Mixed strategy equilibria (msNE) with N players

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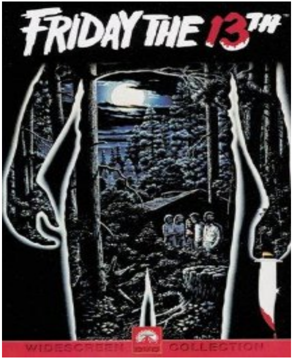
Summarizing...

- We learned how to find msNE in games:
- with 2 players, each with 2 available strategies (2x2 matrix)
  - e.g., matching pennies game, battle of the sexes, etc.
- with 2 players, but each having 3 available strategies (3x3 matrix)
  - e.g., tennis game (which actually reduced to a 2x2 matrix after deleting strictly dominated strategies), and
  - the rock-paper-scissors game, where we couldn't identify strictly dominated strategies and, hence, had to make players indifferent between their three available strategies.
- What about games with 3 players?

More advanced mixed strategy games

What if we have three players, instead of two? (Harrington pp 201-204). "Friday the 13th!"



More advanced mixed strategy games

Jason, Front

Tommy \ *Beth*	Front	Back
Front	0, 0, 0	-4, 1, 2
Back	1, -4, 2	2, 2, -2

Jason, Back

Tommy \ *Beth*	Front	Back
Front	3, 3, -2	1, -4, 2
Back	-4, 1, 2	0, 0, 0

More advanced mixed strategy games

Friday the 13th!

Jason, Front

Tommy \ *Beth*	Front	Back
Front	0, 0, 0	-4, 1, 2
Back	1, -4, 2	2, 2, -2

Jason, Back

Tommy \ *Beth*	Front	Back
Front	3, 3, -2	1, -4, 2
Back	-4, 1, 2	0, 0, 0

1. **First step:** let's check for strictly dominated strategies (none).
2. **Second step:** let's check for psNE (none). The movie is getting interestin!
3. **Third step:** let's check for msNE. (note that all strategies are used by all players), since there are no strictly dominated strategies.

msNE with three players

- Since we could not delete any strictly dominated strategy, then all strategies must be used by all three players.
- In this exercise we need three probabilities, one for each player.
- Let's denote:
  - t the probability that Tommy goes through the front door (first row in both matrices).
  - b the probability that Beth goes through the front door (first column in both matrices).
  - j the probability that Jason goes through the front door (left-hand matrix).

msNE with three players

Let us start with **Jason**, $EU_J(F) = EU_J(B)$, where

$$EU_J(F) = \underbrace{tb0 + t(1-b)2}_{\substack{\text{Tommy goes through} \\ \text{the front door, } t}} + \underbrace{(1-t)b2 + (1-t)(1-b)(-2)}_{\substack{\text{Tommy goes through} \\ \text{the back door, } (1-t)}}$$

$$= -2 + 4t + 4b - 6tb$$

and

$$EU_J(B) = tb(-2) + t(1-b)2 + (1-t)b2 + (1-t)(1-b)0$$

$$= 2t + 2b - 6tb$$

since $EU_J(F) = EU_J(B)$ we have

$$-2 + 4t + 4b - 6tb = 2t + 2b - 6tb \iff \underbrace{t + b = 1}_{\text{Condition (1)}} \qquad (1)$$

msNE with three players

Let us now continue with **Tommy**, $EU_T(F) = EU_T(B)$, where

$$\begin{aligned} EU_T(F) &= bj0 + (1-b)j(-4) + b(1-j)3 + (1-b)(1-j)(1) \\ &= 1 + 2b - 5j + 2bj \end{aligned}$$

and

$$\begin{aligned} EU_T(B) &= bj1 + (1-b)j2 + b(1-j)(-4) + (1-b)(1-j)(0) \\ &= -4b + 2j + 3bj \end{aligned}$$

since $EU_T(F) = EU_T(B)$ we have

$$1 + 2b - 5j + 2bj = -4b + 2j + 3bj \iff \underbrace{7j - 6b + bj = 1}_{\text{Condition (2)}} \quad (2)$$

msNE with three players

- And given that the payoffs for Tommy and Beth are symmetric, we must have that Tommy and Beth's probabilities coincide, $t = b$.
  - Hence we don't need to find the indifference condition $EU_B(F) = EU_B(B)$ for Beth.
  - Instead, we can use Tommy's condition (2) (i.e., $7j - 6b + bj = 1$), to obtain the following condition for Beth:

$$7j - 6t + tj = 1$$

- We must solve conditions (1),(2) and (3).

- First, by symmetry we must have that $t = b$. Using this result in condition (1) we obtain

$$t + b = 1 \implies t + t = 1 \implies t = b = \frac{1}{2}$$

- Using this result into condition (2), we find

$$7j - 6b + bj = 7j - 6\frac{1}{2} + \frac{1}{2}j = 1$$

Solving for j we obtain $j = \frac{8}{15}$.

msNE with three players

- Representing the msNE in Friday the 13th:

$$\left\{ \underbrace{\left(\frac{1}{2}\text{Front}, \ \frac{1}{2}\text{Back}\right)}_{\text{Tommy}}, \underbrace{\left(\frac{1}{2}\text{Front}, \ \frac{1}{2}\text{Back}\right)}_{\text{Beth}}, \underbrace{\left(\frac{8}{15}\text{Front}, \ \frac{7}{15}\text{Back}\right)}_{\text{Jason}} \right\}$$

msNE with three players

- **Just for fun:** What is then the probability that Tommy and Beth scape from Jason?
  - They scape if they both go through a door where Jason is not located.

$$\frac{1}{2}\frac{1}{2}\underbrace{\frac{8}{15}}_{\text{Jason goes Front}} + \frac{1}{2}\frac{1}{2}\underbrace{\frac{7}{15}}_{\text{Jason goes Back}} = \frac{15}{60}$$

- The **first term** represents the probability that both Tommy and Beth go through the Back door (which occurs with $\frac{1}{2}\frac{1}{2} = \frac{1}{4}$ probability) while Jason goes to the Front door.
- The **second term** represents the opposite case: Tommy and Beth go through the Front door (which occurs with $\frac{1}{2}\frac{1}{2} = \frac{1}{4}$ probability) while Jason goes to the Back door.

msNE with three players

- Even if they escape from Jason this time, there is still...

- There are actually NO sequels:
  - Their probability of escaping Jason is then $(\frac{15}{60})^{10}$,about 1 in a million !

Testing the Theory

- A natural question at this point is how we can empirically test, as external observers, if individuals behave as predicted by our theoretical models.
  - In other words, how can we check if individuals randomize with approximately the same probability that we found to be optimal in the msNE of the game?

Testing the Theory

- In order to test the theoretical predictions of our models, we need to find settings where players seek to "surprise" their opponents (so playing a pure strategy is not rational), and where stakes are high.
  - Can you think of any?

Penalty kicks in soccer

Penalty kicks in soccer

His payoffs represent the probability that the kicker does not score (That is why within a given cell, payoffs sum up to one).

Payoffs represent the probability he scores.

		Goalkeeper		
		Left	Center	Right
Kicker	Left	.65, .35	.95, .05	.95, .05
	Center	.95, .05	0, 1	.95, .05
	Right	.95, .05	.95, .05	.65, .35

Penalty kicks in soccer

- We should expect soccer players randomize their decision.
  - Otherwise, the kicker could anticipate where the goalie dives and kick to the other side. Similarly for the goalie.
- Let's describe the kicker's expected utility from kicking the ball left, center or right.

Penalty kicks in soccer

$$
\begin{aligned}
EU_{\text{Kicker}}(Left) &= g_l * 0.65 + g_r * 0.95 + (1 - g_r - g_l) * 0.95 \\
&= 0.95 - 0.3g_l \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
EU_{\text{Kicker}}(Center) &= g_l * 0.95 + g_r * 0.95 + (1 - g_r - g_l) * 0 \\
&= 0.95(g_r + g_l) \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
EU_{\text{Kicker}}(Right) &= g_l * 0.95 + g_r * 0.65 + (1 - g_r - g_l) * 0.95 \\
&= 0.95 - 0.3g_r \qquad (3)
\end{aligned}
$$

Penalty kicks in soccer

- Since the kicker must be indifferent between all his strategies, $EU_{\text{Kicker}}(Left) = EU_{\text{Kicker}}(Right)$

  $$0.95 - 0.3g_l = 0.95 - 0.3g_r \implies g_l = g_r \implies g_l = g_r = g$$

  Using this information in (2), we have

  $$0.95(g + g) = 1.9g$$

  Hence,

  $$\underbrace{0.95 - 0.3g}_{\substack{EU_{\text{Kicker}}(Left) \\ \text{or} \\ EU_{\text{Kicker}}(Right)}} = \underbrace{1.9g}_{EU_{\text{Kicker}}(Center)} \implies g = \frac{0.95}{2.2} = 0.43$$

Penalty kicks in soccer

- Therefore,

$$(\sigma_L, \sigma_C, \sigma_R) = (\underbrace{0.43}_{g_l}, \underbrace{0.14}_{\substack{\text{From the fact that} \\ g_l+g_r+g_c=1}}, \underbrace{0.43}_{\substack{g_r, \\ \text{where } g_l=g_r=g}})$$

- If the set of goalkeepers is similar, we can find the same set of mixed strategies,

$$(\sigma_L, \sigma_C, \sigma_R) = (0.43, 0.14, 0.43)$$

Penalty kicks in soccer

- Hence, the probability that a goal is scored is:
  - Goalkeeper dives left $\longrightarrow$

$$0.43 * (\underbrace{0.43}_{\substack{\text{Kicker}\\\text{aims}\\\text{left}}} *0.65 + \underbrace{0.14}_{\substack{\text{Kicker}\\\text{aims}\\\text{center}}} *0.95 + \underbrace{0.43}_{\substack{\text{Kicker}\\\text{aims}\\\text{right}}} *0.95)$$

  - Goalkeeper dives center $\longrightarrow$

$$+0.14 * (0.43 * 0.95 + 0.14 * 0 + 0.43 * 0.95)$$

  - Goalkeeper dives right $\longrightarrow$

$$+0.43 * (0.43 * 0.95 + 0.14 * 0.95 + 0.43 * 0.65)$$

$= 0.82044$, i.e., a goal is scored with 82% probability.

Penalty kicks in soccer

- Interested in more details?
  - First, read Harrington pp. 199-201.
  - Then you can have a look at the article
    - "Professionals play Minimax" by Ignacio Palacios-Huerta, *Review of Economic Studies*, 2003.
  - This author published a very readable book last year:
    - *Beautiful Game Theory: How Soccer Can Help Economics.* Princeton University Press, 2014.

Summarizing...

- So far we have learned how to find msNE is games:
  - with two players (either with 2 or more available strategies).
  - with three players (e.g., Friday the 13th movie).
- What about generalizing the notion of msNE to games with N players?
  - Easy! We just need to guarantee that every player is indifferent between all his available strategies.

msNE with N players

- **Example: "Extreme snob effect" (Watson).**
- Every player chooses between alternative X and Y (Levi's and Calvin Klein). Every player i's payoff is 1 if he selects Y, but if he selects X his payoff is:
  - 2 if no other player chooses X, and
  - 0 if some other player chooses X as well

- Let's check for a symmetric msNE where all players select Y with probability α. Given that player i must be indifferent between X and Y, $EU_i(X) = EU_i(Y)$, where

$$EU_i(X) = \underbrace{\alpha^{n-1} 2}_{\text{all other } n-1 \text{ players select } Y} + \underbrace{(1 - \alpha^{n-1}) 0}_{\text{Not all other players select Y}}$$

msNE with N players

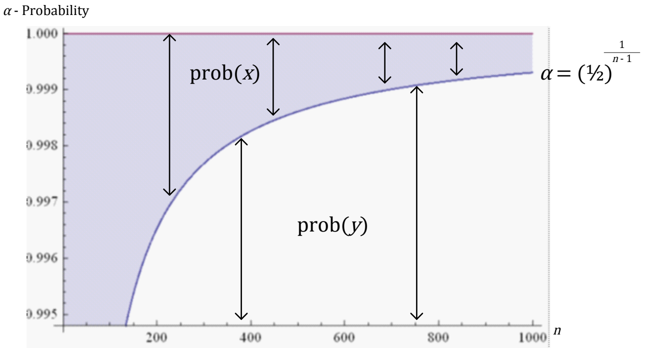
- and $EU_i(Y) = 1$, then $EU_i(X) = EU_i(Y)$ implies

$$\alpha^{n-1}2 = 1 \iff \alpha = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}$$

- **Comparative statics of α, the probability a player selects the "conforming" option Y, $\alpha = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}$:**
  - α increases in the size of the population n.
    - That is, the larger the size of the population, the more likely it is that somebody else chooses the same as you, and as a consequence you don't take the risk of choosing the snob option X. Instead, you select the "conforming" option Y.

msNE with N players

- Probability of choosing strategy Y as a function of the number of individuals, *n*.



$$prob(X) + prob(Y) = 1, prob(X) ... then, (X) = 1 - prob(Y)$$

Another example of msNE with N players

- **Another example with N players: The bystander effect**
- The "bystander effect" refers to the lack of response to help someone nearby who is in need.
  - *Famous example*: In 1964 Kitty Genovese was attacked near her apartment building in New York City. Despite 38 people reported having heard her screams, no one came to her aid.
  - Also confirmed in laboratory and field studies in psychology.

Another example of msNE with N players

- General finding of these studies:
  - A person is less likely to offer assistance to someone in need when the person is in a large group than when he/she is alone.
    - e.g., all those people who heard Kitty Genovese's cries knew that many others heard them as well.
  - In fact, some studies show that the *more* people that are there who could help, the *less* likely help is to occur.
- Can this outcome be consistent with players maximizing their utility level?
  - Yes, let's see how.

Another example of msNE with N players

		Other players	
		All ignore	At least one helps
Player	Helps	<u>a</u>	c
	Ignores	d	<u>b</u>

- where $a > d \longrightarrow$ so if all ignore, I prefer to help the person in need.
- but $b > c \longrightarrow$ so, if at least somebody helps, I prefer to ignore.
- Note that assumptions are not so selfish : people would prefer to help if nobody else does.

Another example of msNE with N players

- msNE:
  - Let's consider a *symmetric msNE* whereby every player i:
    - Helps with probability p, and
    - Ignores with probability $1 - p$.

Another example of msNE with N players

$$EU_i(Help) = \underbrace{(1-p)^{n-1} * a}_{\substack{\text{If everybody} \\ \text{else ignores}}} + \underbrace{\left[1-(1-p)^{n-1}\right] * c}_{\substack{\text{If at least one of the} \\ \text{other } n-1 \text{ players helps}}}$$

$$EU_i(Ignore) = \underbrace{(1-p)^{n-1} * d}_{\substack{\text{If everybody} \\ \text{else ignores}}} + \underbrace{\left[1-(1-p)^{n-1}\right] * b}_{\substack{\text{If at least one of the} \\ \text{other } n-1 \text{ players helps}}}$$

- When a player randomizes, he is indifferent between help and ignore,

$$\begin{aligned} EU_i(Help) &= EU_i(Ignore) \\ & (1-p)^{n-1} * a + \left[1-(1-p)^{n-1}\right] * c \\ &= (1-p)^{n-1} * d + \left[1-(1-p)^{n-1}\right] * b \\ \Longrightarrow & (1-p)^{n-1}(a-c-d+b) = b-c \end{aligned}$$

Another example of msNE with N players

- Solving for p,

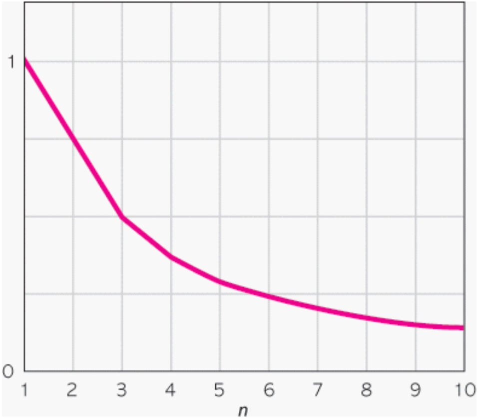
$$
\begin{aligned}
(1-p)^{n-1} &= \frac{b-c}{a-c-d+b} \\
&\implies 1-p = \left(\frac{b-c}{a-c-d+b}\right)^{\frac{1}{n-1}} \\
&\implies p^* = 1-\left(\frac{b-c}{a-c-d+b}\right)^{\frac{1}{n-1}}
\end{aligned}
$$

- Example: $a=4,\ b=3,\ c=2,\ d=1$, satisfying the initial assumptions: $a>d$ and $b>c$

$$
p^* = 1-\left(\frac{3-1}{4-2-1+3}\right)^{\frac{1}{n-1}} = 1-\left(\frac{1}{4}\right)^{\frac{1}{n-1}}
$$

Another example of msNE with N players

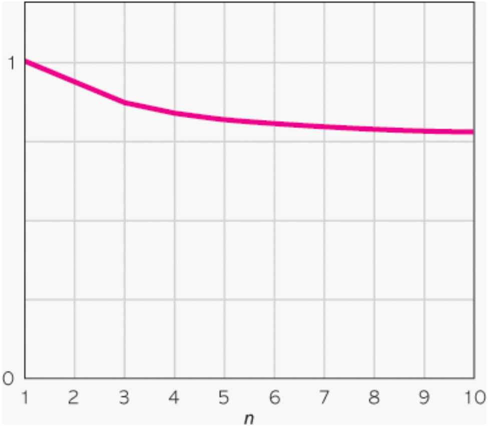
- Probability of a person helping, p^*



More people makes me less likely to help.

Another example of msNE with N players

- Probability that the person in need receives help, $(p^*)^n$



More people actually make it **less** likely that the victim is helped!

- Intuitively, the new individual in the population brings a positive and a negative effect on the probability that the victim is finally helped:
  - **Positive effect**: the additional individual, with his own probability of help, p^*, increases the chance that the victim is helped.
  - **Negative effect**: the additional individual makes more likely, that someone will help the victim,thus leading each individual citizen to reduce his own probability of helping, i.e., p^* decreades in n.
- However, the fact that $(p^*)^n$ decreases in n implies that the negative effect offsets the positive effect.