Hotelling model with quadratic transportation costs[1]

Consider the following model of horizontally differentiated products with two firms. In the first stage, every firm i chooses its location, l_i, in the interval $[0,1]$, where $i = \{1,2\}$. In the second stage, every firm, observing the location pair (l_1, l_2) from the first stage, responds setting a price p_i. In the third stage, given firms' location and prices, consumers buy one unit of the good from either firm 1 or 2. Consumers are, for simplicity, uniformly distributed in the unit line. Assume that consumers suffer quadratic transportation costs, and both firms' marginal production cost is $c > 0$.

Third stage - Finding demand. If a consumer purchases from firm 1, his utility is $r - p_1 - t(x - l_1)^2$, while purchasing from firm 2 yields $r - p_2 - t(x - l_2)^2$. Therefore, the indifferent consumer $\widehat{x}$ solves

$$r - p_1 - t(\widehat{x} - l_1)^2 = r - p_2 - t(x - l_2)^2$$

which yields

$$\widehat{x} = \frac{l_1 + l_2}{2} - \frac{p_1 - p_2}{2t(l_2 - l_1)}.$$

Firm 1's demand is $\widehat{x}$, while firm 2's demand is $1 - \widehat{x}$. When both firms set the same prices, $p_1 = p_2$, then firm 1 simplifies to $\widehat{x} = \frac{l_1 + l_2}{2}$, while firm 2's demand becomes $1 - \widehat{x} = 1 - \frac{l_1 + l_2}{2}$.

Second stage - Prices. *Finding firm 1's best response function.* Firm 1 chooses the price p_1 that solves

$$\max_{p_1} \ (p_1 - c) \underbrace{\left(\frac{l_1 + l_2}{2} - \frac{p_1 - p_2}{2t(l_2 - l_1)} \right)}_{\text{Demand, } \widehat{x}}$$

Differentiating with respect to p_1, we obtain

$$\left(\frac{l_1 + l_2}{2} - \frac{p_1 - p_2}{2t(l_2 - l_1)} \right) + (p_1 - c) \left(-\frac{1}{2t(l_2 - l_1)} \right) = 0$$

Solving for p_1, we find firm 1's best response function

$$p_1(p_2) = \frac{t(l_1 + l_2)(l_2 - l_1) + c}{2} + \frac{1}{2} p_2$$

with vertical intercept at $\frac{t(l_1+l_2)(l_2-l_1)+c}{2}$ and slope $\frac{1}{2}$. Intuitively, when firm 2 increases its price by \$1, firm 1 responds by increasing its own by \$0.5.
In addition, a marginal increase in firm 1's location, l_1, or in firm 2's location, l_2, yields the following change in the above best response functions

$$\frac{\partial p_1(p_2)}{\partial l_1} = -tl_1 < 0 \quad \text{and} \quad \frac{\partial p_1(p_2)}{\partial l_2} = tl_2 > 0,$$

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respectively. Therefore, when firm 1 moves its position rightward, its best response function shifts downward in a parallel fashion, indicating that firm charges less for its product. Intuitively, its position is closer to firm 2's, attenuating product differentiation, and ultimately decreasing the price that firm 1 can charge. In contrast, when firm 2 moves its position rightward, both firms move further away from each other, entailing more differentiated products. In this case, firm 1's best response function shifts upwards, indicating a higher price p_1.

Finding firm 2's best response function. Operating similarly for firm 2, we have that this firm chooses price p_2 to solve

$$\max_{p_2} (p_2 - c)\underbrace{\left(1 - \left(\frac{l_1 + l_2}{2} - \frac{p_1 - p_2}{2t(l_2 - l_1)}\right)\right)}_{\text{Demand, } 1-\widehat{x}}$$

Differentiating with respect to p_2, we obtain

$$\left(1 - \left(\frac{l_1 + l_2}{2} - \frac{p_1 - p_2}{2t(l_2 - l_1)}\right)\right) + (p_2 - c)\left(-\frac{1}{2t(l_2 - l_1)}\right) = 0$$

Solving for p_2, we find firm 2's best response function

$$p_2(p_1) = \frac{t(l_2 - l_1)(2 - l_1 - l_2) + c}{2} + \frac{1}{2}p_1$$

with vertical intercept at $\frac{t(l_2-l_1)(2-l_1-l_2)+c}{2}$ and slope $\frac{1}{2}$, thus exhibiting the same intuition as the positively sloped best response function of firm 1.

As in the case of firm 1, a marginal increase in firm 1's location, l_1, or in firm 2's location, l_2, yields

$$\frac{\partial p_2(p_1)}{\partial l_1} = -t(1 - l_1) < 0 \quad \text{and} \quad \frac{\partial p_2(p_1)}{\partial l_2} = t(1 - l_2) > 0,$$

respectively. Therefore, when firm 1 moves its position rightward, it becomes closer to firm 2, attenuating product differentiation, and ultimately decreasing the price that firm 2 charges (downward shift in firm 2's best response function). In contrast, when firm 2 moves its position rightward, both firms move further away from each other, entailing more differentiated products. In this case, firm 2's best response function shifts upward.

Finding equilibrium prices. Simultaneously solving for p_1 and p_2 in the above best response functions, we find equilibrium prices

$$\begin{aligned} p_1^*(l_1, l_2) &= c + \frac{t}{3}(l_2 - l_1)(2 + l_1 + l_2), \text{ and} \\ p_2^*(l_1, l_2) &= c + \frac{t}{3}(l_2 - l_1)(4 - l_1 - l_2). \end{aligned}$$

Following with our above discussion, note that when both firms locate at the same position, $l_1 = l_2$, equilibrium prices simplify to marginal cost pricing, $p_1^*(l_1, l_2) = p_2^*(l_1, l_2) = c$. Our results also help us examine the case of pricing under exogenous product differentiation (e.g., $l_1 = 0$ and $l_2 = 1$). In this setting, equilibrium prices become

$$\begin{aligned} p_1^*(l_1, l_2) &= c + \frac{t}{3}(1 - 0)(2 + 0 + 1) = c + t, \text{ and} \\ p_2^*(l_1, l_2) &= c + \frac{t}{3}(1 - 0)(4 - 0 - 1) = c + t. \end{aligned}$$

Therefore, second-stage profits are

$$\begin{aligned}\pi_1^*(l_1, l_2) &= \frac{t}{18}(l_2 - l_1)(2 + l_1 + l_2)^2, \text{ and} \\ \pi_2^*(l_1, l_2) &= \frac{t}{18}(l_2 - l_1)(4 - l_1 - l_2)^2.\end{aligned}$$

which also collapse to zero when both firms locate at the same position, $l_1 = l_2$, and to $\pi_1^*(l_1, l_2) = \pi_2^*(l_1, l_2) = \frac{t}{2}$ when firms' locations are exogenously determined at $l_1 = 0$ and $l_2 = 1$.

First stage - Equilibrium location. *Finding firm 1's best response function.* In the first stage, firm 1 anticipates the equilibrium prices that firms charge in the second stage, and chooses its location l_1 to solve

$$\max_{l_1} \frac{t}{18}(l_2 - l_1)(2 + l_1 + l_2)^2$$

Differentiating with respect to l_1, we find

$$-\frac{t}{18}(2 + l_1 + l_2)^2 + \frac{t}{9}(l_2 - l_1)(2 + l_1 + l_2) = 0$$

which, solving for l_1, yields firm 1's best response function

$$l_1(l_2) = -\frac{2}{3} + \frac{1}{3}l_2.$$

Finding firm 2's best response function. Similarly, firm 2 chooses location l_2 to solve

$$\max_{l_2} \frac{t}{18}(l_2 - l_1)(4 - l_1 - l_2)^2$$

Differentiating with respect to l_2, we find

$$-\frac{t}{18}(4 - l_1 - l_2)^2 - \frac{t}{9}(l_2 - l_1)(4 - l_1 - l_2) = 0$$

which, solving for l_2, yields firm 2's best response function

$$l_2(l_1) = \frac{4}{3} + \frac{1}{3}l_1.$$

Simultaneously solving for l_1 and l_2 in the above best response functions, yields

$$l_1 = -\frac{2}{3} + \underbrace{\frac{1}{3}\left(\frac{4}{3} + \frac{1}{3}l_1\right)}_{l_2(l_1)}$$

Solving for l_1, we obtain

$$l_1 = -\frac{1}{4}.$$

Inserting this result into firm 2's best response function, we find that

$$l_2 = 4 + \frac{1}{3}\left(-\frac{1}{4}\right) = \frac{5}{4}.$$

Therefore, firms differentiate their products as much as possible. In the interval $[0, 1]$, they locate at the endpoints of the line, $l_1 = 0$ and $l_2 = 1$.

From our above discussion, we know that these positions yield equilibrium prices $p_1^*(l_1, l_2) = p_2^*(l_1, l_2) = c + t$, and equilibrium profits $\pi_1^*(l_1, l_2) = \pi_2^*(l_1, l_2) = \frac{t}{2}$.