EconS 424 - Randomized Strategies

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- The famous spy, 001, has to choose one of four routes, *a*, *b*, *c*, or *d* (listed in order of speed in good conditions) to ski down a mountain. At the same time, 001's notorious rival, 002, has to choose whether to use (*y*) or to not use (*x*) his valuable explosive device to cause an avalanche, knowing that an avalanche on a fast route is much more dangerous for 001 than an avalanche on a slow route.
- The payoffs of this game are represented on the next slide.

- Payoffs for the Spy Avalanche game:

<div align="center">

002

		x	*y*
	a	12, 0	0, 6
001	*b*	11, 1	1, 5
	c	10, 2	4, 2
	d	9, 3	6, 0

</div>

- Let $\theta_2(x)$ denote the probability that 001 believes 002 selects x. Explain what 001 should do if $\theta_2(x) > \frac{2}{3}$, if $\theta_2(x) < \frac{2}{3}$, and if $\theta_2(x) = \frac{2}{3}$.

Watson, Ch. 11 # 11

- If $\theta_2(x)$ is the probability that 001 believes 002 will select x, then the probability that 001 believes 002 will select y is $1 - \theta_2(x)$. Let's start our analysis by calculating the expected utilities for 001 for an unknown value of $\theta_2(x)$.

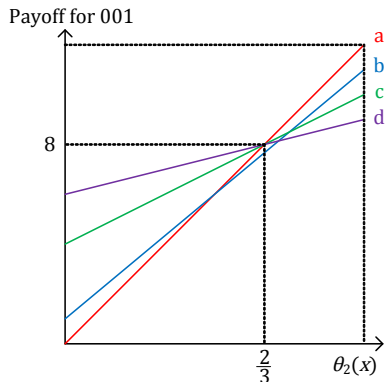$\pi_{001}(a) = \theta_2(x) * 12 + (1 - \theta_2(x)) * 0 = 12\theta_2(x)$

$\pi_{001}(b) = \theta_2(x) * 11 + (1 - \theta_2(x)) * 1 = 10\theta_2(x) + 1$

$\pi_{001}(c) = \theta_2(x) * 10 + (1 - \theta_2(x)) * 4 = 6\theta_2(x) + 4$

$\pi_{001}(d) = \theta_2(x) * 9 + (1 - \theta_2(x)) * 6 = 3\theta_2(x) + 6$

Watson, Ch. 11 # 11

- Representing the payoffs in a graph:

Watson, Ch. 11 # 11

- It will be the most simple to start with the scenario where $\theta_2(x) = \frac{2}{3}$. Plugging that value into our expected payoff functions, we have payoffs for 001 of

$$\pi_{001}(a) = \overbrace{\frac{2}{3}}^{\theta_2(x)} * 12 + \overbrace{\left(1 - \frac{2}{3}\right)}^{1-\theta_2(x)} * 0 = 8$$

$$\pi_{001}(b) = \frac{2}{3} * 11 + \left(1 - \frac{2}{3}\right) * 1 = 7.67$$

$$\pi_{001}(c) = \frac{2}{3} * 10 + \left(1 - \frac{2}{3}\right) * 4 = 8$$

$$\pi_{001}(d) = \frac{2}{3} * 9 + \left(1 - \frac{2}{3}\right) * 6 = 8$$

- As we can see, if $\theta_2(x) = \frac{2}{3}$, agent 001 is indifferent between routes a, c, or d.

Watson, Ch. 11 # 11

- In the case $\theta_2(x) > \frac{2}{3}$, agent 001 should choose route a, since as can be seen in the graph, the payoff function for a is always the highest among the four possible payoffs (i.e., a is the steepest of the three curves that are tied for the best when $\theta_2(x) = \frac{2}{3}$, and will grow the most from an increase in $\theta_2(x)$). Formally,

$$\pi_{001}(a) > \pi_{001}(c) > \pi_{001}(d) \ \forall \theta_2(x) > \frac{2}{3}$$

- Using the same intuition, when $\theta_2(x) < \frac{2}{3}$, agent 001 should choose route d (since it is the flattest of the three, and a decrease in $\theta_2(x)$ will affect it the least). Formally,

$$\pi_{001}(d) > \pi_{001}(c) > \pi_{001}(a) \ \forall \theta_2(x) < \frac{2}{3}$$

  - (You may be wondering why route b was excluded from this ranking. We'll answer why in the next part!)

- Imagine that you are Mr. Cue, the erudite technical advisor to military intelligence. Are there any routes that you would advise 001 to definitely **not** take? Explain your answer.

Watson, Ch. 11 # 11

- It is advised that 001 never take route b. Route b is dominated by a mixture of routes a and c. One such mixture assigns $\frac{2}{3}$ probability on a, and $\frac{1}{3}$ probability on c (Remember that we only need to find *one* probability that this happens). It is easy to see that the expected payoff from this randomized strategy is
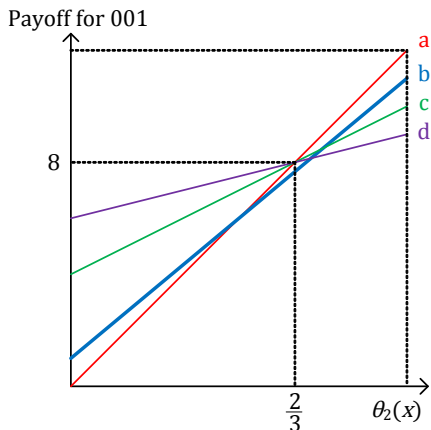
$$
\begin{aligned}
\frac{2}{3}\pi_{001}(a, x) + \frac{1}{3}\pi_{001}(c, x) &= \frac{2}{3} * 12 + \frac{1}{3} * 10 \\
&= 11.33 > 11 = \pi_{001}(b, x)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{2}{3}\pi_{001}(a, y) + \frac{1}{3}\pi_{001}(c, y) &= \frac{2}{3} * 0 + \frac{1}{3} * 4 \\
&= 1.33 > 1 = \pi_{001}(b, y)
\end{aligned}
$$

- We can also show this using our graph. It is easy to see that the payoffs represented by the function $\pi_{001}(b)$ are always less than the payoffs of at least one other function:

- A viewer of this epic drama is trying to determine what will happen. Find a Nash Equilibrium in which one player plays a pure strategy, S_i, and the other player plays a mixed strategy, σ_j.

- First, we want to find which player plays the pure strategy. This will happen when all of the payoffs for the opponent of the player using the pure strategy are equal. This happens when 001 plays the strategy c. By playing c, 002 receives a payoff of 2 regardless of what strategy he plays, and is thus indifferent.

		002	
		x	*y*
	a	12, 0	0, 6
	b	11, 1	1, 5
001	*c*	10, 2	4, 2
	d	9, 3	6, 0

- Further, 002 can mix so that c is the best response for 001 (i.e., make him indifferent between c and his other undominated strategies, a, and d). A mixture of $\frac{2}{3}$ and $\frac{1}{3}$ implies that 001 receives a payoff of 8 from all of his undominated strategies. This Nash Equilibrium is $S_1 = c$ and $\sigma_2 = (\frac{2}{3}, \frac{1}{3})$.

- Find a different mixed-strategy equilibrium in which this same pure strategy, S_i, is assigned zero probability.

Watson, Ch. 11 # 11

- Since b is dominated, we now consider a mixture by 001 over a and d. In finding the Nash Equilibrium above, we noticed that 002's mixing with probability $\left(\frac{2}{3}, \frac{1}{3}\right)$ makes 001 indifferent between a, c, and d.
- Thus, we need only to find a mixture over a and d that makes 002 indifferent between x and y. Let p denote the probability with which 001 plays a, and $1-p$ denote the probability with which he plays d.
- Indifference on the part of 002 is reflected by:

$$
\begin{aligned}
E\pi_{002}(x; a, d) &= E\pi_{002}(y; a, d) \\
p * 0 + (1-p) * 3 &= p * 6 + (1-p) * 0 \\
&\implies p = \frac{1}{3}
\end{aligned}
$$

Watson, Ch. 11 # 11

- Thus, the expected payoffs of agent 002 are:

$$E\pi_{002}(x; a, d) = \overbrace{\frac{1}{3}}^{p} * 0 + \overbrace{\left(1 - \frac{1}{3}\right)}^{1-p} * 3 = 2$$

$$E\pi_{002}(y; a, d) = \frac{1}{3} * 6 + \left(1 - \frac{1}{3}\right) * 0 = 2$$

which means that 002 receives a payoff of 2 whether he chooses x or y. This equilibrium is:

$$\sigma = \left(\left(\frac{1}{3}, 0, 0, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right) \right)$$

- Are there any other equilibria?

- In considering whether there are any more equilibria, it is useful to notice that in both of the above equilibria 002's payoff from choosing x is the same as that from y.
- Thus we should expect that as long as the frequency of a to d is kept the same, 001 could also play c with positive probability.
- Let p denote the probability with which 001 plays a, and let q denote the probability with which he plays c. Since he never plays b, the probability with which d is played is $1 - p - q$.

Watson, Ch. 11 # 11

- Making player 2 indifferent between playing x and y requires that

$$2q + 3(1 - p - q) = 6p + 2q$$

- This implies that any p and q such that $1 = 3p + q$ will work.
  - One such case is $(\frac{1}{9}, 0, \frac{6}{9}, \frac{2}{9})$, implying an equilibrium of
    $\sigma = ((\frac{1}{9}, 0, \frac{6}{9}, \frac{2}{9}), (\frac{2}{3}, \frac{1}{3}))$.

  In this context, the probabilities are

  Prob(a) = p

  Prob(c) = 1 - 3p

  Prob(d) = 1 - p - (1 - 3p) = 2p

- For the game illustrated below, find all mixed-strategy Nash Equilibria.

Player 2

	x	*y*	*z*
a	2, 3	1, 4	3, 2
b	5, 1	2, 3	1, 2
c	3, 7	4, 6	5, 4
d	4, 2	1, 3	6, 1

Player 1

- Let's start by looking for strictly dominated strategies. Note that c strictly dominates a and y strictly dominates z. This, any Nash Equilibrium in mixed strategies must assign zero probability to those dominated strategies. We can then eliminate them, so the reduced game is as shown below.

		Player 2	
		x	y
	b	5, 1	2, 3
Player 1	c	3, 7	4, 6
	d	4, 2	1, 3

- For this reduced game, b strictly dominates d, so the latter can be deleted. The reduced game is as shown below.

		Player 2	
		q x	$1 - q$ y
p	b	$\underline{5}$, 1	2, $\underline{3}$
$1 - p$	c	3, $\underline{7}$	$\underline{4}$, 6

Player 1

- The game has no pure-strategy Nash Equilibria. To find the mixed-strategy Nash Equilibria, let p denote the probability that player 1 chooses b and q denote the probability that player 2 chooses x.

- The expected payoffs for player 1 are:

$$
\begin{aligned}
EU_1(b) &= q * 5 + (1 - q) * 2 = 2 + 3q \\
EU_1(c) &= q * 3 + (1 - q) * 4 = 4 - q
\end{aligned}
$$

- For player 1 to be indifferent between strategies b and c, it must be true that:

$$
\begin{aligned}
EU_1(b) &= EU_1(c) \\
2 + 3q &= 4 - q \\
\implies q &= \frac{1}{2}
\end{aligned}
$$

- The expected payoffs for player 2 are:

$$EU_2(x) = p * 1 + (1 - p) * 7 = 7 - 6p$$
$$EU_2(y) = p * 3 + (1 - p) * 6 = 6 - 3p$$

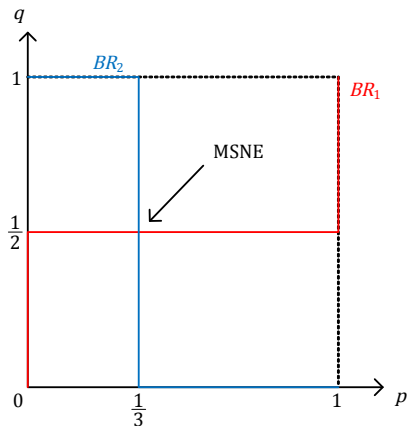- For player 2 to be indifferent between strategies x and y, it must be true that:

$$EU_2(x) = EU_2(y)$$
$$7 - 6p = 6 - 3p$$
$$\implies p = \frac{1}{3}$$

Harrington, Ch. 7 # 6

- Graphically,

Harrington, Ch. 7 # 6

- For player 1, when $q > \frac{1}{2}$, it is better to play b as a pure strategy (i.e., $p = 1$). Otherwise, if $q < \frac{1}{2}$, it is better for player 1 to play the pure strategy c (i.e., $p = 0$).
  - Intuitively, if player 2 is very likely to play x (which occurs when $q = 1$ as indicated in the first column of the matrix), player 1 is better off by responding with b (with a payoff of 5) rather than c (with a payoff of 3).

- Likewise for player 2, when $p > \frac{1}{3}$, it is better to play y as a pure strategy (i.e., $q = 0$). Otherwise, if $p < \frac{1}{3}$, it is better for player 2 to play the pure strategy x (i.e., $q = 1$).
  - Intuitively, if player 1 is very likely to play b (which occurs when $p = 1$ as indicated in the first row of the matrix), player 2 is better off by responding with y (with a payoff of 3) rather than x (with a payoff of 1).

- Finally, the Mixed-Strategy Nash Equilibrium is given by:

$$MSNE = \left\{ \left(0a, \frac{1}{3}b, \frac{2}{3}c, 0d \right) \left(\frac{1}{2}x, \frac{1}{2}y, 0z \right) \right\}$$

- Each of three players is deciding between the pure strategies *go* and *stop*. The payoff to *go* is $\frac{120}{m}$, where m is the number of players that choose *go*, and the payoff to *stop* is 55 (which is received regardless of what the other players do).

- Find all Nash Equilibria in mixed strategies.

Harrington, Ch. 7 # 10

- There are at least *seven* Nash Equilibria in mixed strategies (recall the difference between degenerated and non-degenerated mixed strategies).

- First, there are three asymmetric pure-strategy Nash Equilibria in which two players choose *go* and the other one chooses *stop*. Each player who chooses *go* earns a payoff of $\frac{120}{2} = 60$, which exceeds the payoff of 55 from choosing *stop*. The player who chooses *stop* earns 55, which exceeds the payoff from choosing *go*, which is $\frac{120}{3} = 40$.

Harrington, Ch. 7 # 10

- Now consider a strategy profile in which one player chooses the pure strategy *go* and the other two players symmetrically randomize, choosing *go* with probability p. The mixed strategy equilibrium condition for both mixing players is:

$$EU(go) = EU(stop)$$

$$\underbrace{(1-p)}_{\substack{\text{I go and the} \\ \text{other player stops}}} * 60 + \underbrace{p}_{\substack{\text{I go and the} \\ \text{other player goes}}} * 40 = 55$$

$$\implies p = \frac{1}{4}$$

- The solution for p is the mixed strategy for the other player that makes this player indifferent between *stop* and *go*. It is a Nash Equilibrium for one player to use the pure strategy *go* and each of the other two players to choose *go* with probability $\frac{1}{4}$; this gives us another three Nash Equilibria.

Harrington, Ch. 7 # 10

- Finally, there is a symmetric mixed-strategy Nash Equilibrium in which each player chooses *go* with probability q. The mixed strategy equilibrium condition is defined by:

$$EU(go) = EU(stop)$$

$$\underbrace{(1-q)^2}_{\substack{\text{I go and the} \\ \text{other players stop}}} *120 + \underbrace{2q(1-q)}_{\substack{\text{I go, one player} \\ \text{stops and the} \\ \text{other player goes}}} *60 + \underbrace{q^2}_{\substack{\text{I go and the} \\ \text{other players go}}} *40 = 55$$

$$40q^2 - 120q + 65 = 0$$

- Using the quadratic formula, one finds that q is approximately 0.71. There is then a symmetric Nash Equilibrium in which each player chooses *go* with probability 0.71.

  - It is possible that there are asymmetric Nash Equilibria in which two or more players randomize but with different probabilities.