Reactions to Arrow's impossibility theorem

- Introducing assumptions on individual preferences (section 21.D in MWG):
  - i.e., assuming single-peaked preferences for every individual).
- Using a different approach (section 6.3 in JR):
  - Aggregating the intensity of individual preferences (not only the ranking of alternatives for each individual) into a social welfare function that captures the intensity in social preferences.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - Since preferences are defined as single-peaked with respect to a linear order in X, we first have to define what a linear order means (standard definition in math):
  - A binary relation $\geq$ on the set of alternatives X is a **linear order** on X if it is:
    - reflexive, i.e., $x \geq x$ for every $x \in X$,
    - transitive, i.e., $x \geq y$ and $y \geq z$ implies $y \geq z$, and
    - total, i.e., for any two distinct $x, y \in X$, we have that either $x \geq y$ or $y \geq x$, but not both.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - If the set of alternatives is a subset of the real line, i.e., $X \subset \mathbb{R}$,
  - then the linear order $\geq$ is the natural "greater than or equal to" order of the real numbers.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - The rational preference relation $\succsim$ is single peaked with respect to the linear order $\geq$ on X if there is an alternative $x \in X$ with the property that $\succsim$ is increasing with respect to $\geq$ on the set of alternatives below x, $\{y \in X : x \geq y\}$, and decreasing with respect to $\geq$ on the set of alternatives above x, $\{y \in X : y \geq x\}$. That is,

$$\begin{aligned} \text{If } x &\geq z > y \text{ then } z \succ y, \text{ and} \\ \text{If } y &> z \geq x \text{ then } y \succ z, \end{aligned}$$

  - *In words:* There is an alternative x that represents a peak of satisfaction.
  - Moreover, satisfaction increases as we approach this peak (so there cannot be any other peak of satisfaction).

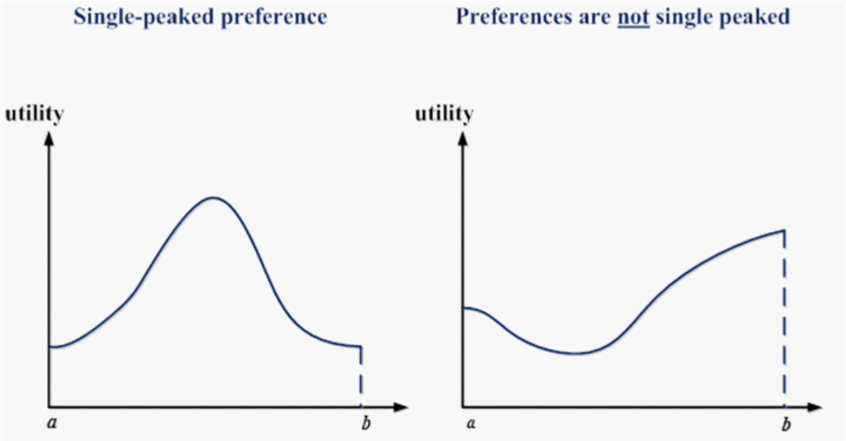
Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - *Example 21.D.4:*
  - Suppose a set of alternatives $X = [a, b] \subset \mathbb{R}$.
  - Then, a preference relation $\succsim$ on X is single peaked if and only if it is *strictly convex*:
    - That is, if and only if, for every alternative $w \in X$, we have that, for any two alternatives y and z weakly preferred to w, i.e., $y \succsim w$ and $z \succsim w$ where $y \neq z$, their linear combination is strictly preferred to w,

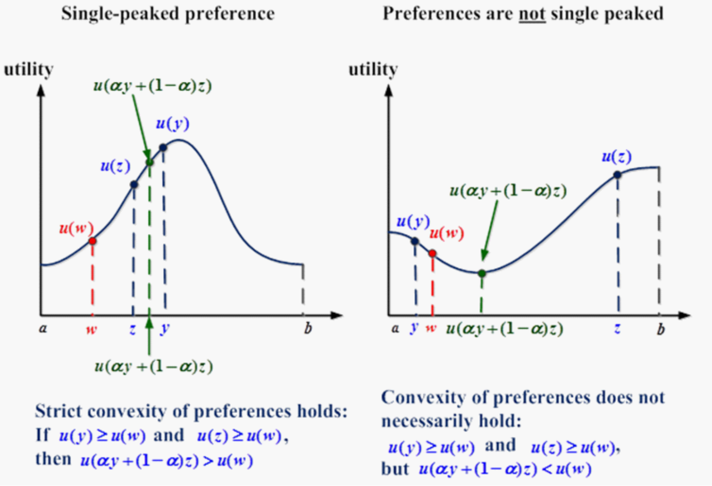
$$\alpha y + (1 - \alpha) z \succ w \quad \text{for all } \alpha \in (0, 1)$$

- Figures of utility functions satisfying/violating the single-peaked property.

Reactions to Arrow's impossibility theorem



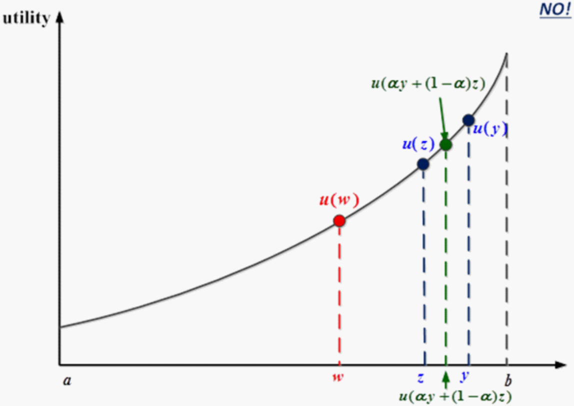
Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem

Is the single-peaked property equivalently to strict concavity on the utility function?

NO!



Here we have a utility function which is convex, yet the single-peaked property holds:
if $u(y) \geq u(w)$ and $u(z) \geq u(w)$, then $u(\alpha y + (1-\alpha)z) < u(w)$

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - We will now restrict our attention to settings in which all individuals have single-peaked preferences with respect to the same linear order $\geq$.
  - Consider pairwise majority voting.
    - Formally, for any pair $\{x, y\} \subset X$, we say $x\widehat{F}\left(\succsim^1, \succsim^2, ..., \succsim^I\right) y$ to be as "x is socially at least as good as y", if the number of agents that strictly prefer x to y is larger or equal to the number of agents that strictly prefer y to x, that is,

$$\text{if } \#\left\{i \in I : x \succ^i y\right\} \geq \#\left\{i \in I : y \succ^i x\right\}$$

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - We will next show that, with single-peaked preferences, the social preferences arising from pairwise majority voting have maximal elements.
    - That is, there are alternatives that cannot be defeated by any other alternatives, i.e, the Condorcet paradox does not hold.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - Before doing that, we need a few definitions:
  - Let x_i denote the maximal alternative for individual i according to his preference $\succsim^i$, i.e., his "peak".
  - Let us next define what we mean by a median agent:
    - Agent $h \in I$ is a median agent for the profile $(\succsim^1, \succsim^2, ..., \succsim^I)$ of single-peaked preferences with respect to the linear order $\geq$ if

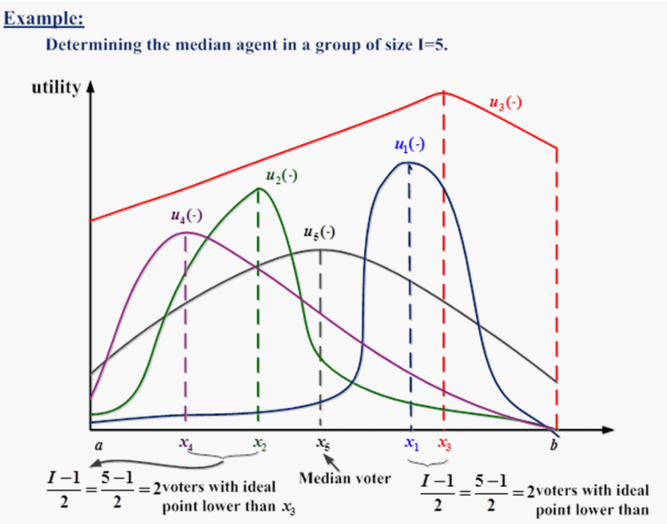
$$\#\{i \in I : x_i \geq x_h\} \geq \frac{I}{2} \text{ and } \#\{i \in I : x_h \geq x_i\} \geq \frac{I}{2}$$

    - That is, the number of individuals whose ideal point is larger than h's ideal point is larger than half of the population.
    - Similarly, the number of individuals whose ideal point is smaller than h's ideal point is larger than half of the population.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
  - A natural conclusion of the definition of a median agent is that:
    - If there are no ties in peaks and the number of individuals is odd,
    - then there are exactly $\frac{I-1}{2}$ individuals with ideal points strictly smaller than x_h, and $\frac{I-1}{2}$ individuals with ideal points strictly larger than x_h.
    - That is, the median agent is unique.

Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
- We are now ready to claim the existence of a Condorcet winner in this setting, and to prove it.
  - Suppose that $\geq$ is a linear order on X and consider a profile of individual preferences $\left(\succsim^1, \succsim^2, ..., \succsim^I\right)$ where, for every individual i, $\succsim^i$ is single peadked with respect to $\geq$.
  - Let $h \in I$ be a median agent with ideal point x_h.
  - Then, $x_h \widehat{F}\left(\succsim^1, \succsim^2, ..., \succsim^I\right) y$ for every alternative $y \in X$.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
- Interpretation of "$x_h \widehat{F}\left(\succsim^1, \succsim^2, ..., \succsim^I\right) y$ for every alternative $y \in X$":
  - The ideal point of the median agent cannot be defeated by majority voting by any other alternative y.
  - When an alternative cannot be defeated by majority voting by any other alternative, we refer to it as a "Condorcet winner".
  - Hence, a Condorcet winner exists when the preferences of all agents are single peaked with respect to the same linear order.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
- Proof of the Condorcet winner result:
  - Take any alternative $y \in X$ and suppose that the ideal point of the median agent, x_h, satisfies $x_h > y$ (the argument is analogous if $y > x_h$).
  - NTS that alternative y cannot defeat x_h, that is,

$$\#\left\{i \in I : x_h \succ^i y\right\} \geq \#\left\{i \in I : y \succ^i x_h\right\}$$

  - Consider now the set of individuals $S \subset I$ with ideal points to the right-hand side of x_h, that is

$$\{i \in I : x_i \geq x_h\}.$$

  - Then, $x_i \geq x_h > y$ for every individual $i \in S$.

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
- Proof (cont'd):
  - Hence, by single-peaked preferences, $x_h \succsim^i y$ for every individual $i \in S$.
    - That is, all individuals in S (i.e., all with ideal points to the right-hand side of x_h) will vote for the ideal point of the median agent, x_h.
  - Finally, because agent h is a median agent, the number of individuals with ideal points to the right-hand side of x_h, i.e., $\#S$, satisfies $\#S \geq \frac{I}{2}$.
  - Therefore,

$$\#\left\{i \in I : x_h \succ^i y\right\} \geq \#S \geq \frac{I}{2} \geq \#\left(I \backslash S\right) \geq \#\left\{i \in I : y \succ^i x_h\right\}$$

Reactions to Arrow's impossibility theorem

- **Single-peaked preferences:**
- The existence of a Condorcet winner guarantees that we don't run into cyclicality
  - That is, the order in which pairs of alternatives are confronted in pairwise majority voting does not affect the final outcome.
  - However, the previous assumptions don't guarantee transitivity.
  - Let's see one example in which a Condorcet winner exists, yet transitivity in the social preference relation is violated.

Reactions to Arrow's impossibility theorem

- **Example of instransitive social preferences:**
- Consider a set of alternatives $X = \{x, y, z\}$ and $I = 4$ individuals.
- Consider the following profile of individual preferences

$$\begin{aligned} x &\succ {}^1 y \succ^1 z \quad \text{for individual 1,} \\ z &\succ {}^2 y \succ^2 x \quad \text{for individual 2,} \\ x &\succ {}^3 z \succ^3 y \quad \text{for individual 3, and} \\ y &\succ {}^4 x \succ^4 z \quad \text{for individual 4} \end{aligned}$$

Reactions to Arrow's impossibility theorem

- **Example of instransitive social preferences:**
- We thus have that

$$\begin{aligned}\#\left\{i \in I : x \succ^{i} y\right\} &= \#\left\{i \in I : y \succ^{i} x\right\} = \\ \#\left\{i \in I : z \succ^{i} y\right\} &= \#\left\{i \in I : z \succ^{i} y\right\} = 2\end{aligned}$$

  which implies that x is socially indifferent to y and, similarly, y is socially indifferent to z.
- We can then write $z\widehat{F}\left(\succsim^{1}, \succsim^{2}, \succsim^{3}, \succsim^{4}\right) y$ and $y\widehat{F}\left(\succsim^{1}, \succsim^{2}, \succsim^{3}, \succsim^{4}\right) x$.

Reactions to Arrow's impossibility theorem

- **Example of instransitive social preferences:**
- For transitivity, we would need that $z\widehat{F}\left(\succsim^1, \succsim^2, \succsim^3, \succsim^4\right)x$.
- Can we obtain this result? No:
  - Since $\#\left\{i \in I : x \succ^i z\right\} = 3$ and $\#\left\{i \in I : z \succ^i x\right\} = 1$ implies

$$x\widehat{F}\left(\succsim^1, \succsim^2, \succsim^3, \succsim^4\right)z.$$

    which is the opposite of what we NTS.
- Hence, in this case majority voting fails to generate a transitive social welfare functional.

Reactions to Arrow's impossibility theorem

- **Example of instransitive social preferences:**
- Social preferences are nonetheless acyclic since:
  - A pairwise majority voting between x and y yields a tie (2 vote for each);
  - A pairwise majority voting between y and z yields a tie (2 vote for each);
  - A pairwise majority voting between z and x yields alternative z being the winner (the Condorcet winner) with three votes against one.
- Hence, alternative z is the Condorcet winner:
- Trying to confront z against another alternative, such as x or y, yields either of these alternatives being defeated by z (or a tie, but they never defeat z).

Reactions to Arrow's impossibility theorem

- **How can we guarantee transtivity in the swf?**
- We need to impose two additional conditions:
  - Preference relation of every individual i must be strict (no indifference between alternatives is allowed); and
  - The number of individuals I is odd.

Reactions to Arrow's impossibility theorem

- **Proof of transitivity in the swf:**
  - Consider a set $X = \{x, y, z\}$, where $x\widehat{F}\left(\succsim^1, \succsim^2, ..., \succsim^I\right) y$ and $y\widehat{F}\left(\succsim^1, \succsim^2, ..., \succsim^I\right) z$.
  - Then, x defeats y, and y defeats x.
  - Since individual preferences are strict and I is odd, there must be one alternative in X that is not defeated by any other alternative in X.
  - Such alternative can be neither y (which is defeated by x) nor z (which is defeated by y).
  - Hence, such alternative has to be x, and we can thus cnclude that $x\widehat{F}\left(\succsim^1, \succsim^2, ..., \succsim^I\right) z$, as required to prove transitivity.

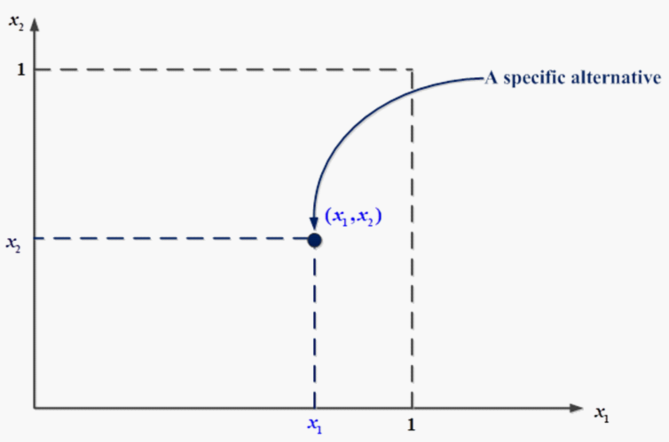
Reactions to Arrow's impossibility theorem

- Hence, imposing the assumption of single-peaked preferences helped us obtain an acyclic social ranking, thus avoiding the Condorcet paradox.
- But, our discussion considered that $X \subset \mathbb{R}$, i.e., the set of alternatives was unidimensional.
  - What if, for instance, we are considering a policy issue in which individual preferences rank alternatives according to two dimensions?
  - Bad news: cyclicality emerges again, even if we assume convex preferences.

Reactions to Arrow's impossibility theorem

- Consider that the space of alternatives is bidimensional and, in particular, given by the unit square, i.e., $X = [0, 1]^2$.
- A specific alternative is, hence, represented now by a pair $x = (x_1, x_2)$.
- See next figure.

Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem

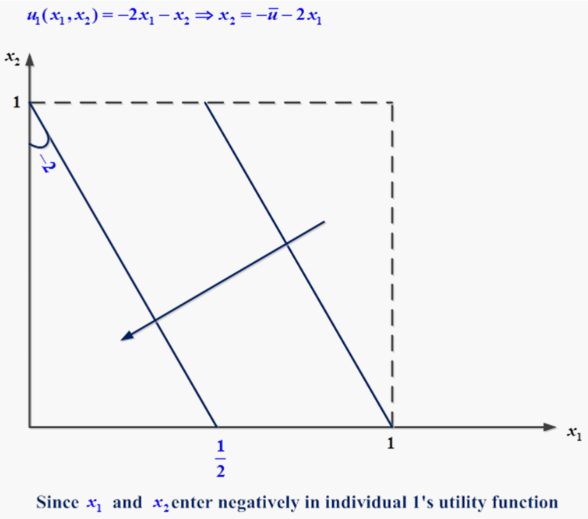
- Consider three individuals with the following utility functions:

$$
\begin{aligned}
u_1(x_1, x_2) &= -2x_1 - x_2, \\
u_2(x_1, x_2) &= x_1 + 2x_2, \text{ and} \\
u_3(x_1, x_2) &= x_1 - x_2.
\end{aligned}
$$

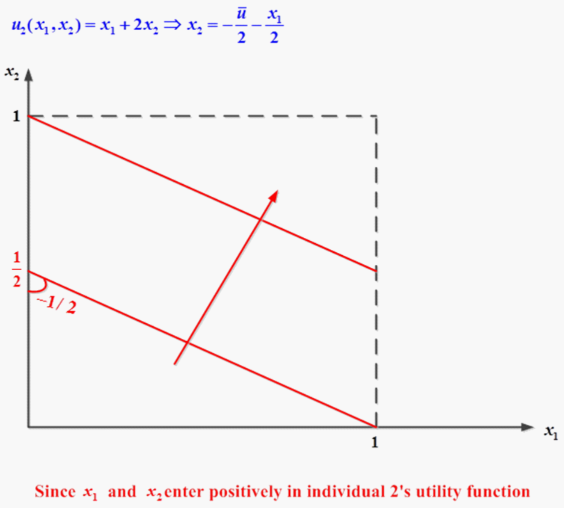
- We can represent their indifference curves in the unit square, by solving for x_2, for a given utility level $\overline{u}$,

$$
\begin{aligned}
x_2 &= -\overline{u} - 2x_1, \\
x_2 &= \frac{\overline{u} - x_1}{2}, \text{ and} \\
x_2 &= x_1 - \overline{u}.
\end{aligned}
$$

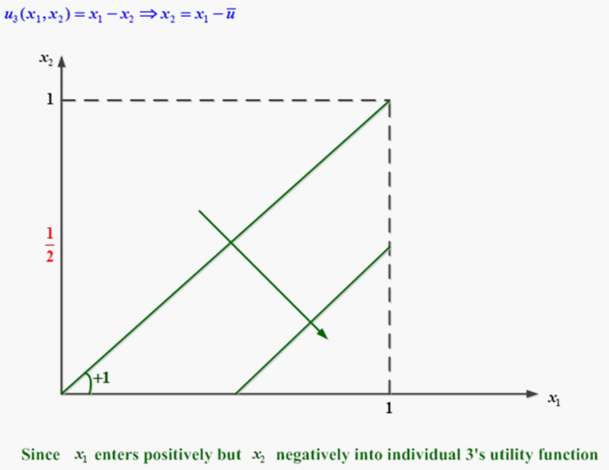
Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem

- Preferences are all convex.
- Yet, no pair $x = (x_1, x_2)$ can be Condorcet winner.
- To show that, we need to show that, for every pair $x = (x_1, x_2)$, we can find another pair $y = (y_1, y_2)$ which is preferred by at least two of the three individuals.

Reactions to Arrow's impossibility theorem

- Consider the following three cases:
  - **Case 1:** If $x = (0, x_2)$, then a pair $y = \left(\frac{1}{2}, x_2\right)$ is preferred by agents 2 and 3 to x.

$$
\begin{array}{ccc}
u_1(x) = -x_1 & > & u_1(y) = -1 - x_1 \\
u_2(x) = 2x_2 & < & u_2(y) = \frac{1}{2} + x_2 \\
u_3(x) = -x_2 & < & u_3(y) = \frac{1}{2} - x_2
\end{array}
$$

Reactions to Arrow's impossibility theorem

- Consider the following three cases:
  - **Case 2:** If $x = (x_1, 1)$, then a pair $y = \left(x_1, \frac{1}{2}\right)$ is preferred by agents 1 and 3 to x.

$$\begin{array}{ccc} u_1(x) = -2x_1 - 1 & < & u_1(y) = -2x_1 - \frac{1}{2} \\ u_2(x) = x_1 + 2 & > & u_2(y) = x_1 + 1 \\ u_3(x) = x_1 - 1 & < & u_3(y) = x_1 - \frac{1}{2} \end{array}$$

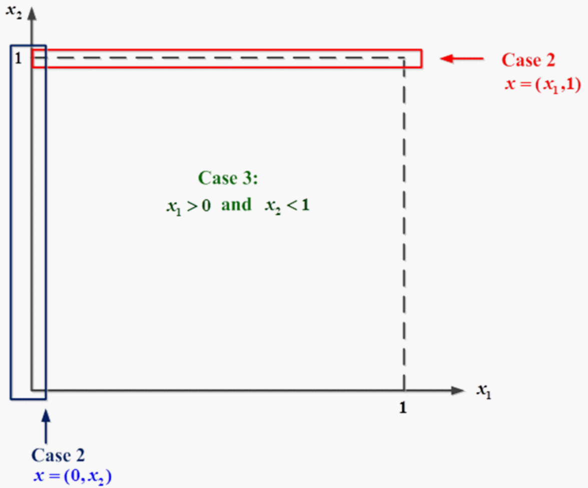
Reactions to Arrow's impossibility theorem

- Consider the following three cases:
  - **Case 3:** If $x_1 > 0$ and $x_2 < 1$, then a pair $y = (x_1 - \varepsilon, x_2 + \varepsilon)$, where $\varepsilon > 0$, is preferred by agents 1 and 2 to x.

$$
\begin{array}{lcc}
u_1(x) = -2x_1 - x_2 & < & u_1(y) = -2(x_1 - \varepsilon) - (x_2 + \varepsilon) = \\
 & & = -2x_1 - x_2 + \varepsilon \\
u_2(x) = x_1 + 2x_2 & < & u_2(y) = (x_1 - \varepsilon) + 2(x_2 + \varepsilon) = \\
 & & = x_1 + 2x_2 + \varepsilon \\
u_3(x) = x_1 - x_2 & > & u_3(y) = (x_1 - \varepsilon) - (x_2 + \varepsilon) = \\
 & & = x_1 - x_2 - 2\varepsilon
\end{array}
$$

Reactions to Arrow's impossibility theorem

We have thus spanned the unit square:



Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Let's not dispair:
  - We have just found one counterexample in which the bidimensionality of the alternatives in X yields cyclicality.
  - But, we can find other settings in which, despite alternatives being multidimensional, cyclicality doesn't arise.
  - More generally, under which conditions can we guarantee that cyclicality does not emerge?

Reactions to Arrow's impossibility theorem

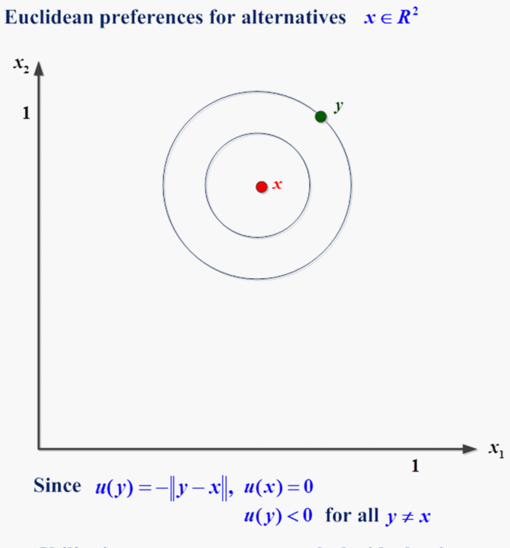
- **Multidimensional alternatives:**
- Consider alternatives with n-dimensions, $x \in \mathbb{R}^n$
- Individual preferences are represented by utility function

$$u(y) = -\|y - x\|$$

where x denotes the ideal point of this individual. Hence, the utility of vector y is given by the Euclidean distance from his ideal point x.

  - You can think about x as the "peak" of the utility mountain of this individual, where the level sets of the mountain are circles.
  - Figure for the case in which $X = \mathbb{R}^2$.

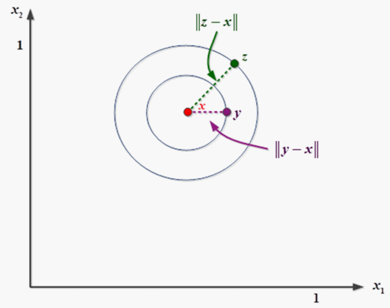
Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Hence, given two alternatives y and z, an individual with ideal point x will prefer the one closer to x (where "closer to" is defined by the Euclidean distance).
  - Figure for the case in which $X = \mathbb{R}^2$.

Reactions to Arrow's impossibility theorem



This individual prefers alternative y to z, since

$$u(y) = -\|y - x\| > u(z) = -\|z - x\|$$

$$-\left[(y_1 - x_1)^2 + (y_2 - x_2)^2\right]^{1/2} > -\left[(z_1 - x_1)^2 + (z_2 - x_2)^2\right]^{1/2}$$

<u>Example</u>: x=(1,1), y=(2,0.8), z=(3,3). Then

$$-\left[(2-1)^2 + (0.8-1)^2\right]^{1/2} > -\left[(3-1)^2 + (3-1)^2\right]^{1/2} \Leftrightarrow -\left[1 + 0.04\right]^{1/2} > -\sqrt{8}$$

$$1.01 < 2.82$$

Reactions to Arrow's impossibility theorem

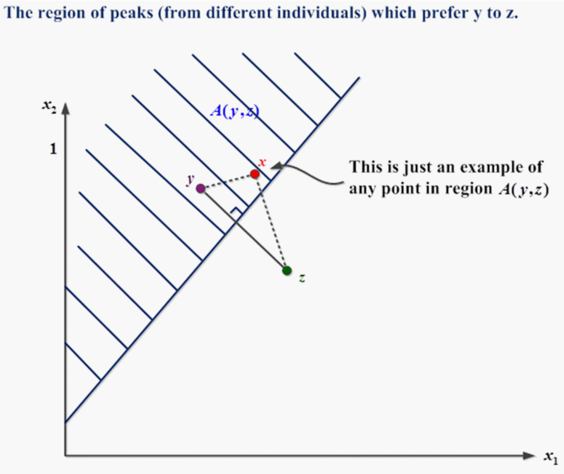
- **Multidimensional alternatives:**
- Thus, the region peaks of different individuals that prefer y to z is

$$A(y, z) = \{x \in \mathbb{R}^n : \|x - y\| < \|x - z\|\}$$

That is, all those individuals whose ideal points, x, are closer to y than to z.

  - In the next figure, the boundary of $A(y, z)$ is given by a line (generally, it could be a hyperplane if $n > 2$)
  - This line is perpendicular to the segment connecting y and z, and passing through its midpoint.

Reactions to Arrow's impossibility theorem

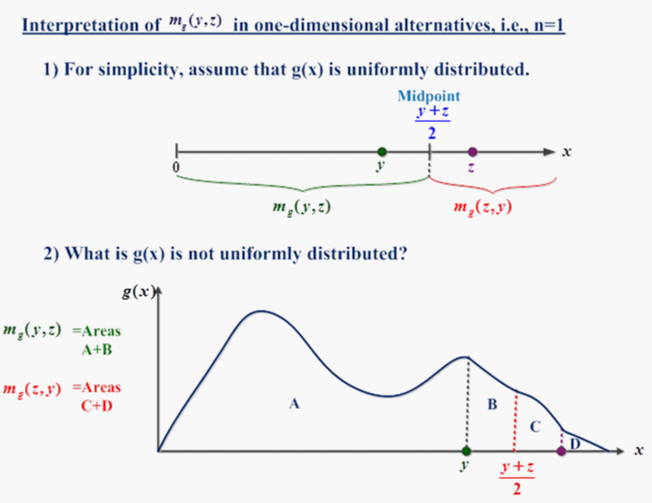


Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Consider now a continuum of individuals, each of them with the above preferences.
- The ideal points, $x \in \mathbb{R}^n$, are distributed with density function $g(x)$.
- Then, for any two alternatives y and z, the fraction of the population that prefers y to z is

$$\int_{A(y,z)} g(z)dz \equiv m_g(y,z)$$

Reactions to Arrow's impossibility theorem



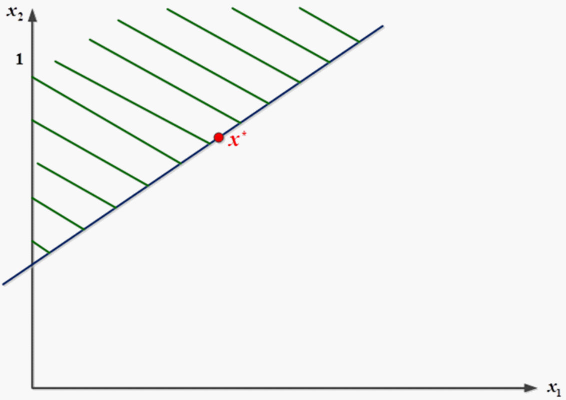
Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Let us now show under which conditions there can be a Condorcet winner, i.e., an alternative x^* that cannot be defeated by any other alternative y.
  - 1st line of implication:
    - If alternative x^* is a median, then x^* is a Condorcet winner.
  - 2nd line of implication:
    - If alternative x^* is a Condorcet winner, then x^* is a median.

Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Let us now show under which conditions there can be a Condorcet winner.
    - Suppose there is an alternative $x^* \in \mathbb{R}^n$ such that any halfspace through x^* divides $\mathbb{R}^n$ into two half-spaces, each having a total mass of $\frac{1}{2}$ according to the density $g(\cdot)$.
    - See figure

Reactions to Arrow's impossibility theorem

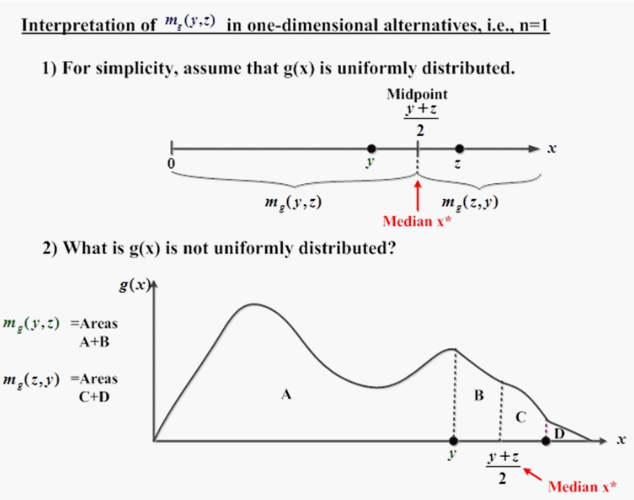


Half of the population is to the left of x*(shaded area), and half is to the right (unshaded area).

Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Point x^* then leaves exactly half of the population to the left (in the Euclidean sense) and the other half to the right:
  - As a consequence, point x^* is referred to as "median."
  - It coincides with the usual notion of median in the case of $n = 1$ (see next figure).

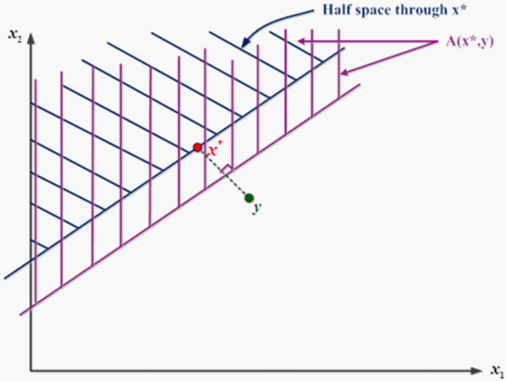
Reactions to Arrow's impossibility theorem



Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- A median x^* in the above sense is a Condorcet winner:
  - Point x^* cannot be defeated by any other alternative $y \neq x^*$.
  - In particular, $A(x^*, y)$ becomes larger than the half-space through x^*. (See next figure).
  - Therefore, $m_g(x^*, y) \geq \frac{1}{2}$, thus being defeated by point x^*.

Reactions to Arrow's impossibility theorem



In order to construct A(x*,y), which represents the set of individuals ' peaks that prefer x* to y, we

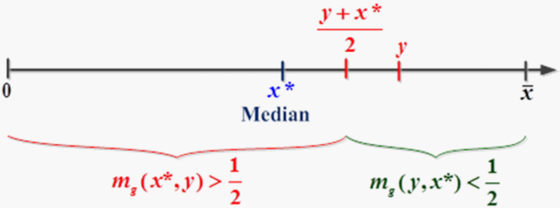
- first need to plot a line between x* and y,
- then we plot a hyper-plane (a line in this case) perpendicular to the segment connecting x* and y, and passing through its midpoint
- the area to the left of this hyper-plane represent A(x*,y)

Reactions to Arrow's impossibility theorem

Application to n=1

If x* is a median, then x* is a Condorcet winner

Any other alternative $y \neq x^*$ would be different by x*



Reactions to Arrow's impossibility theorem

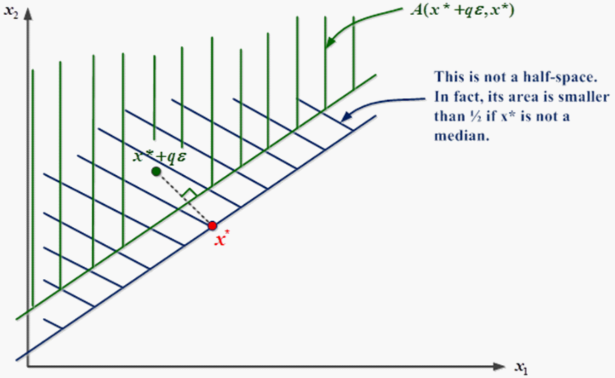
- **Multidimensional alternatives:**
- A median x^* in the above sense is a Condorcet winner:
  - Conversely, if x^* is not a median, then it cannot be a Condorcet winner. x^* is a median$\Leftarrow x^*$ is a Condorcet winner.
  - Specifically, we can move x^* in any direction q such that we give rise to a half-space larger than $\frac{1}{2}$.
  - More formally, there exists a direction $q \in \mathbb{R}^n$ such that the mass of the half-space

$$\{z \in \mathbb{R}^n : q \cdot z > q \cdot x^*\} \quad \text{is larger than } \frac{1}{2}$$

  - In other words, point $x^* + q\varepsilon$ defeats point x^*; see next figure.
    - That is, if x^* is not a median, it cannot be a Condorcet winner.

Reactions to Arrow's impossibility theorem

What if x* is not a median?

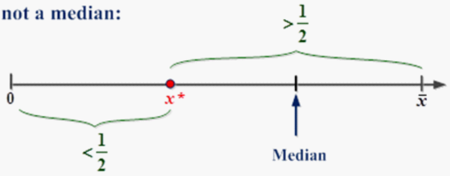


Hence, $m_g(x^* + q\varepsilon, x^*) = \int_{A(x^*+q\varepsilon,x^*)} q(z)dz > \frac{1}{2}$. That is, the region $A(x^* + q\varepsilon, x^*)$ must contain more than half of the population if x* was not a median.

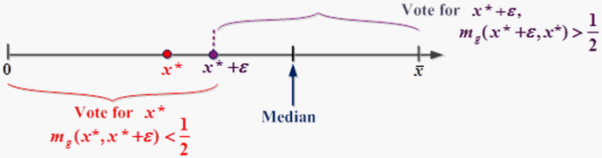
Reactions to Arrow's impossibility theorem

This result is easily illustrated in the more familiar setting of n=1 and g(x) being uniformly distributed:

- **If x* is not a median:**



- **There are alternatives, such as $x^*+\varepsilon$, which would defeat x*:**



Hence x* cannot be a Condorcet winner.

- **However, if x*=median, then we can not find alternatives to x* that would defeat x*.**

Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Notice what we just proved:
  - Consider a density $g(\cdot)$ describing the probability distribution of ideal points for each individual in the population.
  - If this density $g(\cdot)$ provides us with a median x^* that divides the Euclidean space into two regions of equal area...
  - then we can claim that such median is a Condorcet winner.
- That's ok, but the most demanding requirement is the second.
  - We can prove how restrictive this result is, even if we assume a uniform distribution.
  - Let's consider two cases:
    - One that generates a median, and one that doesn't.

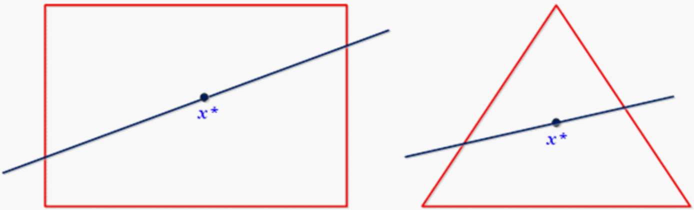
Reactions to Arrow's impossibility theorem

Uniform distribution over a rectangle:

Point x* is the median, since every plane through x* divides the rectangle into two equal areas.

Uniform distribution over a triangle:

There is no median (a point for which every plane through the point divides the triangle into two equal areas).



Reactions to Arrow's impossibility theorem

- **Multidimensional alternatives:**
- Caplin and Nalebuff (1988) tackled this problematic result and brought us the now famous "64% majority rule":
  - They showed that, for a uniform distribution (and, more generally, for any density function satisfying logarithmic concavity) there are always points (which they referred to as "generalized medians")...
  - with the property that a hyperplane through the point divides $\mathbb{R}^n$ into two regions, each of them with a mass larger than $\frac{1}{e} \simeq 0.36$.
- What does that mean?
  - These points cannot be defeated by any other alternative if the majority required is not $\frac{1}{2}$ of the votes, but any number larger than $1 - \frac{1}{e} \simeq 0.64$.

Reactions to Arrow's impossibility theorem - II

- **Second reaction:**
- Allowing for intensity of individual preferences to enter into social preferences.
  - We will do that by using a social welfare function

$$W\left(u^1(\cdot), u^2(\cdot), ..., u^I(\cdot)\right)$$

  - We first need to impose two assumptions on $W(\cdot)$:
    - Utility-level invariant, and
    - Utility-difference invariant.

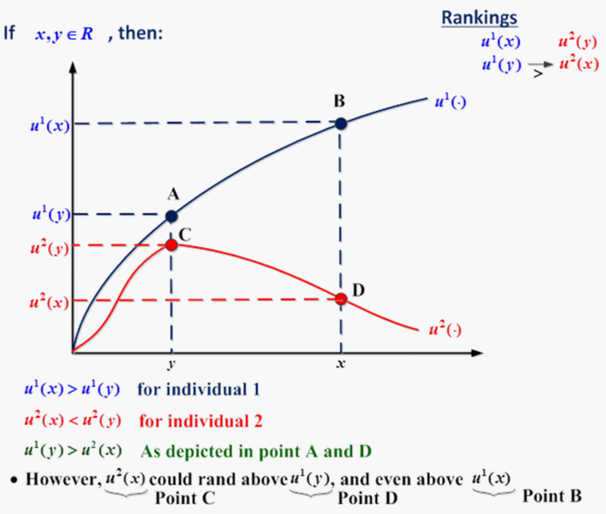
Reactions to Arrow's impossibility theorem - II

- **Utility-level invariance:**
  - *Motivation:* Consider that $u^1(x) > u^1(y)$ for individual 1, and $u^2(x) < u^2(y)$ for individual 2.
  - In addition, assume that $u^1(y) > u^2(x)$, i.e., individual 1 is better off at his least-preferred state than individual 2 is.
  - Then,

$$u^1(x) > u^1(y) > u^2(x)$$

    where $u^2(y)$ must be larger than $u^2(x)$, but could rank above/below $u^1(x)$ or $u^1(y)$. (Figure.)

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

- **Utility-level invariance:**
  - Assume that, in this context, society seeks to make its least well off individual as well off as possible. That is,

$$\max_{x,y} \left\{ \min\left\{u^1(x), u^1(y)\right\}, \min\left\{u^2(x), u^2(y)\right\}\right\} = \max_{x,y} \left\{u^1(y), u^2(x)\right\}$$

and since $u^1(y) > u^2(x)$, alternative y is socially preferred to x.

Reactions to Arrow's impossibility theorem - II

- **Utility-level invariance:**
  - Now, consider strictly increasing transformations $\psi^1(\cdot)$ and $\psi^2(\cdot)$ producing the same individual rankings

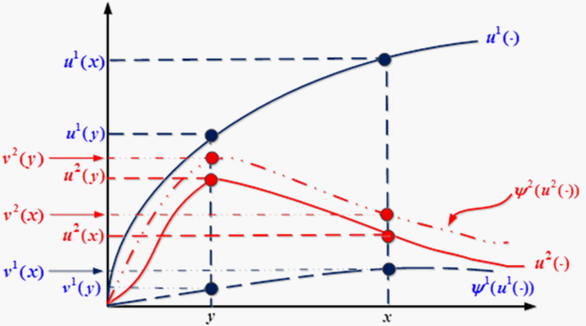
$$
\begin{aligned}
v^1(x) &\equiv \psi^1(u^1(x)) > \psi^1(u^1(y)) \equiv v^1(y), \text{ and} \\
v^2(x) &\equiv \psi^2(u^2(x)) < \psi^2(u^2(y)) \equiv v^2(y)
\end{aligned}
$$

    but altering the ranking across individuals, i.e., $v^1(y) < v^2(x)$.
  - In this setting, society would identify alternative x as socially preferred to y.
  - But this new social ranking is troublesome: We have not changed the individual rankings over alternatives, yet the social ranking changed. (Figure.)
  - In order to avoid this possibility, we only need to avoid different monotonic transformations for individual 1 and 2. That's what utility-level invariance guarantees (i.e., $\psi^1 = \psi^2$).

Reactions to Arrow's impossibility theorem - II

Continuing with the above example:

ψ^1 shifts $u^1(\cdot)$ downwards while ψ^2 shifts $u^2(\cdot)$ upwards.



While the individual ranking is unaffected, i.e., $v^1(x) > v^1(y)$ and $v^2(x) < v^2(y)$, the ranking between $v^1(y)$ and $v^2(x)$ is affected.

Reactions to Arrow's impossibility theorem - II

- **Utility-level invariance:**
  - *Definition:* A social welfare function $W(\cdot)$ is **utility-level invariant** if it is invariant to arbitrary, but common, strictly increasing transformations ψ applied to every individual's utility function.
  - That is, for every profile of individual preferences $\mathbf{u} \equiv \left(u^1(\cdot), u^2(\cdot), ..., u^I(\cdot)\right)$, where $\mathbf{u}(x) \equiv \left(u^1(x), u^2(x), ..., u^I(x)\right)$ and $\mathbf{u}(y) \equiv \left(u^1(y), u^2(y), ..., u^I(y)\right)$ denote the profile of individual utility levels from any two alternatives $x \neq y$,

$$\text{if } W(\mathbf{u}(x)) > W(\mathbf{u}(y)) \text{ then } W(\psi(\mathbf{u}(x))) > W(\psi(\mathbf{u}(y)))$$

    under a common strictly increasing transformation $\psi(\cdot)$, where $\psi(\mathbf{u}(x)) \equiv \left(\psi\left(u^1(x)\right), \psi\left(u^2(x)\right), ..., \psi\left(u^I(x)\right)\right)$ and similarly for $\psi(\mathbf{u}(y))$.

Reactions to Arrow's impossibility theorem - II

- **Utility-difference invariance:**
  - Let us now move to a second type of information often used in making social choices:
    - The utility that each individual gains/losses when he moves from an alternative y to another alternative x.
    - That is, $u^1(x) - u^1(y)$ for individual 1, which in this example was considered positive, and
    - $u^2(x) - u^2(y)$ for individual 2, which in this example is negative.

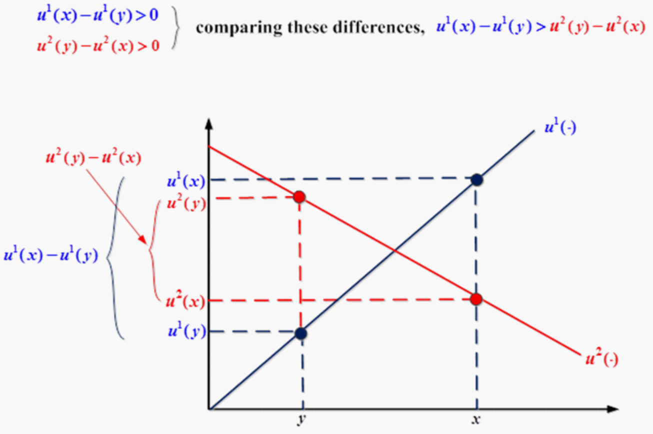
Reactions to Arrow's impossibility theorem - II

- **Utility-difference invariance:**
  - A common comparison is then whether individual 1's gain, $u^1(x) - u^1(y)$ of moving to x is larger than individual 2's loss, $u^2(y) - u^2(x)$.

$$u^1(x) - u^1(y) > u^2(y) - u^2(x)$$

  - Figure

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

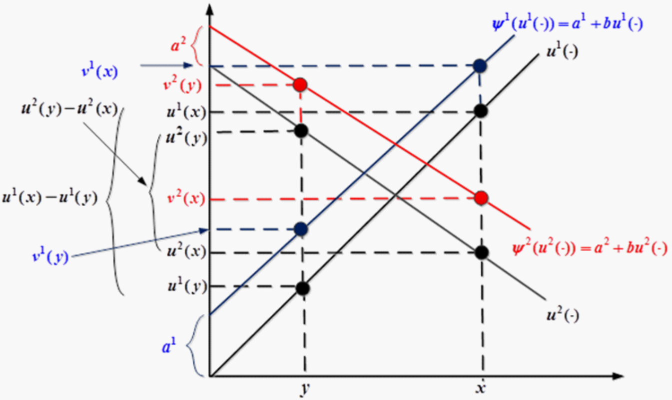
- **Utility-difference invariance:**
  - For the swf to preserve this information, we need that monotonic transformations are linear, i.e.,

$$\psi^i\left(u^i(x)\right) = a^i + bu^i(x)$$

    where $b > 0$ is common to all individuals.
  - Figure.

Reactions to Arrow's impossibility theorem - II

After applying the monotonic transformations, the difference $v^1(x) - v^1(y)$ is still larger than $v^2(y) - v^2(x)$.



Reactions to Arrow's impossibility theorem - II

- **Utility-difference invariance:**
  - *Definition:* A social welfare function $W(\cdot)$ is **utility-difference invariant** if it is invariant to strictly increasing transformations of the form
$$\psi\left(u^i(x)\right) = a^i + bu^i(x),$$
where $b > 0$ is common to all individuals.

Reactions to Arrow's impossibility theorem - II

- **Two more assumptions on the SWF:**
  - **Anonymity.** Let $\mathbf{u}(x)$ and $\widetilde{\mathbf{u}}(x)$ be two utility vectors, where $\widetilde{\mathbf{u}}(x)$ has been obtained from $\mathbf{u}(x)$ after a permutation of its elements. Then,

$$W(\mathbf{u}(x)) = W(\widetilde{\mathbf{u}}(x))$$

  - *Interpretation*:
    - The social ranking of alternatives should not depend on the identity of the individuals involved, but only on the levels of utility each alternative entail.

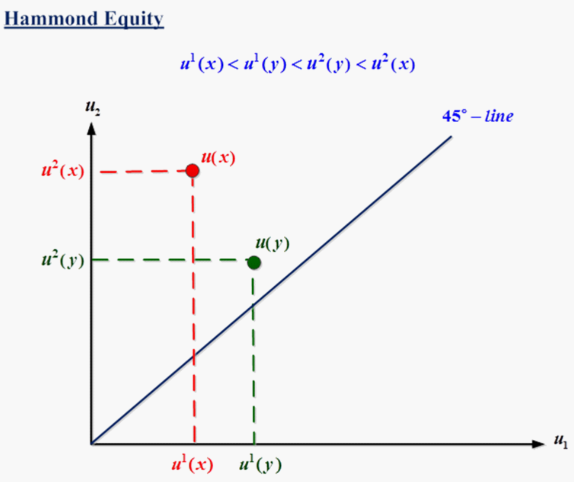
Reactions to Arrow's impossibility theorem - II

- **Two more assumptions on the SWF:**
  - **Hammond Equity.** Let $\mathbf{u}(x)$ and $\mathbf{u}(y)$ be the utility vectors of two distinct alternatives x and y, where $u^k(x) = u^k(y)$ for every individual k except for two individuals: i and j. If

$$u^i(x) < u^i(y) < u^j(y) < u^j(x)$$

    then $W(\mathbf{u}(y)) \geq W(\mathbf{u}(x))$.
  - *Interpretation*:
    - Society has a preference towards the alternative that produces the smallest variance in utilities across individuals (alternative y in this case).
    - Seems reasonable in some cases, but critizable in orders: for instance, $u^i(x) = 1 < u^i(y) = 1.1 < u^j(y) = 1.2 < u^j(x) = 100$.

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

- We can now show that some well-known SWF, such as the Rawlsian and the utilitarian, can be characterized by some of the properties we just mentioned:
  - Utility-level invariance,
  - Utility-difference invariance,
  - Anonymity (A), and
  - Hammond Equity (HE),

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
  - Welfare is given by that of the worst-off member, that is, $W(x) = \min\left\{u^1(x), ..., u^I(x)\right\}$
  - *Theorem 6.2 in JR:*
    - A strictly increasing and continuous swf W satisfies HE if and only if it can be represented with the Rawlsian form, $W(x) = \min\left\{u^1(x), ..., u^I(x)\right\}$.
  - *As a corollary:*
    - Moreover, W satisfies A and is utility-level invariant.
  - Let's prove these results.

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
  - *Proof*:
  - 1st line of implication:
    - If W is continuous, strictly increasing, and satisfies HE, then W must be Rawlsian.
  - 2nd line of implication:
    - If W is Rawlsian, then W is continuous, strictly increasing, and satisfies HE.

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
  - *Proof*: Suppose that W is continuous, strictly increasing and satisfies HE.
  - We then NTS that W takes the form

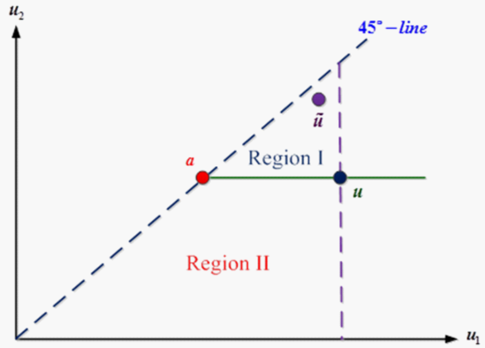
$$W(x) = \min \left\{ u^1(x), ..., u^I(x) \right\}$$

  - That is, $W(x) \geq W(y)$ if and only if

$$\min \left\{ u^1(x), ..., u^I(x) \right\} \geq \min \left\{ u^1(y), ..., u^I(y) \right\}$$

  - Consider the next figure.

Reactions to Arrow's impossibility theorem - II



Point $\tilde{u}$ satisfies $u^2 < \tilde{u}^2 < \tilde{u}^1 < u^1 \rightarrow$ Indeed, $\tilde{u}^1 < u^1$ for individual 1,
$u^2 < \tilde{u}^2$ for individual 2, and
$\tilde{u}^2 < \tilde{u}^1$ across individuals (since $\tilde{u}$ lies below $45^\circ - line$

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- *Proof:*
- Here is what we are planning to do:
  - The social indifference curve of a Rawlsian swf must be a right angle (and all kinks are crossed by a ray from the origin).
  - We must then show that, starting from any arbitrary point **a** on the 45-degree line:
    - All points in a horizontal ray starting from the 45-degree line, and
    - all points in a vertical ray starting from the 45-degree line,
    - must yield the same social welfare as in point **a**.

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
  - Consider the next figure.
  - Choose an arbitrary point **a** on the 45-degree line, and point **u** on the ray extending from *a* to the right.
  - We seek to show that $W(\mathbf{u}) = W(\mathbf{a})$.
  - Define region I and II.
  - Consider a point $\widetilde{\mathbf{u}}$ in region I. Note that

$$u^2 < \widetilde{u}^2 < \widetilde{u}_1 < u^1$$

  - Graphically, note that point $\widetilde{\mathbf{u}}$ is closer to the 45-degree line than **u** is, thus reducing utility dispersion across individuals; as depicted in the figure.

Reactions to Arrow's impossibility theorem - II

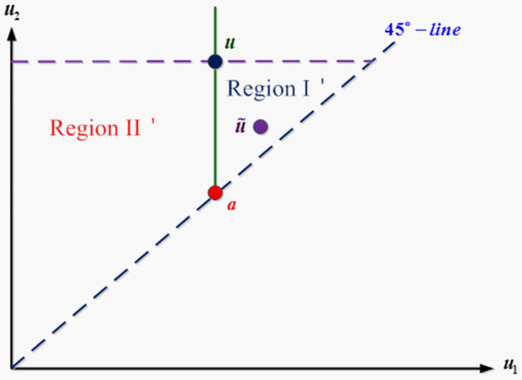
- **The Rawlsian SWF**
  - Since point $\widetilde{\mathbf{u}}$ implies a smaller utility dispersion than $\mathbf{u}$ society prefers, according to HE, point $\widetilde{\mathbf{u}}$, i.e., $W(\widetilde{\mathbf{u}}) \geq W(\mathbf{u})$.
  - This argument is true for any point $\widetilde{\mathbf{u}}$ in region I, i.e., $W(I) \geq W(\mathbf{u})$.
  - What about region II?
    - We must have that $W(II) < W(\mathbf{u})$ since W is strictly increasing and all points in region II are to the southwest of $\mathbf{u}$.
    - Hence,

$$W(I) \geq W(\mathbf{u}) > W(II)$$

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
  - What about the points on the frontier between regions I and II, such as point **a**?
    - By continuity of the swf W, since $W(I) \geq W(\mathbf{u})$ in region I and $W(\mathbf{u}) > W(\mathbf{u})$ in region II, $W(\mathbf{u}) = W(\mathbf{a})$, as we wished to show.
  - We can extend the same argument, but now starting from a ray that extends from **a** upwards (rather than rightwards).
    - That is, we have just examined the welfare at points below the 45-degree line, but a similar argument applies for points above the 45-degree line.
    - See figure.

Reactions to Arrow's impossibility theorem - II



For individual 1, $\tilde{u}^1 > u^1$

For individual 2, $\tilde{u}^2 < u^2$

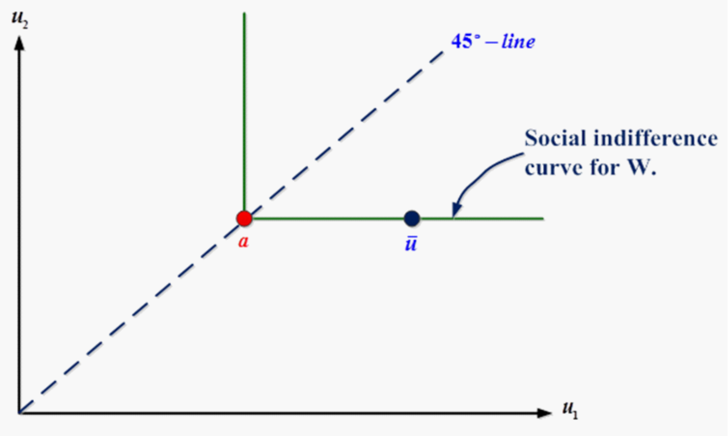
Across individuals, since $\tilde{u}$ **lies above the** $45^\circ - line$, $\tilde{u}^2 > \tilde{u}^1$,

thus implying $u^2 > \tilde{u}^2 > \tilde{u}^1 > u^1$

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
  - Because W is strictly increasing, no other points can yield the same social welfare than **a** other than the two rays we just examined.
    - That is, the union of the two rays provides us with the social indifference curve for W. (See figure.)
    - Therefore, W has the same indifference map as teh function $\min\left\{u^1(x), ..., u^I(x)\right\}$.

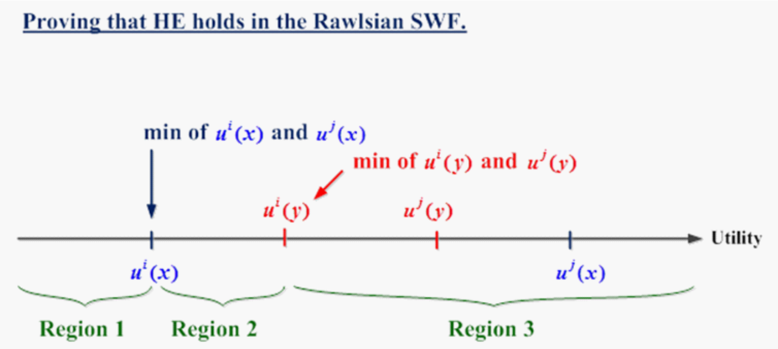
Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- *Other direction:* If $W(x) = \min\left\{u^1(x), ..., u^I(x)\right\}$ then HE holds.
  - Let's we apply the definition of HE: if $u^k(x) = u^k(y)$ for every individual k except for two individuals: i and j, and assume that
  $$u^i(x) < u^i(y) < u^j(y) < u^j(x)$$
  - Figure.
  - We now NTS that the alternative with the smaller utility dispersion is socially preferred, i.e., $W(\mathbf{u}(y)) \geq W(\mathbf{u}(x))$.

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- Then, $u^k(x) = u^k(y)$ lies in either of the following regions:
  - *Region 1, where* $u^k(x) = u^k(y) < u^i(x)$.
    - Then $W(\mathbf{u}(x)) = u^k(x)$ and $W(\mathbf{u}(y)) = u^k(y)$, and
    - Society is indifferent between alternatives y and x, i.e., $W(\mathbf{u}(y)) = W(\mathbf{u}(x))$, which is allowed according to the HE property (recall that we seek to show $W(\mathbf{u}(y)) \geq W(\mathbf{u}(x))$).

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- Then, $u^k(x) = u^k(y)$ lies in either of the following regions:
  - *Region 2, where* $u^i(x) < u^k(x) = u^k(y) < u^i(y)$.
    - Then $W(\mathbf{u}(x)) = u^i(x)$ and $W(\mathbf{u}(y)) = u^k(y)$, and
    - Society prefers alternative y to x, i.e., $W(\mathbf{u}(y)) > W(\mathbf{u}(x))$, thus satisfying the HE property.
    - Intuitively, alternative y yields a smaller utility dispersion than x does.

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- Then, $u^k(x) = u^k(y)$ lies in either of the following regions:
  - *Region 3, where $u^i(y) < u^k(x) = u^k(y)$.*
    - Then $W(\mathbf{u}(x)) = u^i(x)$ and $W(\mathbf{u}(y)) = u^i(y)$, and
    - Society prefers alternative y to x, i.e., $W(\mathbf{u}(y)) > W(\mathbf{u}(x))$, thus satisfying the HE property.
    - Intuitively, alternative y yields a smaller utility dispersion than x does.

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- *Corollary:* $W(x) = \min\left\{u^1(x), ..., u^I(x)\right\}$ satisfies anonymity, and is utility-level invariant.
  - Anonymity is obvious. Take a utility vector $u^1(x), ..., u^I(x)$, where
  $$\min\left\{u^1(x), ..., u^I(x)\right\} = u^k(x)$$
  - Now perform a permutation on the identities of individuals, and apply the min on their utility levels again. The min is still $u^k(x)$.

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- *Corollary:* $W(x) = \min\left\{u^1(x), ..., u^I(x)\right\}$ satisfies anonymity, and is utility-level invariant.
  - What about utility-level invariance?
    - Let's first define what we need to show.
    - Consider a strictly increasing transformation common to all individuals $\psi : \mathbb{R} \to \mathbb{R}$.
    - If $W(\mathbf{u}(x)) \geq W(\mathbf{u}(y))$ then the social ranking is preserved after applying a common strictly increasing transformation to all individuals' utility function, i.e.,

$$W\left(\psi\left(u^1(x)\right), ..., \psi\left(u^I(x)\right)\right) \geq \psi\left(W\left(u^1(x), ..., u^I(x)\right)\right)$$

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- Let us now show utility-level invariance.
  - Define a strictly increasing transformation common to all individuals $\psi : \mathbb{R} \to \mathbb{R}$. Then,

$$W\left(\psi\left(u^1(x)\right), ..., \psi\left(u^I(x)\right)\right) = \psi\left(W\left(u^1(x), ..., u^I(x)\right)\right)$$

  - *Example*: $\psi\left(u^i(x)\right) = \alpha + \beta u^i(x)$, then

$$\psi\left(W\left(u^1(x), ..., u^I(x)\right)\right) = \alpha + \beta \min\left\{u^1(x), ..., u^I(x)\right\}$$

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- Let us now show utility-level invariance.
  - Therefore,

$$W\left(\psi\left(u^1(x)\right),...,\psi\left(u^I(x)\right)\right) \geq W\left(\psi\left(u^1(y)\right),...,\psi\left(u^1(y)\right)\right)$$

implies

$$\psi\left(W\left(u^1(x),...,u^I(x)\right)\right) \geq \psi\left(W\left(u^1(y),...,u^I(y)\right)\right)$$

which is equivalent to

$$W\left(u^1(x),...,u^I(x)\right) \geq W\left(u^1(y),...,u^I(y)\right)$$

as required by utility-level invariance.

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- What about utility-difference invariance, UDI?
  - It does not necessarily hold.
  - To see this, consider a counterexample, where

$$\begin{aligned} W(\mathbf{u}(x)) &= \min\left\{u^1(x), u^2(x)\right\} = u^1(x) = 10, \text{ and} \\ W(\mathbf{u}(y)) &= \min\left\{u^1(y), u^2(y)\right\} = u^2(y) = 5 \end{aligned}$$

Hence, $W(\mathbf{u}(x)) > W(\mathbf{u}(y))$

Reactions to Arrow's impossibility theorem - II

- **The Rawlsian SWF**
- What about utility-difference invariance, UDI?
  - We now apply the linear, but potentially asymmetric, strictly increasing transformation $\psi^i\left(u^i(x)\right) = a^i + bu^i(x)$, where $b > 0$.
  - Consider for instance $b = 1$, $a^1 = 1$ and $a^2 = 150$. We then obtain

$$\begin{aligned} W(\psi^i(\mathbf{u}(x))) &= \min\left\{1 + u^1(x), 1 + u^2(x)\right\} = 1 + u^1(x) = 11, \\ W(\psi^i(\mathbf{u}(y))) &= \min\left\{150 + u^1(y), 150 + u^2(y)\right\} \\ &= 150 + u^2(y) = 155 \end{aligned}$$

    which implies that the social ranking between alternatives x and y is reverted to $W(\mathbf{u}(x)) > W(\mathbf{u}(y))$.
  - Hence, UDI doesn't necessarily hold for the Rawlsian swf.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- This is probably the most commonly used swf in economics.

$$W(x) = u^1(x) + u^2(x) + ... + u^I(x) = \sum_{i=1}^{I} u^i(x)$$

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
  - *Theorem 6.3 in JR:*
    - A strictly increasing and continuous swf W satisfies A and utility-difference invariance if and only if it can be represented with the utilitarian form, $W(x) = \sum_{i=1}^{I} u^i(x)$.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
- Here is what we need to show:
  - 1st line of implication:
    - If W is utilitarian, then A and UDI holds.
  - 2nd line of implication:
    - If A and UDI holds, then W must be utilitarian.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - When W takes the utilitarian form, A holds since the utility level of each individual receives the same weight.
    - That is, a permutation on the identities of individuals will not alter the social ranking of alternatives.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - When W takes the utilitarian form, utility-difference invariance holds as well. In particular,

    $$\text{if} \quad W(x) = u^1(x) + u^2(x) \geq u^1(y) + u^2(y) = W(y),$$

    then

    $$\begin{aligned} & \left(a^1 + bu^1(x)\right) + \left(a^2 + bu^2(x)\right) \\ \geq \quad & \left(a^1 + bu^1(y)\right) + \left(a^2 + bu^2(y)\right) \end{aligned}$$

    also needs to hold.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - This inequality collapses to

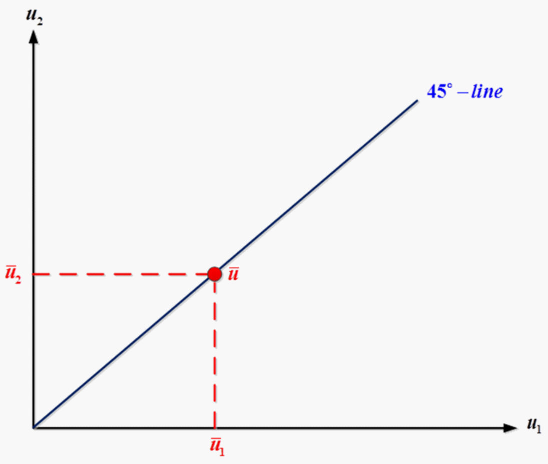
$$b\left[u^1(x) + u^2(x)\right] \geq b\left[u^1(y) + u^2(y)\right]$$

    which is satisfied since $u^1(x) + u^2(x) \geq u^1(y) + u^2(y)$, and $b > 0$ by definition.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - We now need to show the oppose line of implication: a strictly increasing and continuous swf satisfying A and utility-difference invariance can only be represented with the utilitarian form.
  - Consider the next figure.
  - Take a point $\mathbf{u}$ on the 45-degree line.

Reactions to Arrow's impossibility theorem - II



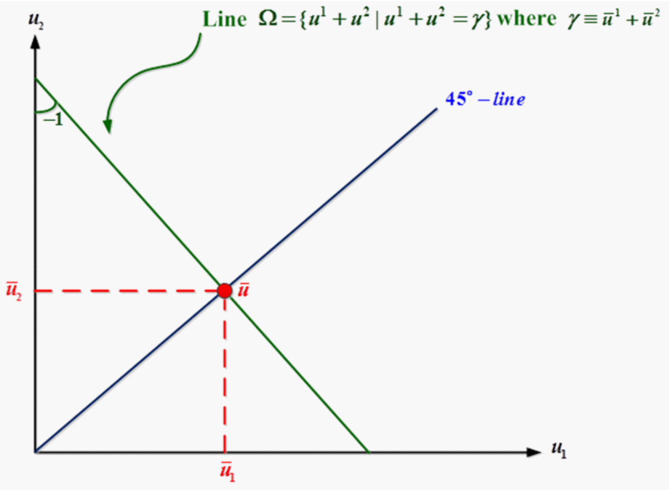
Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - Sum the two components in point $\overline{\mathbf{u}}$, i.e., $\overline{u}^1 + \overline{u}^2 \equiv \gamma$.
  - Consider the set of points for which the sum of their two components, $u^1 + u^2$, yields exactly γ.

$$\Omega = \left\{ u^1 + u^2 \mid u^1 + u^2 = \gamma \right\}$$

  - These are all the points in the line that crosses $\overline{\mathbf{u}}$ and has a slope of -1.

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

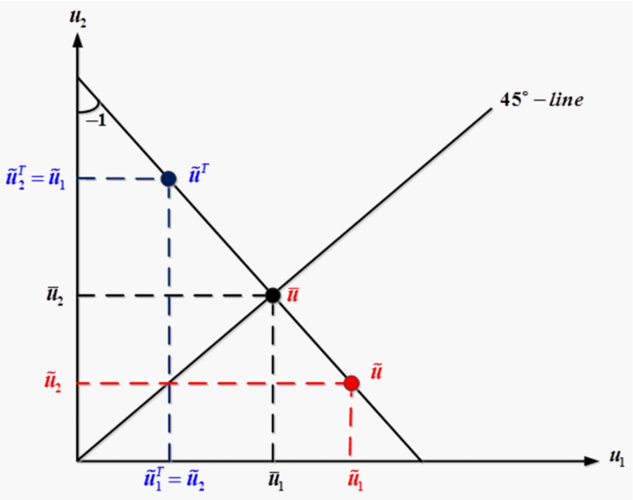
- **The Utilitarian SWF**
- *Proof:*
- Here is what we are planning to do:
  - The social indifference curve of a utilitarian swf must be linear, i.e., $u^2 = W - u^1$.
  - We must then show that all points in line Ω yield the same social welfare as in point $\bar{\mathbf{u}}$.

$$W(\Omega) = W(\bar{\mathbf{u}}).$$

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - Choose any point in line Ω, distinct from $\overline{\mathbf{u}}$, such as $\widetilde{\mathbf{u}}$.
  - Point $\widetilde{\mathbf{u}}^T$ is just a permutation of $\widetilde{\mathbf{u}}$, i.e., if $\widetilde{\mathbf{u}} = (\widetilde{u}^1, \widetilde{u}^2)$ point $\widetilde{\mathbf{u}}^T$ becomes $\widetilde{\mathbf{u}}^T = (\widetilde{u}^2, \widetilde{u}^1)$.

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - By condition A, points $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}}^T$ must be ranked the same way relative to $\overline{\mathbf{u}}$.
  - Note that we are not saying that societies with swf that satisfy A and UDI are indifferent between points $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}}^T$; we don't know that yet.
    - We only say that, if $W(\widetilde{\mathbf{u}}) \geq W(\overline{\mathbf{u}})$, then such social ranking is maintained for point $\widetilde{\mathbf{u}}^T$, i.e., $W(\widetilde{\mathbf{u}}^T) \geq W(\overline{\mathbf{u}})$.
    - Likewise, if $W(\overline{\mathbf{u}}) \geq W(\widetilde{\mathbf{u}})$, then such social ranking is maintained for point $\widetilde{\mathbf{u}}^T$, i.e., $W(\overline{\mathbf{u}}) \geq W(\widetilde{\mathbf{u}}^T)$.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - Suppose that $W(\overline{\mathbf{u}}) > W(\widetilde{\mathbf{u}})$.
  - Under UDI, this social ranking must be unaffected by linear transformations of the form $\psi^i\left(u^i(\cdot)\right) = a^i + bu^i(\cdot)$.
  - Let $b = 1$ and $a^i = \overline{u}^i - \widetilde{u}^i$, i.e.,

$$\psi^i\left(u^i(x)\right) = \underbrace{\overline{u}^i(x) - \widetilde{u}^i(x)}_{a^i} + u^i(x)$$

  - Applying this transformation to $\widetilde{\mathbf{u}}$ yields $\psi^i\left(\widetilde{u}^i(x)\right) = \overline{u}^i(x) - \widetilde{u}^i(x) + \widetilde{u}^i(x) = \overline{u}^i(x)$, i.e.,

$$\left(\psi^1\left(\widetilde{u}^1\right), \psi^2\left(\widetilde{u}^2\right)\right) = \overline{\mathbf{u}}$$

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - Applying this transformation to $\overline{\mathbf{u}}$ yields $\psi^i\left(\overline{u}^i(x)\right) = \overline{u}^i(x) - \widetilde{u}^i(x) + \overline{u}^i(x) = 2\overline{u}^i(x) - \widetilde{u}^i(x)$
  - However, since point $\overline{\mathbf{u}}$ lies on the 45-degree line, $2\overline{u}^i(x) = \widetilde{u}^i(x) + \widetilde{u}^j(x)$.
  - Using this property in our above result yields a transformation of

$$\psi^i\left(\overline{u}^i(x)\right) = 2\overline{u}^i(x) - \widetilde{u}^i(x) = \underbrace{\left[\widetilde{u}^i(x) + \widetilde{u}^j(x)\right]}_{2\overline{u}^i(x)} - \widetilde{u}^i(x) = \widetilde{u}^j(x)$$

  - That is,

$$\left(\psi^1\left(\overline{u}^1\right), \psi^2\left(\overline{u}^2\right)\right) = \widetilde{\mathbf{u}}^T$$

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - Therefore, point $\tilde{\mathbf{u}}$ is transformed into $\overline{\mathbf{u}}$, and point $\overline{\mathbf{u}}$ is transformed into $\tilde{\mathbf{u}}^T$.
  - Thus, if $W(\overline{\mathbf{u}}) > W(\tilde{\mathbf{u}})$, as we originally assumed, then UDI implies that $W(\tilde{\mathbf{u}}^T) > W(\overline{\mathbf{u}})$.
    - Hence, $W(\tilde{\mathbf{u}}^T) > W(\overline{\mathbf{u}})$ and $W(\overline{\mathbf{u}}) > W(\tilde{\mathbf{u}})$, which implies $W(\tilde{\mathbf{u}}^T) > W(\tilde{\mathbf{u}})$, thus violating A.
    - Therefore, our initial assumption $W(\overline{\mathbf{u}}) > W(\tilde{\mathbf{u}})$ cannot hold.

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- *Proof:*
  - A similar argument applies if we, instead, start our proof assuming that $W(\overline{\mathbf{u}}) < W(\widetilde{\mathbf{u}})$.
  - We can therefore conclude that $W(\overline{\mathbf{u}}) = W(\widetilde{\mathbf{u}})$ which, together with A, implies that

$$W(\overline{\mathbf{u}}) = W(\widetilde{\mathbf{u}}) = W(\widetilde{\mathbf{u}}^T)$$

  - Since point $\widetilde{\mathbf{u}}$ was chosen arbitrarily in the line Ω, we can claim that the social welfare at point $\overline{\mathbf{u}}$ is the same as any point along the line Ω, i.e.,

$$W(\overline{\mathbf{u}}) = W(\Omega)$$

Reactions to Arrow's impossibility theorem - II

- **The Utilitarian SWF**
- Note that droping the requirement of A, we can expand our previous results to any "generalized utilitarian" swf of the form

$$W(x) = \sum_{i=1}^{I} \alpha^i u^i(x)$$

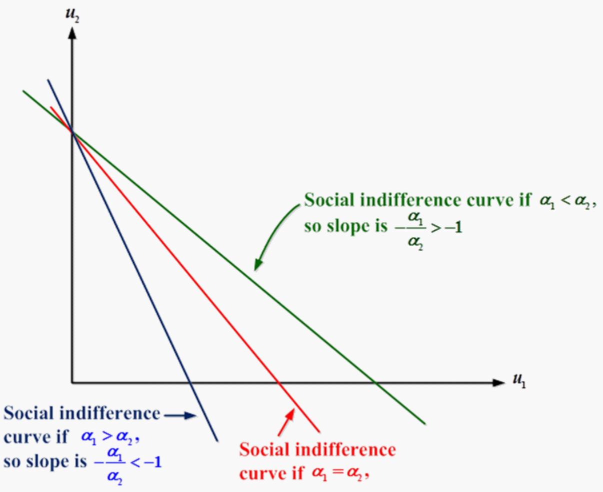
where $\alpha^i > 0$ represents the weight society assigns to individual i.

- *Example:* For the case of two individuals, $W = \alpha^1 u^1 + \alpha^2 u^2$, which yields a social indifference curve of

$$u^2 = \frac{W}{\alpha^2} - \frac{\alpha^1}{\alpha^2} u^1,$$

thus being still a straight, negatively sloped line, but the slope is now $-\frac{\alpha^1}{\alpha^2}$.

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- In the analysis of certain policies, i.e., moving from x to y, we might be interested in percentage change in utility for each individual, $\frac{u^i(x)-u^i(y)}{u^i(x)}$, and
- whether such a percentage is large for individual i than for j.

$$\frac{u^i(x) - u^i(y)}{u^i(x)} > \frac{u^j(x) - u^j(y)}{u^j(x)}$$

- If we seek to maintain the ranking of percentage changes across individuals invariant to monotonic transformations on the utility functions...
  - we need monotonic transformations to be *linear* and *common* among individuals, $\psi(u^i) = bu^i$, where $b > 0$ for all i.

Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- Applying $\psi(u^i) = bu^i$, we obtain

$$\frac{bu^i(x) - bu^i(y)}{bu^i(x)} > \frac{bu^j(x) - bu^j(y)}{bu^j(x)}$$

which reduces to

$$\frac{u^i(x) - u^i(y)}{u^i(x)} > \frac{u^j(x) - u^j(y)}{u^j(x)}$$

- Hence, when the swf is invariant to arbitrary, but linear and common, strictly increasing transformations of the form we say that the swf is utility-percentage invariant.

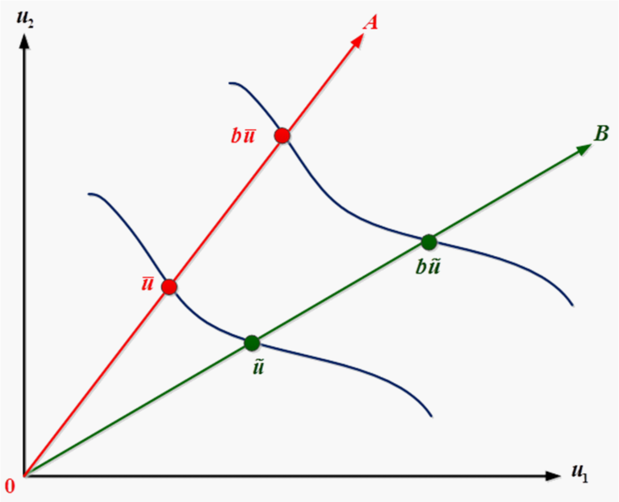
Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- As a consequence, if a swf satisfies utility-percentage invariance, it must also satisfy:
  - Utility-level invariance, since for that we need that the strictly increasing transformations are common across individuals, i.e., $\psi^i(\cdot) = \psi^j(\cdot)$ for any two individuals $i \neq j$; and
  - Utility-difference invariance, since for that we need that the strictly increasing transformation for each individual to be linear, i.e., $\psi^i(u^i) = a^i + bu^i$ where $b > 0$.
  - That is, UPI is a special case of ULI and of UDI.

Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- UPI allows for whole class of swf, whereby the Rawlsian and utilitarian are just special cases.
- Let's start demonstrating that UPI yields homothetic social indifference curves.
    - *Proof*:
    - Consider the following figure.
    - Choose an arbitrary point $\bar{\mathbf{u}}$.

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
  - Since W is strictly increasing, the social indifference curve must be negatively sloped.
    - Now choose a point through ray OA, i.e., $b\bar{\mathbf{u}}$, where $b > 0$.

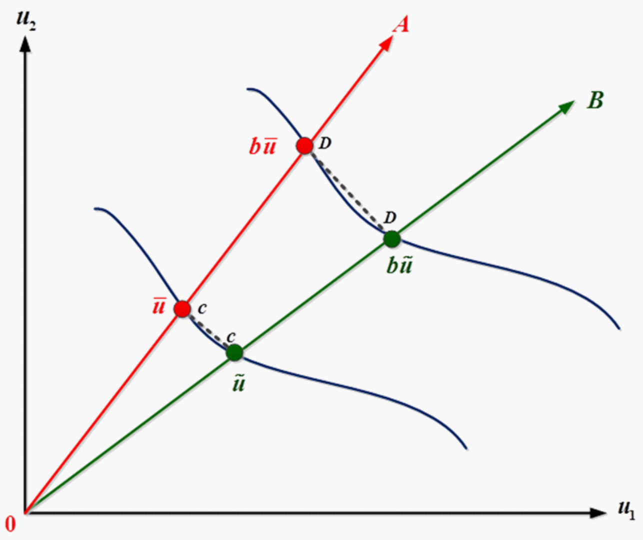
Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
  - Select now another point, $\widetilde{\mathbf{u}}$, lying on the same social indifference curve, i.e., $W(\widetilde{\mathbf{u}}) = W(\overline{\mathbf{u}})$.
    - Following a similar argument as above, choose a point through ray OB, i.e., $b\widetilde{\mathbf{u}}$, where $b > 0$.
    - By the UPI requirement, $W(b\widetilde{\mathbf{u}}) = W(b\overline{\mathbf{u}})$, so points $b\overline{\mathbf{u}}$ and $b\widetilde{\mathbf{u}}$ must lie on the same social indifference curve.

Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
  - We NTS homotheticity of the social indifference curve:
    - The tangent at point $\overline{\mathbf{u}}$ must coincide with that in point $b\overline{\mathbf{u}}$, and
    - The tangent at point $\widetilde{\mathbf{u}}$ must coincide with that in point $b\widetilde{\mathbf{u}}$.
  - The slope of chord CC approximates the slope of the tangent at $\overline{\mathbf{u}}$, whereas
    - the slope of chord DD approximates the slope of the tangent at $b\overline{\mathbf{u}}$.
    - (This, of course, happens when points $\overline{\mathbf{u}}$ and $\widetilde{\mathbf{u}}$ are close.)

Reactions to Arrow's impossibility theorem - II



Reactions to Arrow's impossibility theorem - II

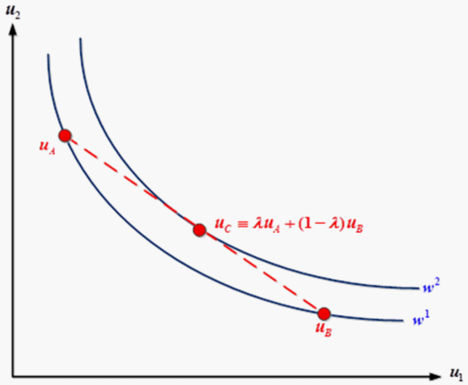
- **Flexible form SWF**
  - Since points $b\bar{\mathbf{u}}$ and $b\tilde{\mathbf{u}}$ have both been increased by the same factor b, the slope of chord CC coincides with that of DD.
  - If we choose a point $\tilde{\mathbf{u}}$ closer and closer to $\bar{\mathbf{u}}$, the slope of chords CC and DD still coincide,
    - but their slopes better approximates that of the tangent through each point.
  - In the limit, the slope of the social indifference curve at point $\bar{\mathbf{u}}$ coincides with that at point $b\bar{\mathbf{u}}$, proving homotheticity.

Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- We have just showed that UPI yields homothetic social indifference curves.
- But, what's the effect of impossing other common assumptions on the swf in the shape of social indifference curves?
  - *Anonymity:* Social indifference curves become "mirror images" above and below the 45-degree line.
  - *Quasiconcavity:* Similarly as in consumer theory, this assumption on the swf implies that social indifference curves are bowed-in towards the origin.
    - Intuitively, society prefers "balanced" utility vectors to "unbalanced" ones, i.e., preference for equality.

Reactions to Arrow's impossibility theorem - II

Quasiconcavity of the SWF



- At u_A individual 2 is extremely well, relative to individual 1
- At u_B individual 1 is extremely well, relative to individual 2
- At the linear combination of u_A and u_B society reaches a linear social welfare than the unequal utility vector u_A or u_B alone.

Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- We can encompass all previous forms of swf into the following CES:

$$W(x) = \sum_{i=1}^{I} \left[\left(u^i(x) \right)^{\rho} \right]^{\frac{1}{\rho}}$$

where $0 \neq \rho < 1$.

  - Hence, the constant elasticity of social substitution between the utility of any two individuals, σ, can be expressed as $\sigma = \frac{1}{1-\rho}$.

- This swf satisfies three properties mentioned above (A, WP, and quasiconcavity).

Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- This swf also satisfies a property we discussed in EconS 501:
  - *Strong separability*: The MRS_{u^i,u^j} only depends on u^i and u^j, but not on u^k for any other individual $k \neq i,j$.
  - In particular, MRS_{u^i,u^j} of this CES swf is

$$MRS_{u^i,u^j} = -\left(\frac{u^i}{u^j}\right)^{\rho-1}$$

Reactions to Arrow's impossibility theorem - II

- **Flexible form SWF**
- Figures in next slide with three cases of CES swf, as parameter ρ decreases:
  - $\rho \rightarrow 1$ (linear social indifference curves, i.e., utilitarian swf),
  - $-\infty < \rho < 1$ (curvy social indifference curves),
  - $\rho \rightarrow -\infty$ (right-angel social indifference curves, i.e., Rawlsian swf).

Reactions to Arrow's impossibility theorem - II