Semi-separating equilibria

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More about PBEs

- So far, all the exercises we considered had at least one PBE.
- What if we check for all...
  - separating strategy profiles, and
  - all pooling strategy profiles,
  - and we find that none can be sustained as a PBE!
- Then, we will allow players to use mixed strategies
  - that is, the first mover will randomize his messages, and
  - the second mover will randomize his responses.

More about PBEs

- This randomization is common in games where the sender does not want to reveal his type, as in...
  - you got it: Poker games.
- Let's analyze this randomization in a **Poker game**.
  - But, first, let's show that indeed no separating or pooling strategy profile can be supported as a PBE,
  - and hence we will need to allow for randomizations.
- See the additional **handout**: "*Handout on PBE-III: Semi-separating equilibrium*"

Brinkmanship

- Let's analyze another situation where semi-separating equilibria emerge.
- **Brinkmanship:**
  - "The ability to get to the verge without getting into war"
  - according to John Foster Dulles (secretary of state in the 1950s).
- Let's consider US-North Korea relationships.
  - Harrington, pp. 343-346.

Brinkmanship - Time structure

- First, nature determines whether North Korean (former) president, Kim Jong-Il, is sane or crazy.

Is Kim Jong-Il sane?

Or is he crazy?

Brinkmanship - Time structure

- You can easily adapt this setting to the new president, Kim-Jong Un.

Is Kim-Jong Un sane?

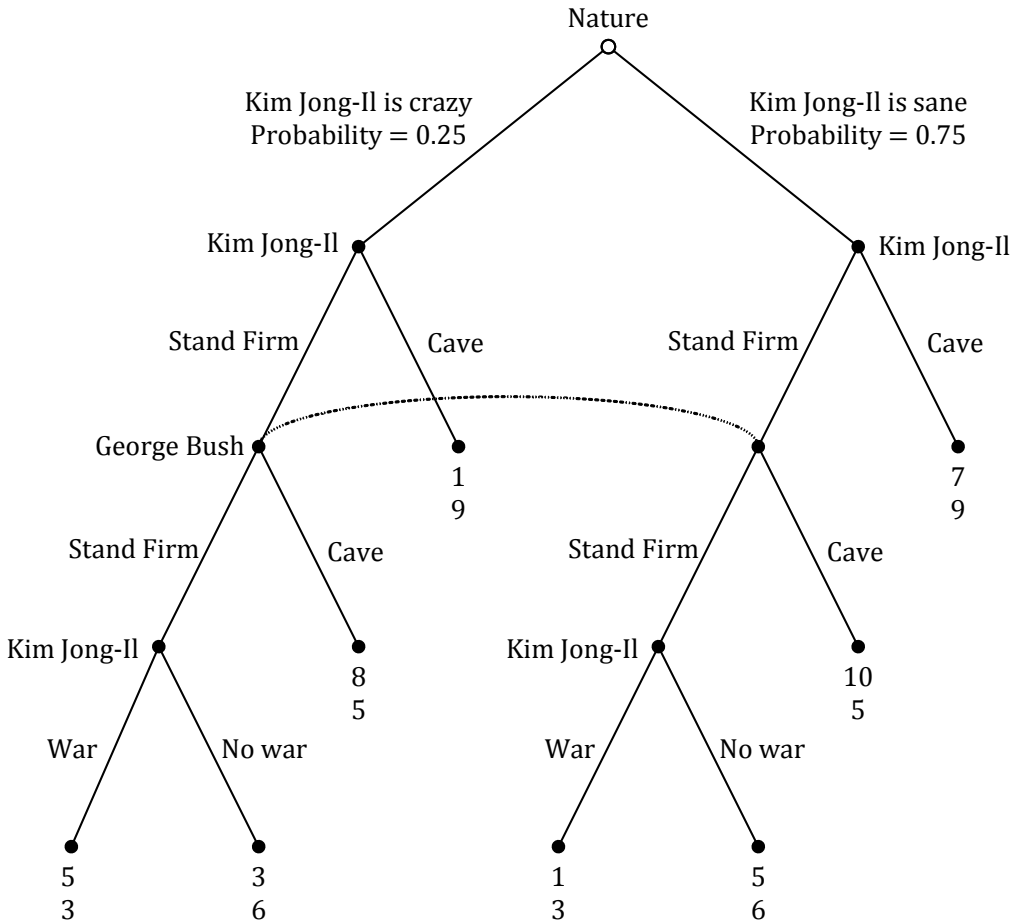
Or is he crazy?

- Perhaps Dennis Rodman will know!

Brinkmanship - Time structure

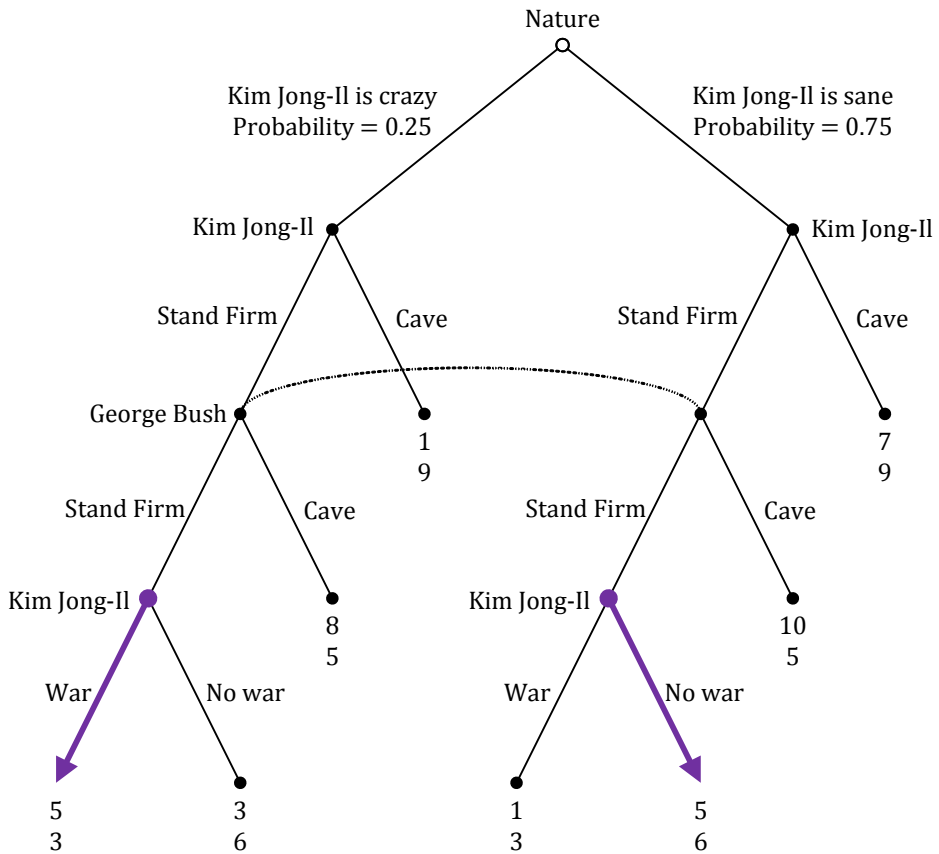
- After learning his mental state, Kim Jong-Il decides whether to Stand Firm in developing nuclear weapons or to Cave to the demands of the US and the UN.
  - If he caves, then the nuclear crisis is over (and the game too).
  - If he stands firm, then the US must decide whether it will Stand firm or Cave.
    - If the US caves, the game is over.
    - If the US stands firm, then North Korea decides whether or not to take a hostile action (such as launching nuclear weapons).

Brinkmanship



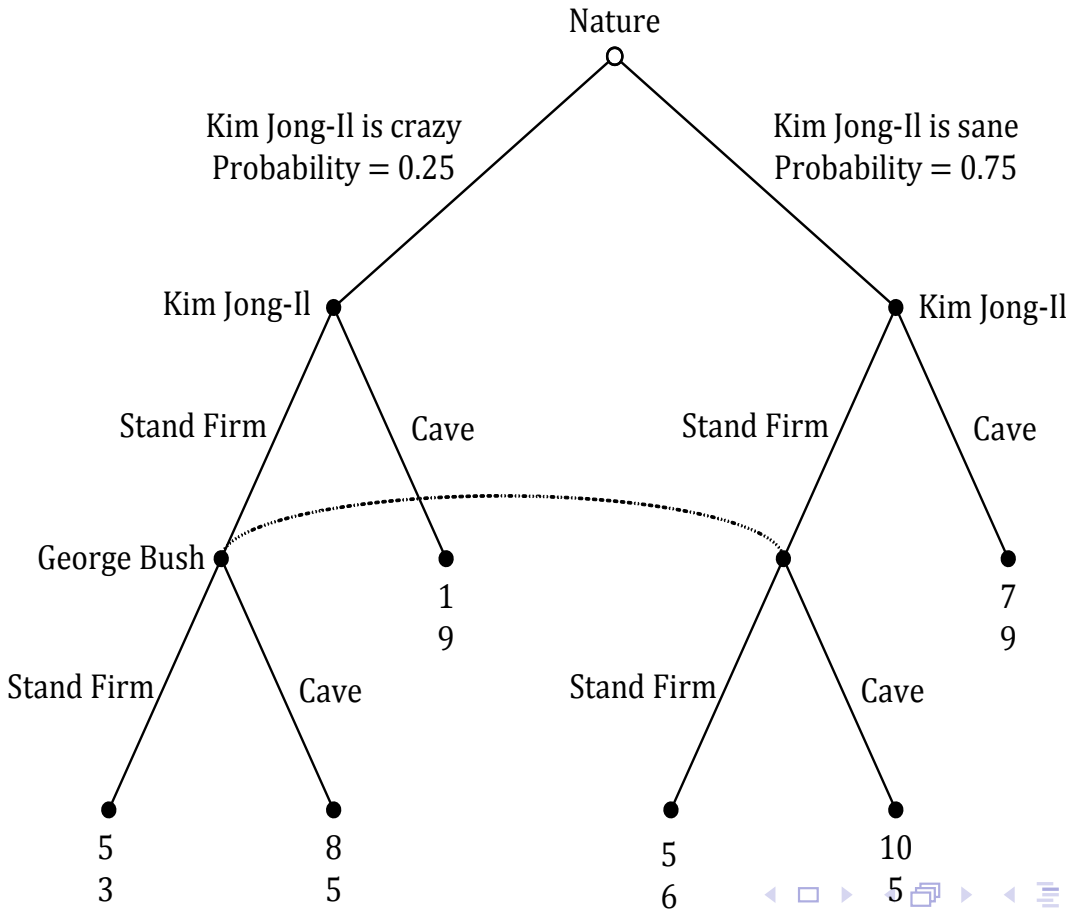
Brinkmanship

- We can identify two proper subgames, and anticipate how Kim-Jong Il will behave if the game progresses until that last stage.



Brinkmanship

- We can therefore simplify our game by depicting the equilibrium payoffs in the two proper subgames we identified above.



Brinkmanship

- Let us first show that
  - No separating strategy profiles,
  - or pooling strategy profiles,

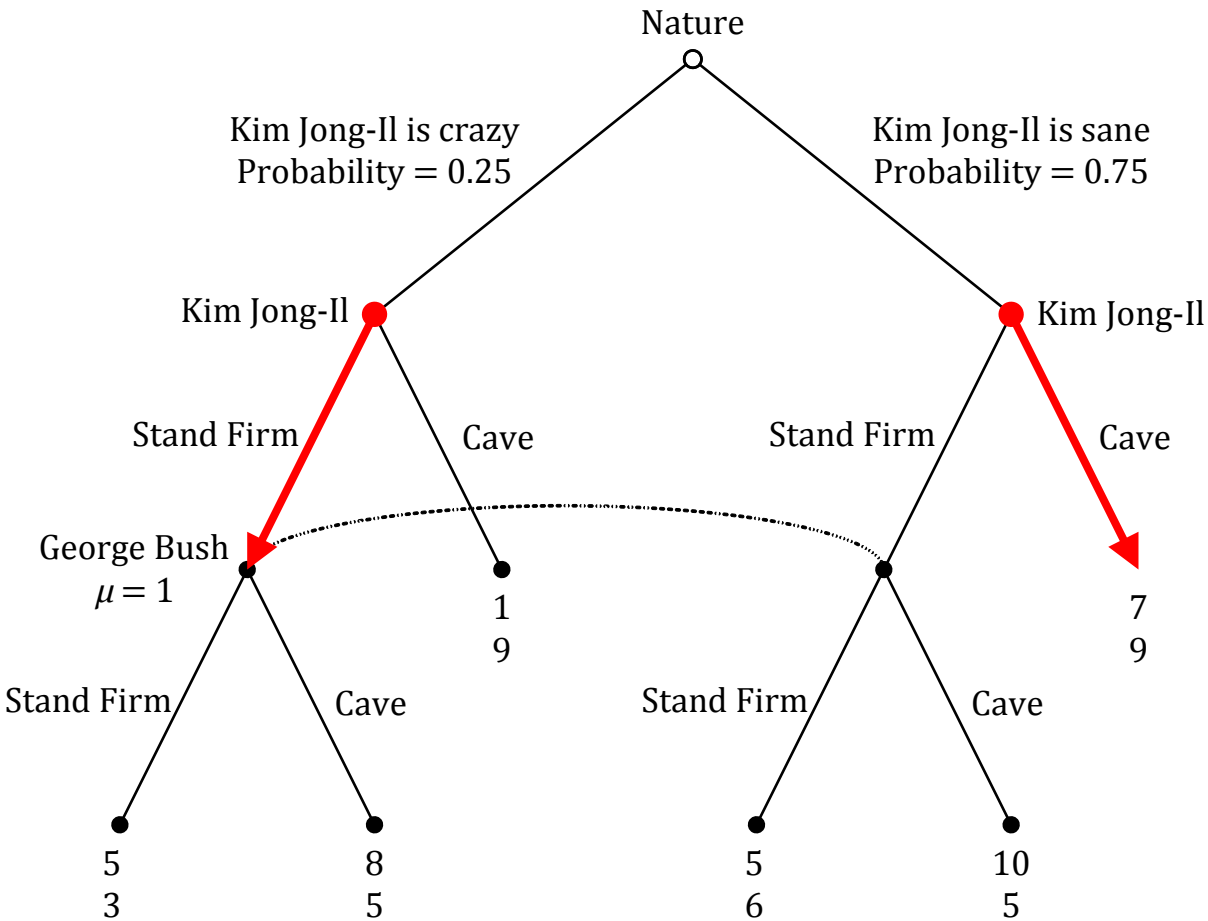
  can be sustained as a PBE.

- (Later on we will then examine if a semi-separating strategy profile, where players are allowed to randomize, can be supported as a PBE.)

Brinkmanship

- Let us first check the "natural" candidate of separating equilibrium **(Stand,Cave)**, where
  - The crazy Kim-Jong Il stands firm, and
  - The sane Kim-Jong Il caves.
- As usual, we shade the branches that correspond to this strategy profile in the next slide.

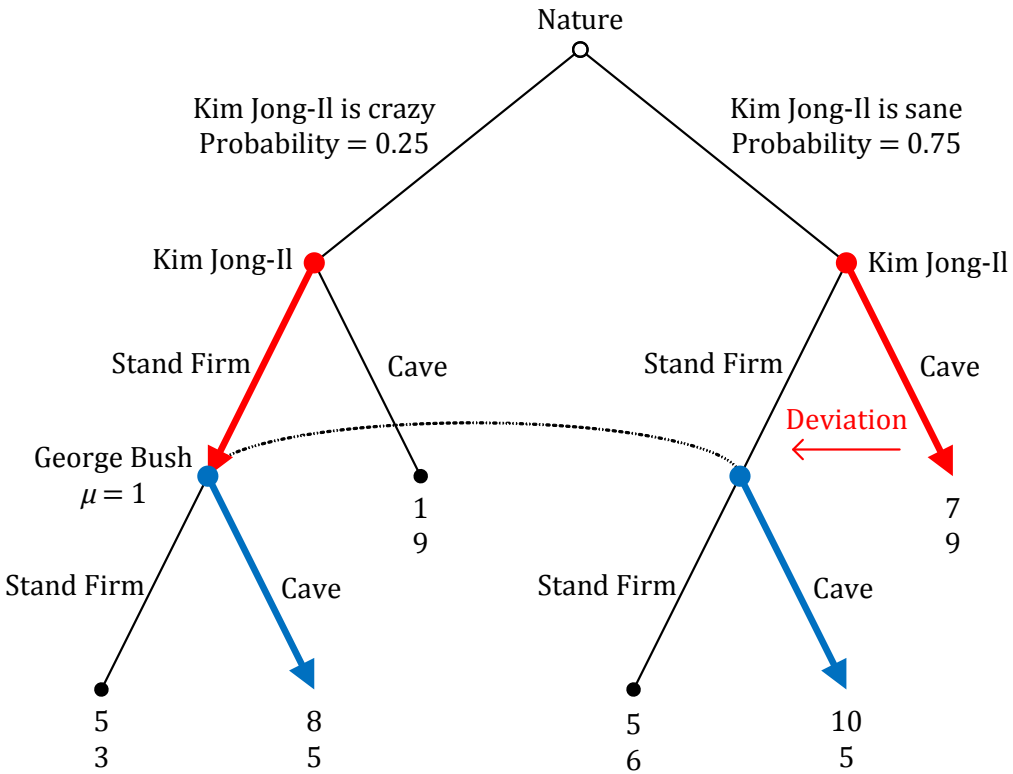
Brinkmanship



Brinkmanship

- **Separating (Stand,Cave):**
  - *Beliefs*: $\mu = 1$.
  - *George Bush's optimal response*:
    - Cave since 5>3 (Kim is crazy!).
    - Shade "Cave" in the figure.$\longrightarrow$
  - *Kim's optimal messages*:
    - If Crazy, Stand firm (as prescribed) since 8>1.
    - If Sane, Stand firm (violation) since 10>7
- This strategy profile **cannot** be sustained as a PBE.

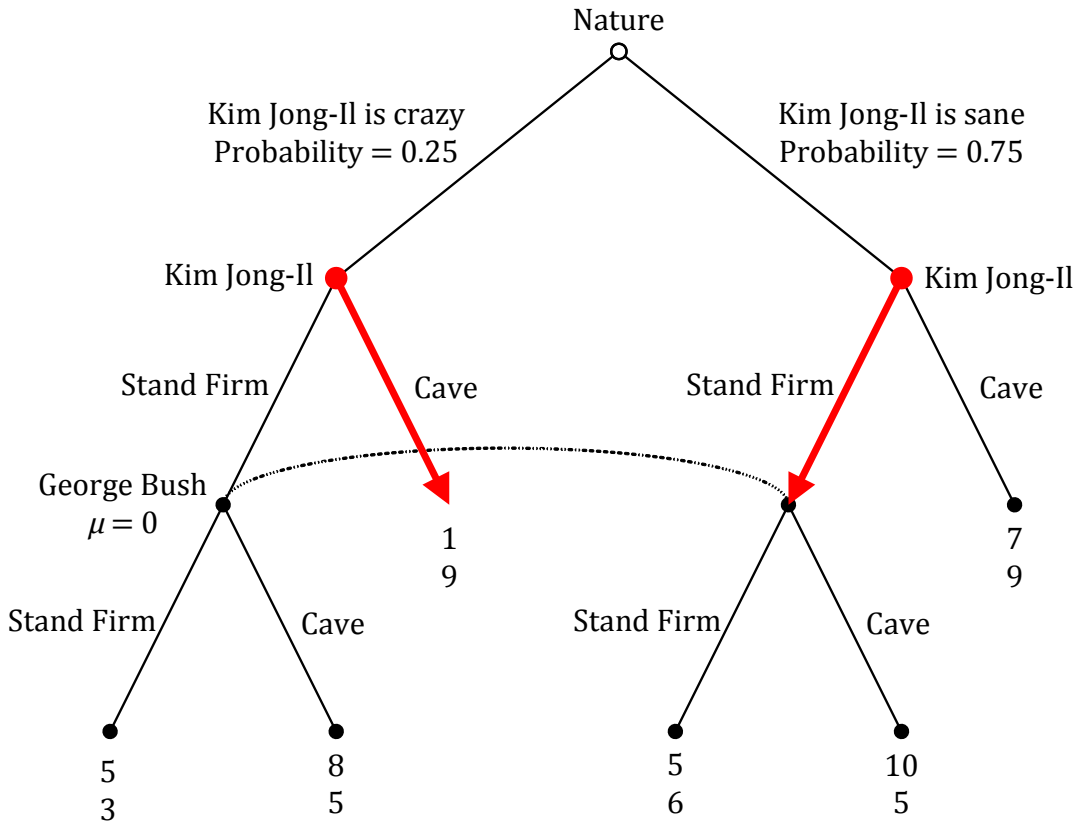
Brinkmanship



Brinkmanship

- Let us next check the "crazy" candidate of separating equilibrium **(Cave,Stand)**, where
  - The crazy Kim-Jong Il caves, while
  - The sane Kim-Jong Il stands firm.
- As usual, we shade the branches that correspond to this strategy profile in the next slide.

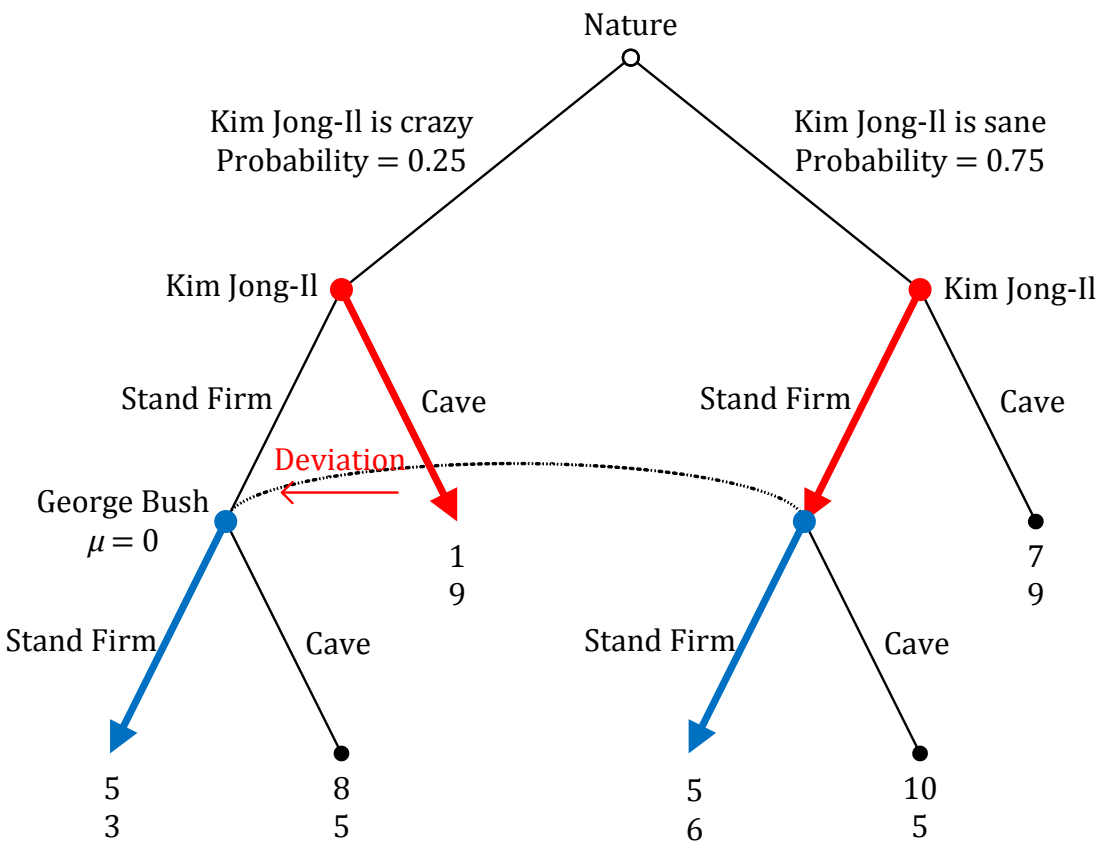
Brinkmanship



Brinkmanship

- **Separating (Cave, Stand):**
  - *Beliefs*: $\mu = 0$.
  - *George Bush's optimal response*:
    - Stand firm since 6>5.
    - Shade "Stand firm" in the figure.$\longrightarrow$
  - *Kim's optimal messages*:
    - If Crazy, Stand firm (violation) since 5>1.
  - This strategy profile **cannot** be sustained as a PBE.

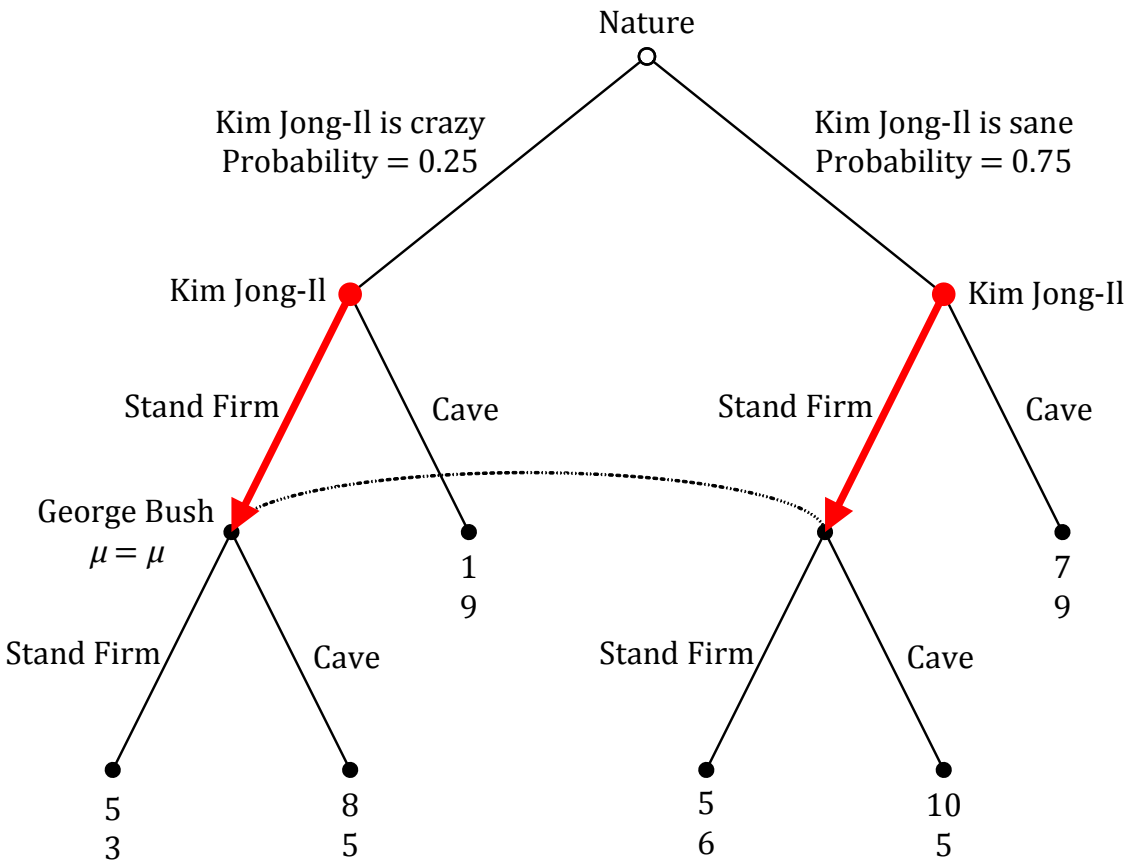
Brinkmanship



Brinkmanship

- We can now start checking the existence of pooling equilibria.
- Let us first check the pooling strategy profile **(Stand,Stand)**, where
  - The crazy Kim-Jong Il stands firm, and
  - The sane Kim-Jong Il also stands firm.
- As usual, we shade the branches that correspond to this strategy profile in the next slide.

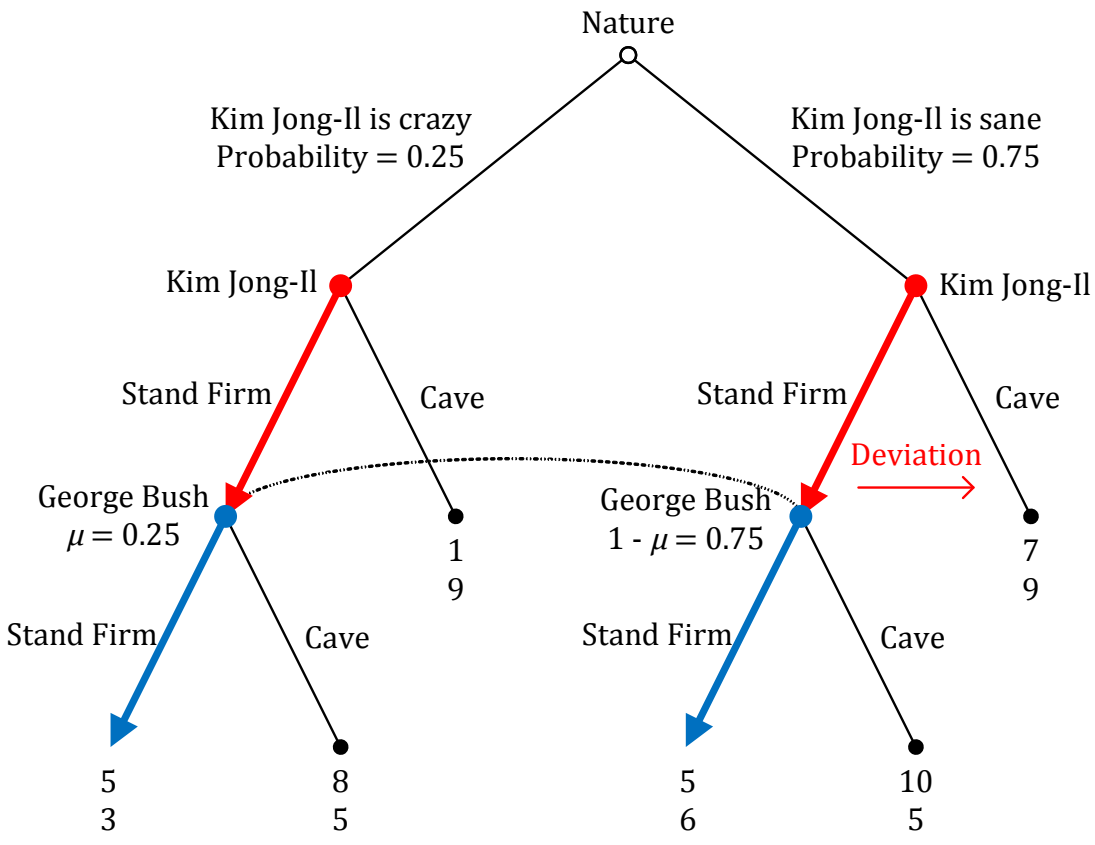
Brinkmanship



Brinkmanship

- **Pooling (Stand,Stand):**
  - *Beliefs*: $\mu = \frac{0.25 \times 1}{(0.25 \times 1) + (0.75 \times 1)} = 0.25$ (=priors).
  - *George Bush's optimal response*:
    - Stand firm since $EU_{Bush}(Stand) > EU_{Bush}(Cave)$
    - $EU_{Bush}(Stand) = 0.25 \times 3 + 0.75 \times 6 = 5.25$
    - $EU_{Bush}(Cave) = 0.25 \times 5 + 0.75 \times 5 = 5$
    - Shade "Stand firm" in the figure.$\longrightarrow$
  - *Kim's optimal messages*:
    - If Crazy, Stand firm (as prescribed) since 5>1.
    - If Sane, Cave (violation) since 7>5
- This strategy profile **cannot** be sustained as a PBE.

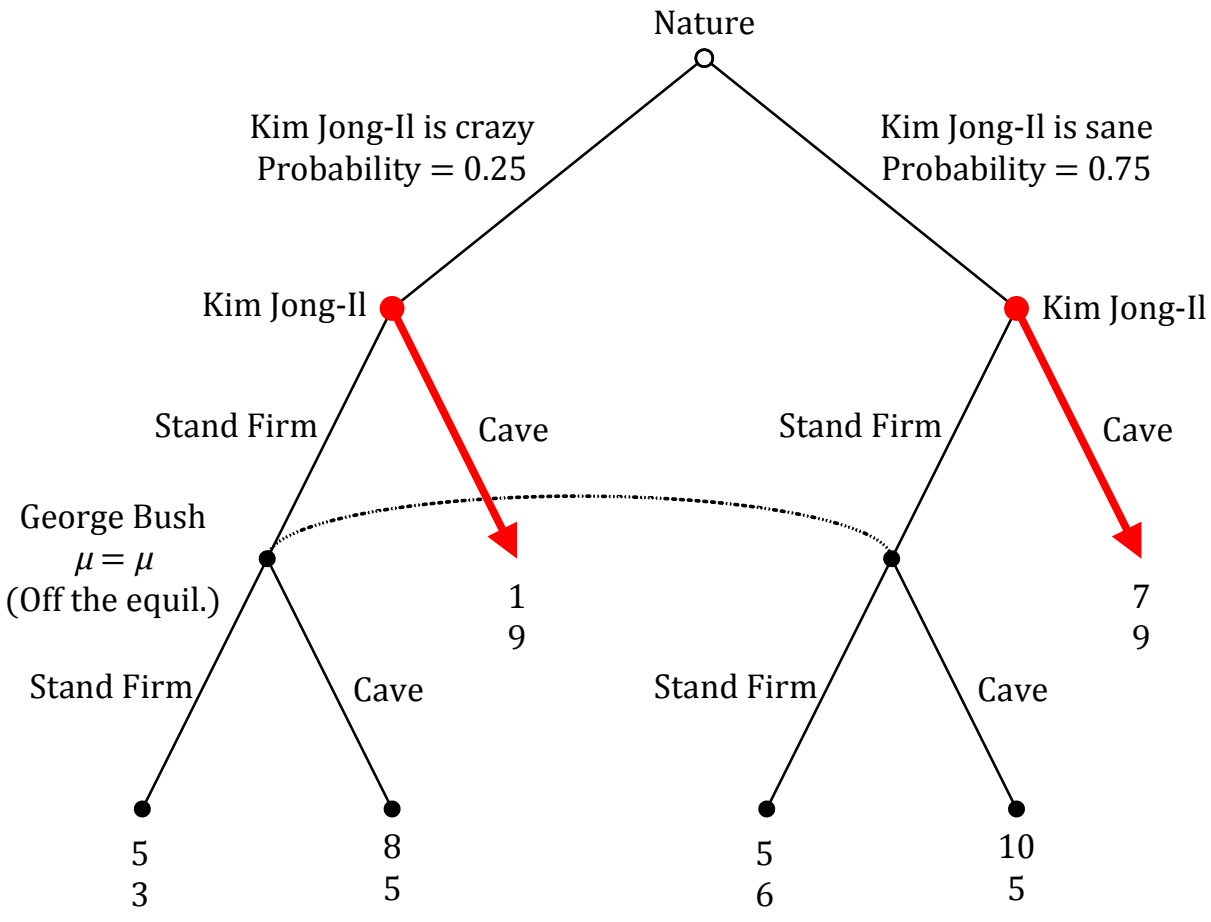
Brinkmanship



Brinkmanship

- Let us finally check the other type of pooling strategy profile, **(Cave,Cave)**, where
  - The crazy Kim-Jong Il caves, and
  - The sane Kim-Jong Il also caves.
- As usual, we shade the branches that correspond to this strategy profile in the next slide.

Brinkmanship



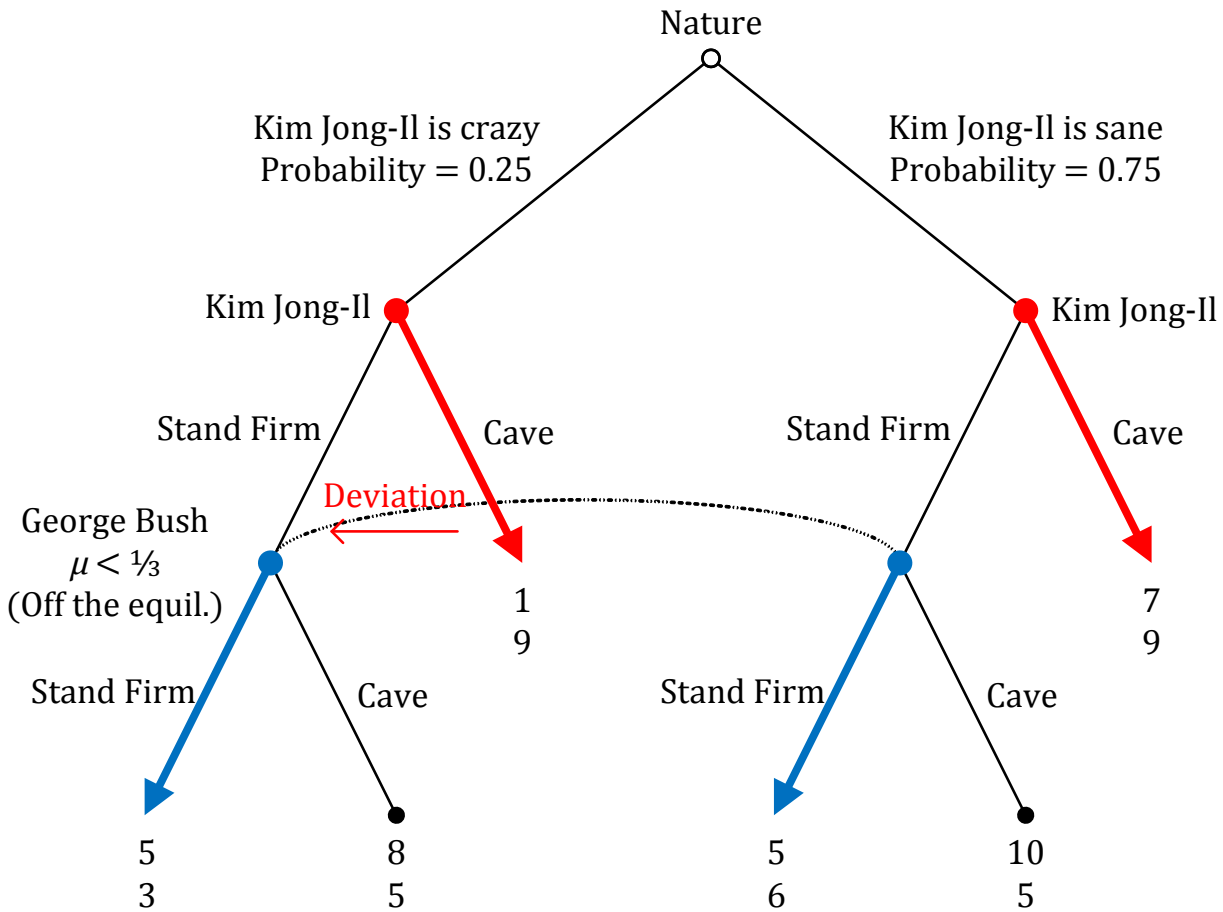
Brinkmanship

- **Pooling (Cave,Cave):**
  - *Beliefs*: $\mu = \frac{0.25 \times 0}{(0.25 \times 0) + (0.75 \times 0)} = \frac{0}{0}$,
    - i.e., $\mu \in [0, 1]$ (off-the-equil.)
  - *George Bush's optimal response*:
    - Stand firm since $EU_{Bush}(Stand) > EU_{Bush}(Cave)$
    - $EU_{Bush}(Stand) = \mu \times 3 + (1 - \mu) \times 6 = 6 - 3\mu$
    - $EU_{Bush}(Cave) = \mu \times 5 + (1 - \mu) \times 5 = 5$
    - Hence, Bush Stands firm iff $6 - 3\mu > 5 \iff \mu < \frac{1}{3}$.
  - We will hence need to construct two cases for Kim's optimal messages ($\mu < \frac{1}{3}$ and $\mu \geq \frac{1}{3}$).

Brinkmanship

- **Pooling (Stand,Stand):**
  - *Kim's optimal messages*:
    - **Case 1,** $\mu < \frac{1}{3}$ (Bush responds Standing firm):
    - (For a visual reference, see figure in the next slide)
    - If Crazy, Kim Stands firm (violation) since 5>1.
  - This strategy profile **cannot** be sustained as a PBE.

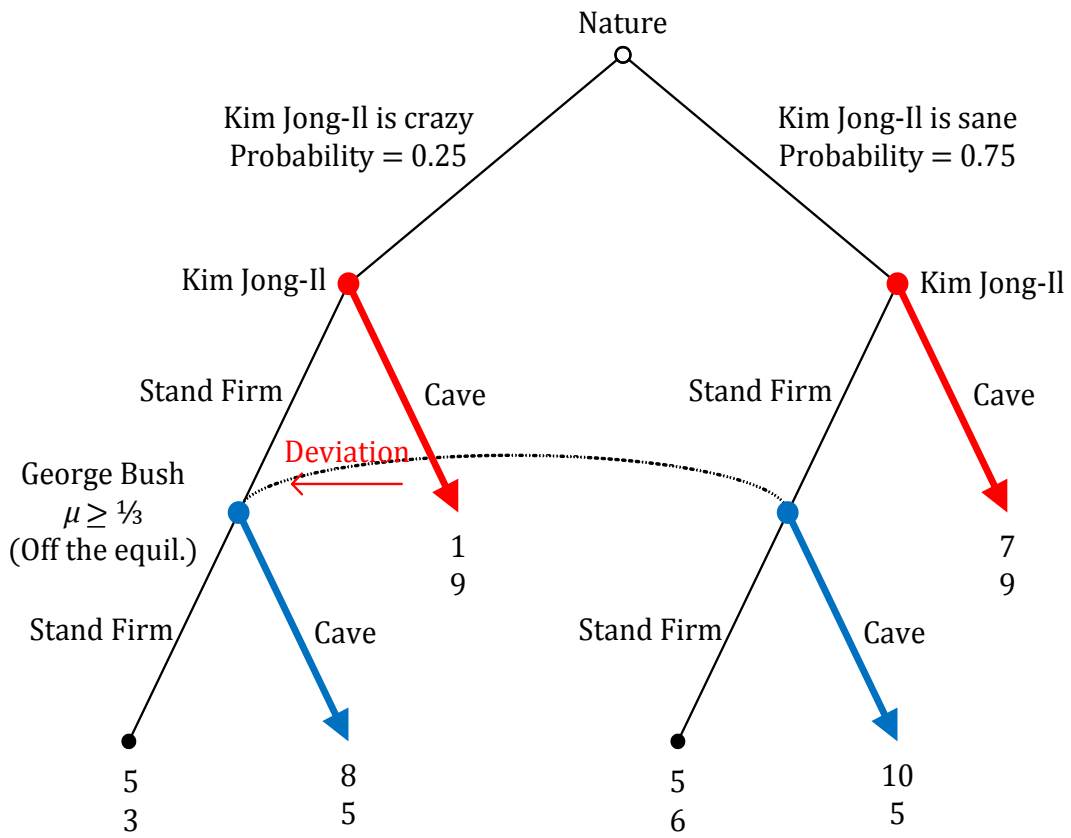
Brinkmanship



Brinkmanship

- **Pooling (Stand,Stand):**
  - *Kim's optimal messages*:
    - **Case 2**, $\mu \geq \frac{1}{3}$ (Bush responds Caving):
    - (For a visual reference, see figure in the next slide)
    - If Crazy, Kim Stands firm (violation) since 8>1.
  - This strategy profile **cannot** be sustained as a PBE either.

Brinkmanship



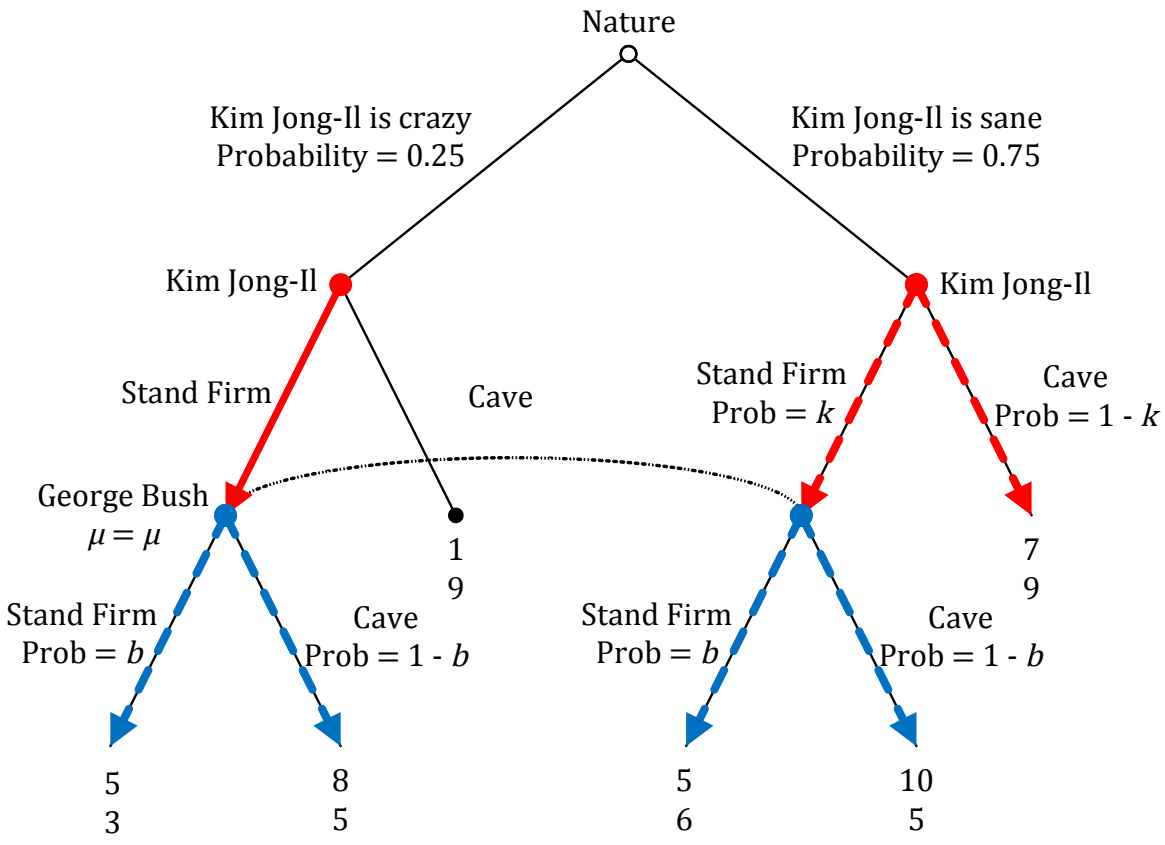
Brinkmanship

- No separating or pooling strategy profile can be sustained as a PBE...
  - when we restrict players to only use pure strategies.
- Let's next allow them to randomize!
  - Semi-separating PBE.

Brinkmanship

- **Let us consider the following Semi-separating PBE:**
- *Kim Jong-il's strategy:*
  - If crazy, choose stand firm.
  - If sane, choose stand firm with probability k.
- *George Bush's strategy:*
  - Choose stand firm with probability b.
- (Strategies depicted in the figure of the next slide)$\longrightarrow$

Brinkmanship



Brinkmanship

- **Beliefs:**
- After observing that NK's president *stands firm*, George Bush's beliefs about the mental state of Kim Jong-il are

$$\mu = \frac{0.25 \times 1}{0.25 \times 1 + 0.75 \times k} = \frac{0.25}{0.25 + 0.75k}$$

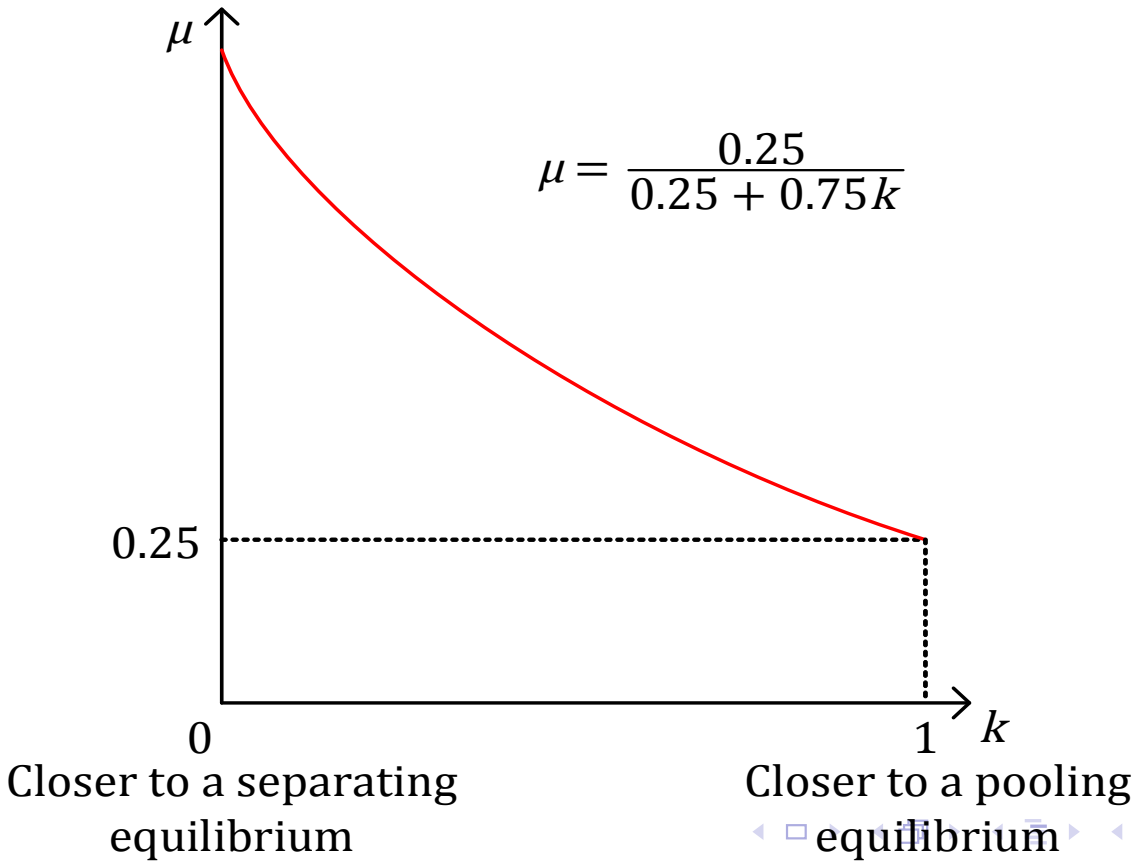
- After observing that NK's president *caves*, George Bush's beliefs are

$$\mu = \frac{0.25 \times 0}{0.25 \times 0 + 0.75 \times (1 - k)} = 0$$

  - Hence, George Bush puts a probability of 1 on the NK president being sane.

Brinkmanship

- George Bush's beliefs μ that Kim Jong-Il is sane
  - After observing that he stood firm, as a function of the probability k that Kim Jong-Il decides to stand firm when he is sane.



Brinkmanship

- **Second mover's strategy (George Bush):**
- After observing that NK's president stands firm, George Bush randomizes iff

$$5 = \frac{0.25}{0.25 + 0.75k} 3 + \left(1 - \frac{0.25}{0.25 + 0.75k}\right) 6$$

  - By caving, GB receives a payoff of 5, regardless of the true type of NK president.
  - By standing firm, he might induce a war (if dealing with a crazy type) resulting in a payoff of 3, or prevent the war (if dealing with a sane type) resulting in a payoff of 6.
- Solving for the only unknown, k, we obtain $k = 0.67$.

Brinkmanship

- **First mover's strategy (Kim Jong-il):**
- If he is crazy, his payoff from standing firm is higher than from caving, regardless of Bush's response. So NK president stands firms when crazy, as prescribed in this PBE.
- If he is sane, he gets a payoff of 7 from caving, and a payoff of 10 or 5, depending on Bush's reaction. Then, he randomizes with probability k such that

$$7 = b5 + (1 - b)10$$

  - And solving for b gives $b = 0.6$.

Brinkmanship

- **Summarizing**:
- Kim Jong-il's strategy:
  - If crazy, then choose stand firm, and if both North Korea and the US choose stand firm, then choose war.
  - If sane, then choose stand firm with probability $k = 0.67$, and if both North Korea and the US choose stand firm, then choose no war.
- George Bush's strategy:
  - Choose stand firm with probability $b = 0.6$, given beliefs

$$\mu = \frac{0.25}{0.25 + 0.75k} = \frac{0.25}{0.75} = 0.33$$

Brinkmanship

- Summary of the semi-separating PBE: