2)}{2(n+1)^2} - n \cdot F$$

Endogenous Entry

- ***Example*** (continued):
  - The number of firms entering the market in equilibrium, n^e, is that solves $\pi(n^e) = 0$,

    $$\frac{1}{(n^e+1)^2} - F = 0 \iff n^e = \frac{1}{\sqrt{F}} - 1$$

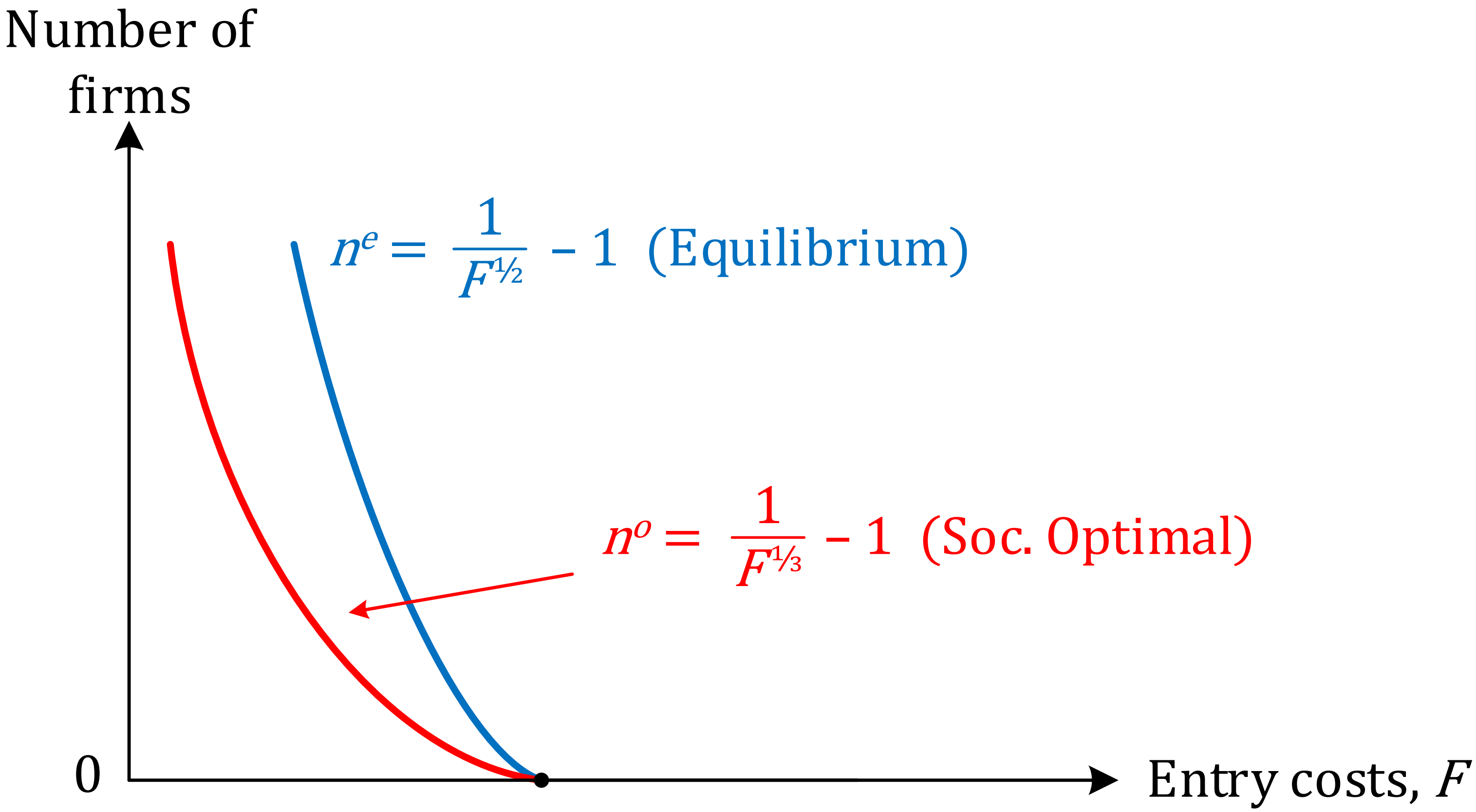
    whereas the number of firms maximizing social welfare, i.e., n^o, is that solves $W'(n^o) = 0$,

    $$W'(n^o) = \frac{1}{(n^o+1)^3} - F = 0 \iff n^o = \frac{1}{\sqrt[3]{F}} - 1$$

    where $n^e \geq n^o$ for all admissible values of F, i.e., $F \in [0,1]$.

Endogenous Entry

- ***Example*** (continued):



Repeated Interaction

Repeated Interaction

- In all previous models, we considered firms interacting during one period (i.e., ***one-shot game***).
- However, firms compete during several periods and, in some cases, during many generations.
- In these cases, a firm's actions during one period might affect its rival's behavior in future periods.
- More importantly, we can show that under certain conditions, the strong competitive results in the Bertrand (and, to some extent, in the Cournot) model can be avoided when firms interact repeatedly along time.
- That is, collusion can be supported in the repeated game even if it could not in the one-shot game.

Repeated Interaction: Bertrand Model

- Consider two firms selling homogeneous products.
- Let $p_{j,t}$ denote firm j's pricing strategy at period t, which is a function of the history of all price choices by the two firms, $H_{t-1} = \{p_{1,t}, p_{2,t}\}_{t=1}^{t-1}$.
- Conditioning $p_{j,t}$ on the full history of play allows for a wide range of pricing strategies:
  - setting the same price regardless of previous history
  - retaliation if the rival lowers its price below a "threshold level"
  - increasing cooperation if the rival was cooperative in previous periods (until reaching the monopoly price p^m)

Repeated Interaction: Bertrand Model

- ***Finitely repeated game***:
  - Can we support cooperation if the Bertrand game is repeated for a finite number of T rounds?
    - No!
  - To see why, consider the last period of the repeated game (period T):
    - Regardless of previous pricing strategies, every firm's optimal pricing strategy in this stage is to set $p_{i,T}^{*} = c$, as in the one-shot Bertrand game.

Repeated Interaction: Bertrand Model

- Now, move to the previous to the last period ($T - 1$):
    - Both firms anticipate that, regardless of what they choose at $T - 1$, they will both select $p_{i,T}^{*} = c$ in period T. Hence, it is optimal for both to select $p_{i,T-1}^{*} = c$ in period $T - 1$ as well.
- Now, move to period (T −2):
    - Both firms anticipate that, regardless of what they choose at $T - 2$, they will both select $p_{i,T}^{*} = c$ in period T and $p_{i,T-1}^{*} = c$ in period $T - 1$. Thus, it is optimal for both to select $p_{i,T-2}^{*} = c$ in period $T - 2$ as well.
- The same argument extends to all previous periods, including the first round of play $t = 1$.
- Hence, both firms behave as in the one-shot Bertrand game.

Repeated Interaction: Bertrand Model

- ***Infinitely repeated game***:
  - – Can we support cooperation if the Bertrand game is repeated for an infinite periods?
    - Yes! Cooperation (i.e., selecting prices above marginal cost) can indeed be sustained using different pricing strategies.
  - – For simplicity, consider the following pricing strategy

$$p_{jt}(H_{t-1}) = \begin{cases} p^m \text{ if all elements in } H_{t-1} \text{ are } (p^m, p^m) \text{ or } t = 1 \\ c \;\; \text{otherwise} \end{cases}$$

- In words, every firm j sets the monopoly price p^m in period 1. Then, in each subsequent period $t > 1$, firm j sets p^m if both firms charged p^m in all previous periods. Otherwise, firm j charges a price equal to marginal cost.

Repeated Interaction: Bertrand Model

- This type of strategy is usually referred to as ***Nash reversion strategy*** (NRS):
  - firms cooperate until one of them deviates, in which case firms thereafter revert to the Nash equilibrium of the unrepeated game (i.e., set prices equal to marginal cost)
- Let us show that NRS can be sustained in the equilibrium of the infinitely repeated game.
- We need to demonstrate that firms do not have incentives to deviate from it, during any period $t > 1$ and regardless of their previous history of play.

Repeated Interaction: Bertrand Model

- Consider any period $t > 1$, and a history of play for which all firms have been cooperative until $t - 1$.
- By cooperating, firm j's profits would be $(p^m - c)\frac{1}{2}\, x(p^m)$, i.e., half of monopoly profits $\frac{\pi^m}{2}$, in all subsequent periods

$$\frac{\pi^m}{2} + \delta\frac{\pi^m}{2} + \delta^2\frac{\pi^m}{2} + \cdots$$

$$= (1 + \delta + \delta^2 + \cdots)\frac{\pi^m}{2} = \frac{1}{1-\delta}\frac{\pi^m}{2}$$

where $\delta \in (0,1)$ denotes firms' discount factor

Repeated Interaction: Bertrand Model

- If, in contrast, firm j deviates in period t, the optimal deviation is $p_{j,t} = p^m - \varepsilon$, where $\varepsilon > 0$, given its rival still sets a price $p_{k,t} = p^m$.
- This allows firm j to capture all market, and obtain monopoly profits π^m during the deviating period.
- A deviation is detected in period $t + 1$, triggering a NRS from firm k (i.e., setting a price equal to marginal cost) thereafter, and entailing a zero profit for both firms.
- The discounted stream of profits for firm j is then

$$\pi^m + \delta 0 + \delta^2 0 + \cdots = \pi^m$$

Repeated Interaction: Bertrand Model

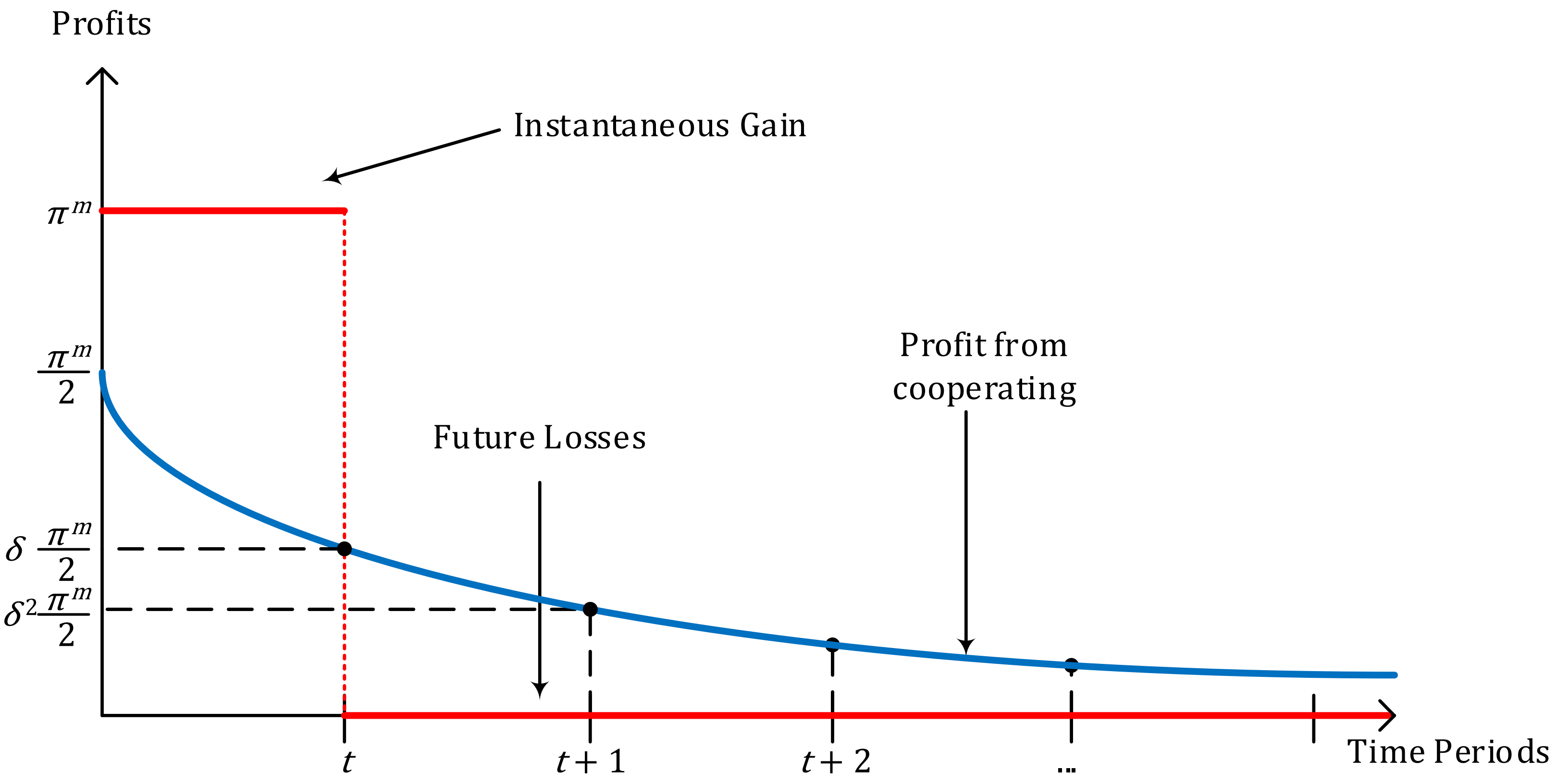
- Hence, firm j prefers to stick to the NRS at period t if

$$\frac{1}{1-\delta}\frac{\pi^m}{2} > \pi^m \iff \delta > \frac{1}{2}$$

- That is, cooperation can be sustained as long as firms assign a sufficiently high value to future profits.

Repeated Interaction: Bertrand Model

- Instantaneous gains and losses from cooperation and deviation



Repeated Interaction: Bertrand Model

- What about firm *j*'s incentives to use NRS after a history of play in which some firms deviated?
  - NRS calls for firm *j* to revert to the equilibrium of the unrepeated Bertrand model.
  - That is, to implement the punishment embodied in NRS after detecting a deviation from any player.
- By sticking to the NRS, firm *j*'s discounted stream of payoffs is

$$0 + \delta 0 + \cdots = 0$$

Repeated Interaction: Bertrand Model

- By deviating from NRS (i.e., setting a price $p_j = p^m$ while its opponent sets a punishing price $p_k = c$), profits are also zero in all periods.
- Hence, firm j has incentives to carry out the threat
  - That is, setting a punishing price of $p_j = c$, upon observing a deviation in any previous period.
- As a result, the NRS can be sustained in equilibrium, since both firms have incentives to use it, at any time period $t > 1$ and irrespective of the previous history of play.

Repeated Interaction: Bertrand Model

- ***Example***:
  - Consider an industry with only 2 firms, a linear demand $Q = 5000 - 100p$, and constant and average marginal costs of $c = \$10$.
  - If one-shot Bertrand game is played, firms would
    - charge a price of $p = c = \$10$
    - sell a total quantity of 4000 units
    - earn zero economic profits
  - If, in contrast, firms collude to fix prices at the monopoly price, can such collusion be sustained?

Repeated Interaction: Bertrand Model

- ***Example*** (continued):
  - Monopoly price is determined by solving the firms' joint PMP

$$\max_{p} \ (p-10)\cdot Q=(p-10)\,(5000-100p)$$

  - FOC:

$$5000-200p+1000=0$$

  - Solving for p yields the monopoly price $p^m=30$.
  - The aggregate output is $Q=2000$ (i.e., 1000 units per firm) and the corresponding profits are $\pi^m=\$40{,}000$ ($20,000 per firm).

Repeated Interaction: Bertrand Model

- ***Example*** (continued):
  - Collusion at the monopoly price is sustainable if

$$\frac{\pi^m}{2}\frac{1}{1-\delta} \geq \pi^m + \frac{\delta}{1-\delta} \cdot 0$$

  - Since $\pi^m = \$40{,}000$, the inequality reduces to

$$20000\frac{1}{1-\delta} \geq 40000 \quad \Leftrightarrow \quad \delta \geq \frac{1}{2}$$

Repeated Interaction: Bertrand Model

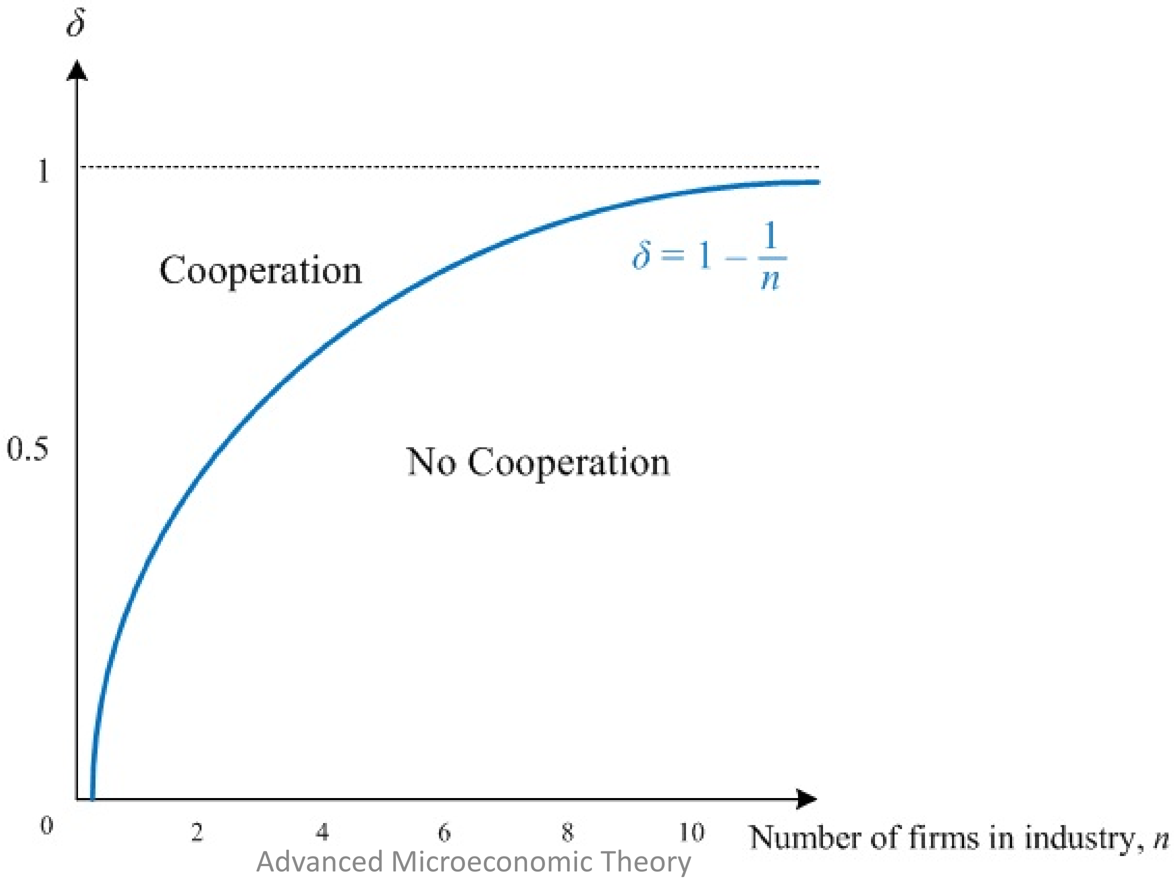
- ***Example*** (continued):
  - What would happen if there were n firms?
  - Each firm's share of the monopoly profit stream under collusion would be $\frac{\pi^m}{n} = \frac{40000}{n}$.
  - Collusion at the monopoly price is sustainable if

$$\frac{40000}{n}\frac{1}{1-\delta} \geq 40000 \Leftrightarrow \delta \geq 1 - \frac{1}{n} \equiv \bar{\delta}$$

  - Hence, as the number of firms in the industry increases, it becomes more difficult to sustain cooperation.

Repeated Interaction: Bertrand Model

- ***Example*** (continued): minimal discount factor sustaining collusion



Repeated Interaction: Cournot Model

- We can extend a similar analysis to the Cournot model of quantity competition with two firms selling homogeneous products.
- For simplicity, consider the following NRS for every firm j

$$q_{jt}(H_{t-1}) = \begin{cases} \frac{q^m}{2} \text{ if all elements in } H_{t-1} \text{ equal } \left(\frac{q^m}{2}, \frac{q^m}{2}\right) \text{ or } t = 1 \\ q_j^{Cournot} \text{ otherwise} \end{cases}$$

  - In words, firm j's strategy is to produce half of the monopoly output $\frac{q^m}{2}$ in period $t = 1$. Then, in each subsequent period $t > 1$, firm j continues producing $\frac{q^m}{2}$ if both firms produced $\frac{q^m}{2}$ in all previous periods. Otherwise, firm j reverts to the Cournot equilibrium output.

Repeated Interaction: Cournot Model

- Let us show that NRS can be sustained in the equilibrium of the infinitely repeated game.
- If firm j uses the NRS in period t, it obtains half of monopoly profits, $\frac{\pi^m}{2}$, thereafter, with a discounted stream of profits of $\frac{\pi^m}{2}\frac{1}{1-\delta}$.
- But, what if firm j deviates from this strategy? What is its optimal deviation?
  - Since firm k sticks to the NRS, and thus produces $\frac{q^m}{2}$ units, we can evaluate firm j's BRF $q_j(q_k)$ at $q_k = \frac{q^m}{2}$, or $q_j\left(\frac{q^m}{2}\right)$.

Repeated Interaction: Cournot Model

- For compactness, let $q_j^{dev} \equiv q_j\left(\frac{q^m}{2}\right)$ denote firm j's optimal deviation.
- This yields profits of

$$\pi_j^{dev} \equiv p\left(q_j^{dev}, \frac{q^m}{2}\right) \times q_j^{dev} - c_j \times q_j^{dev}$$

- By deviating firm j obtains following stream of profits

$$\pi_j^{dev} + \delta\pi_j^{Cournot} + \delta^2\pi_j^{Cournot} + \cdots$$

$$= \pi_j^{dev} + \frac{\delta}{1-\delta}\pi_j^{Cournot}$$

Repeated Interaction: Cournot Model

- Hence, firm j sticks to the NRS as long as

$$\frac{1}{1-\delta}\frac{\pi^m}{2} > \pi_j^{dev} + \frac{\delta}{1-\delta}\pi_j^{Cournot}$$

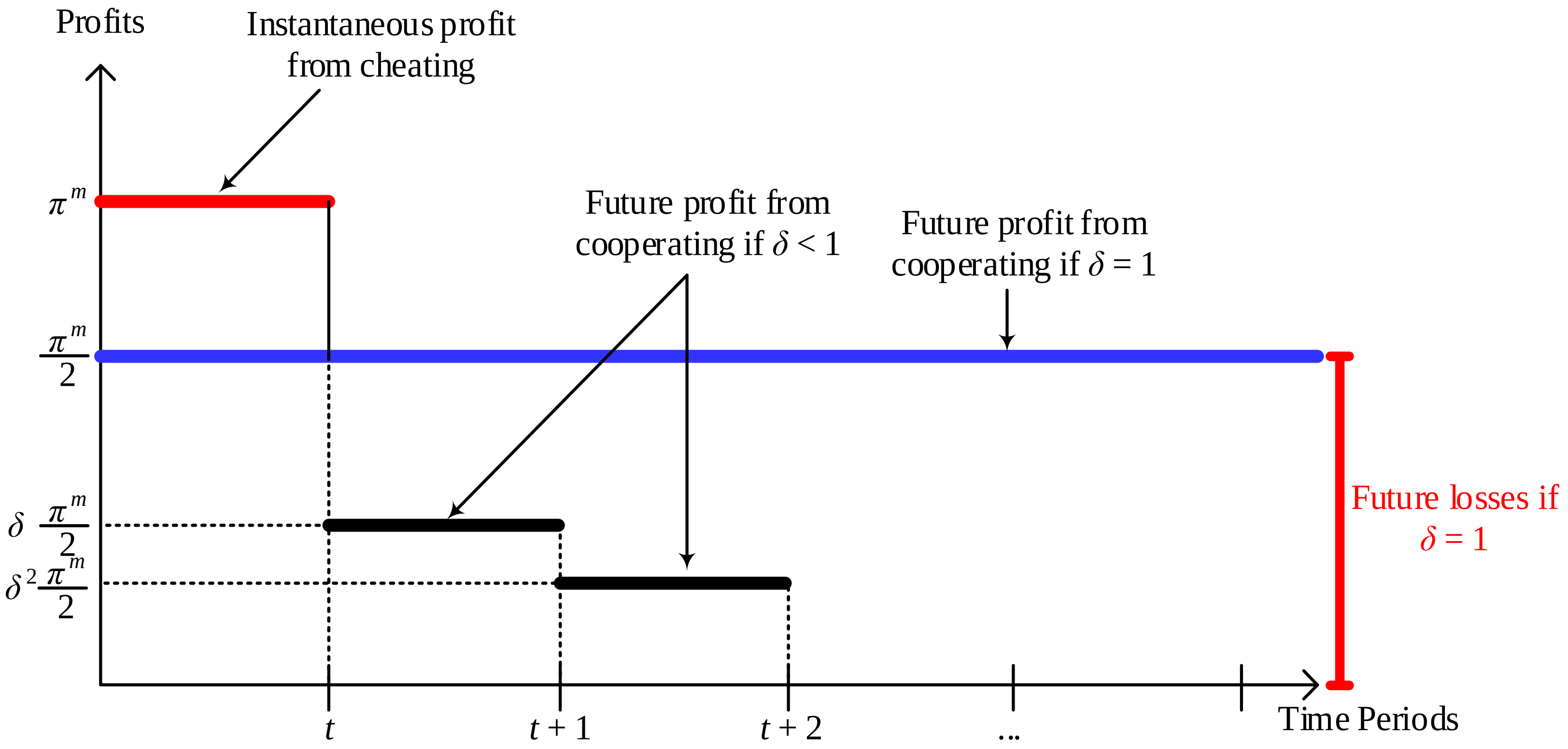
- Multiplying both sides by $(1-\delta)$ and solving for δ we obtain

$$\delta > \frac{\pi_j^{dev} - \frac{\pi^m}{2}}{\pi_j^{dev} - \pi_j^{Cournot}} \equiv \bar{\delta}$$

- Intuitively, every firm j sticks to the NRS as long as it assigns a sufficient weight to future profits.

Repeated Interaction: Cournot Model

- Instantaneous gains and losses from deviation



Repeated Interaction: Cournot Model

- What about firm *j*'s incentives to use NRS after a history of play in which some firms deviated?
  - NRS calls for firm *j* to revert to the equilibrium of the unrepeated Cournot model.
  - That is, to implement the punishment embodied in NRS after detecting a deviation from any player.
- By sticking to the NRS, firm *j*'s discounted stream of payoffs is $\frac{1}{1-\delta}\frac{\pi^{Cournot}}{2}$.
- By deviating from $q_j^{Cournot}$, while firm k produces $q_k^{Cournot}$, firm *j*'s profits, $\hat{\pi}$, are lower than $\pi_j^{Cournot}$ since firm *j*'s best response to its rival producing $q_k^{Cournot}$ is $q_j^{Cournot}$.

Repeated Interaction: Cournot Model

- Firm j sticks to the NRS after a history of deviations since

  $$\pi^{Cournot} + \delta\pi^{Cournot} + \cdots > \hat{\pi} + \delta\pi^{Cournot} + \cdots$$

  which holds given that $\pi^{Cournot} > \hat{\pi}$.

- Hence, no need to impose any further conditions on the minimal discount factor sustaining cooperation, $\bar{\delta}$.

Repeated Interaction: Cournot Model

- ***Example***:
  - Consider an industry with 2 firms, a linear inverse demand $p(q_1, q_2) = a - b(q_1 + q_2)$, and constant and average marginal costs of $c > 0$.
  - Firm i's PMP is

$$\max_{q_i} \left(a - b(q_i + q_j)\right)q_i - cq_i$$

  - FOCs: $$a - 2bq_i - bq_j - c = 0$$
  - Solving for q_i yields firm i's BRF

$$q_i(q_j) = \frac{a-c}{2b} - \frac{q_j}{2}$$

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - Solving the two BRFs simultaneously yields

$$q_i^{Cournot} = \frac{a-c}{3b}$$

with corresponding price of

$$p = a - b\left(\frac{a-c}{3b} + \frac{a-c}{3b}\right) = \frac{a+2c}{3}$$

and equilibrium profits of

$$\pi_i^{Cournot} = \left(\frac{a+2c}{3}\right)\frac{a-c}{3b} - c\left(\frac{a-c}{3b}\right) = \frac{(a-c)^2}{9b}$$

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - If, instead, each firm produced half of monopoly output, $q_i^m = \frac{q^m}{2} = \frac{a-c}{4b}$, they would face a corresponding price of $p^m = \frac{a+c}{2}$ and receive half of the monopoly profits $\pi_i^m = \frac{\pi^m}{2} = \frac{(a-c)^2}{8b}$.
  - In this setting, the optimal deviation of firm i is found by plugging q_i^m into its BRF

$$q_i^{Dev} = q_i(q_j^m) = \frac{a-c}{2b} - \frac{1}{2}\frac{a-c}{4b} = \frac{3(a-c)}{8b}$$

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - This yields price of

$$p = a - b\left(\frac{3(a-c)}{8b} + \frac{a-c}{4b}\right) = \frac{3a+5c}{8}$$

and profits of

$$\pi_i^{Dev} = \left(\frac{3a+5c}{8}\right)\frac{3(a-c)}{8b} - c\left(\frac{3(a-c)}{8b}\right) = \frac{9(a-c)^2}{64b}$$

for the deviating firm, and

$$\hat{\pi} = \left(\frac{3a+5c}{8}\right)\frac{(a-c)}{4b} - c\left(\frac{(a-c)}{4b}\right) = \frac{3(a-c)^2}{32b}$$

for the non-deviating firm.

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - Cooperation is sustainable if

$$\frac{1}{1-\delta}\frac{\pi^m}{2} > \pi_j^{dev} + \frac{\delta}{1-\delta}\pi_j^{Cournot}$$

or, in our case,

$$\frac{1}{1-\delta}\frac{(a-c)^2}{8b} > \frac{9(a-c)^2}{64b} + \frac{\delta}{1-\delta}\frac{(a-c)^2}{9b} \iff \delta > \frac{9}{17}$$

  - For the non-deviating firm, we have $\pi_i^{Cournot} > \hat{\pi}$
    - That is, if the rival firm defects, the non-defecting firm will obtain a larger profit by reverting to the Cournot output level.

Repeated Interaction: Cournot Model

- *Extensions*:
  - Temporary reversions to the equilibrium of the unrepeated game
  - (Temporary) punishments that yield even lower payoffs
  - Less “pure” forms of co-operations
  - Antitrust regulation and imperfect monitoring