Dominance Solvable Games

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EconS 503

Solution Concepts

- The first solution concept we will introduce is that of deleting dominated strategies.
- Intuitively, we seek to delete from the set of strategies off every player those strategies that can never be beneficial for him regardless of the strategies selected by his opponents.
- Let's apply this solution concept to the standard prisoner's dilemma game and then we will define it more formally.

Dominated strategies in a PD game

- Prisoner's Dilemma. Let's first put ourselves in the shoes of player 1 (in rows)

		P_2	
		C	NC
P_1	C	-5,-5	0,-15
	NC	-15,0	-1,-1

- NC is strictly dominated for player 1, since

$$
\begin{aligned}
u_1(C, s_2) &> u_1(NC, s_2) \text{ for all } s_2 = \{C, NC\} \\
i.e., -5 &> -15 \text{ if } s_2 = C \\
0 &> -1 \text{ if } s_2 = NC
\end{aligned}
$$

Dominated strategies in a PD game

- A similar argument applies to Player 2

		P_2	
		C	NC
P_1	C	-5,-5	0,-15
	NC	-15,0	-1,-1

- *NC* is strictly dominated for player 2, since

$$\begin{aligned} u_2(C, s_1) &> u_2(NC, s_1) \text{ for all } s_1 = \{C, NC\} \\ i.e., -5 &> -15 \text{ if } s_1 = C \\ 0 &> -1 \text{ if } s_1 = NC \end{aligned}$$

- Hence, the only undominated (remaining) strategy for both player 1 and player 2 is to confess.

Dominated strategies - Definition

- **Definition of strictly dominated strategy**:
  - A strategy s_i^* is STRICTLY dominated by another strategy s_i' if the latter does strictly better than s_i^* against **every** strategy of the other players.

$$u_i\left(s_i', s_{-i}\right) > u_i\left(s_i^*, s_{-i}\right) \text{ for all } s_{-i} \in S_{-i}$$

- **Definition of weakly dominated strategy**:
  - A strategy s_i^* is WEAKLY dominated by another strategy s_i' if the latter does *at least as well* as s_i^* against every strategy of the other players, and against some strategy it does *strictly better*.

$$\begin{aligned} u_i\left(s_i', s_{-i}\right) &\geq u_i\left(s_i^*, s_{-i}\right) \text{ for all } s_{-i} \in S_{-i} \\ u_i\left(s_i', s_{-i}\right) &> u_i\left(s_i^*, s_{-i}\right) \text{ for at least one } s_{-i} \in S_{-i} \end{aligned}$$

Dominated Strategies

- Note that this definition is quite demanding:
  - A player must find that one of his available strategies yields a larger payoff than other of his available strategies, *regardless* of what his opponents select.
- For instance, in the prisoner's dilemma game, one of the players could say:
  - "I don't care about what my opponent selects, my payoff is always higher with C than with NC."
- It makes sense to never use a strategy that is strictly dominated: there must be another strategy that yields a larger payoff regardless of the strategy that my opponent selects.

Dominated Strategies

- For some games, we will be able to find strictly dominated strategies, and delete them, as they are never going to be used by rational players.
    - We will be able to identify strictly dominated strategies for each player, which allows us to delete them from the matrix...
        - Ultimately leaving us with a more reduced matrix.

- If after several rounds of deleting strictly dominated strategies, we identify that there are no more strictly dominated strategies that we can delete from the matrix, then the remaining cell (or cells) are our equilibrium prediction.
- Since in order to find this equilibrium (or equilibria) we iteratively delete those strategies that are strictly dominated, this procedure is referred as "Iterative Deletion of Strictly Dominated Strategies" (IDSDS).

Dominated Strategies

- Importantly, the application of IDSDS implies what we described the first day of class as "*Common Knowledge of rationality*":
  - Every player is rational:
    - A rational player would never use a strategy that yields a lower payoff than other available strategies, regardless of the strategy his opponent selects. That is, a rational player can actually delete those strategies that are strictly dominated from her set of available strategies.
  - Every player knows that every player is rational:
    - A rational player knows that he is competing against a rational player. Hence, player A can anticipate that player B would never be using strictly dominated strategies.
    - In other words, when player A considers which strategy to select, he does so by first deleting all strictly dominated strategies of player B from the matrix, since player B would never use them. This helps player A consider a reduced matrix with fewer cells to examine.

Dominated Strategies

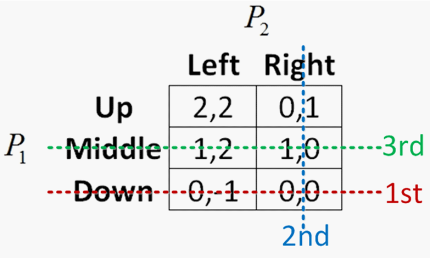
- "Common knowledge of rationality"(cont'd):
- Every player knows that every player knows that every player is rational:
  - By a similar argument, player A knows that player B has already deleted the strategies that are strictly dominated for player A, and that player B considers the reduced matrix once these strictly dominated strategies have been deleted.
- Every player knows that every player knows . . . ad infinitum.

Dominated Strategies

- For other games, however, we won't be able to find strictly dominated strategies.
  - The application of IDSDS doesn't have a bite, and all cells in the game are the "most precise" equilibrium prediction we can claim.
    - What a s. . . #%& (I mean imprecise) equilibrium prediction!
  - Don't despair:
    - We will discuss other solution concepts later on that will allow us to identify more precise equilibrium predictions.

Can we use IDSDS to solve a game?

- Consider the following 3×2 matrix:



- (Up,Left) is the only remaining strategy pair surviving IDSDS.
- These steps on the top of the same matrix can be confusing the first time we see them.
- Let's do one step at a time→

- More "step-by-step" presentation of the application of IDSDS to the previous example:

		P_2	
		Left	**Right**
	Up	2,2	0,1
P_1	**Middle**	1,2	1,0
	~~**Down**~~	~~0,1~~	~~0,0~~

1^{st} step

- First, player 1's utility satisfies:
  - $u_1(\text{Middle}, s_2) > u_1(\text{Down}, s_2)$ for any strategy s_2 that player 2 selects.
  - Hence, "DOWN" is strictly dominated for player 1, and we can delete it since he will never use it.
- Next step$\longrightarrow$

- Hence, the remaining matrix after the first step of deleting a strictly dominated strategies is the following 2×2 matrix:

P_2

P_1		Left	Right
	Up	2,2	0,1
	Middle	1,2	1,0

2[nd] step

- Secondly, player 2's utility satisfies:
  - $u_2(\text{Left}, s_1) > u_2(\text{Right}, s_1)$ for any s_1 chosen by player 1.
  - Hence, "Right" is a strictly dominated strategy for player 2, and we can delete it since he will never select it
- Next step$\longrightarrow$

- The remaining matrix after two steps of applying IDSDS is:

		P_2
		Left
P_1	**Up**	2,2
3[rd] step	~~**Middle**~~	~~1,2~~

- In particular, player 1's utility satisfies:
  - $u_1\left(\text{Up}, s_2\right) > u_1\left(\text{Middle}, s_2\right)$, i.e., $2 > 1$, s_2: only "Left".
  - Hence, "Middle" is a strategy dominated strategy for player 1, and we can delete it.
- Therefore, the only cell surviving IDSDS is that corresponding to strategy profile (Up,Left) with corresponding payoff $(2, 2)$.

The "Team project" game

- Harrington pp. 68-69.
- Consider that two random students in a class are grouped together in a project.
- Each student must independently choose whether to exert a high, moderate or low effort.
- Student types conform to the usual stereotypes:
    - Jocks (reaching for a C),
    - Frat boys and sorority girls (reaching for a B+), and
    - Nerds (reaching for an A).
- As put by Harrington: Is there anyone we haven't offended?

Team Project Between a Nerd and a Frat Boy

- We start applying IDSDS by putting ourselves in the shoes of the Nerd (column player): Low and Moderate effort are strictly dominated by High effort.

		Nerd		
		Low	*Moderate*	*High*
Frat boy	*Low*	0,1	2,2	6,3
	Moderate	1,2	4,3	5,4
	High	2,3	3,4	3,5

Strictly dominated for the Frat boy

Strictly dominated For the Nerd!

$u_{Nerd}(High, s_1) > u_{Nerd}(Low, s_1)$

$\Longrightarrow u_{Nerd}(High, s_1) > u_{Nerd}(Moderate, s_1)$

For any s_1 selected by the Frat boy

Team Project Between a Nerd and a Frat Boy

- The Frat boy, anticipating that the Nerd will exert a High effort (his only surviving strategy) deletes Moderate and High effort as they are strictly dominated by Low effort.
- Hence, only (Low,High) survives the application of IDSDS.

Natural question at this point: Does the order of deletion matter?

- No!
- That's great news: in your application of IDSDS, it does not matter which strategy you start deleting first,(whether you start identifying strictly dominated strategies for the row or the column player) you will end up finding the same strategy profile (or profiles).
- Check it with your classmate: start applying IDSDS in a large matrix (3×3 for instance), you will end up with the same equilibrium prediction.

The Doping Game

- Harrington, pp. 73-75
- So far we analyzed strictly dominated strategies with only two players.
- What if we have **three players**?

Ben chooses *steroids*

Carl

Maurice		Steroids	No steroids
	Steroids	2,3,3	3,1,5
	No steroids	1,4,5	5,2,6

Ben chooses *no steroids*

Carl

Maurice		Steroids	No steroids
	Steroids	3,4,1	4,2,2
	No steroids	5,5,2	6,6,4

The Doping Game

- First, we check if Ben(the matrix player) has some strictly dominated strategy.

Ben chooses *steroids*

Maurice \ Carl	*Steroids*	*No steroids*
Steroids	2,3,③	3,1,⑤
No steroids	1,4,⑤	5,2,⑥

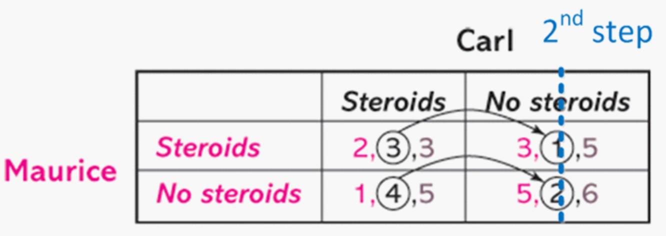
Ben chooses *no steroids*

Maurice \ Carl	*Steroids*	*No steroids*
Steroids	3,4,①	4,2,②
No steroids	5,5,②	6,6,④

- We compare $u_3(\text{steroids}, s_1, s_2)$ against $u_3(\text{No steroids}, s_1, s_2)$ where s_1 and s_2 are fixed across matrices.
- No steroids is a strictly dominated strategy for Ben, as it yields a lower payoff than steroids, for every profile (s_1, s_2) of the other two athletes.
- We can then delete "No steroids" from Ben by deleting the right hand matrix.

The Doping Game

- We are hence left with the left-hand matrix:
- We can now move to Carl (column player) and search whether he has strictly dominated strategies.



	Steroids	No steroids
Steroids	2,(3),3	3,(1),5
No steroids	1,(4),5	5,(2),6

- Similarlly as for Ben, Carl finds "No steroids" to be strictly dominated by "steroids", and hence we delete column two from the matrix.

The Doping Game

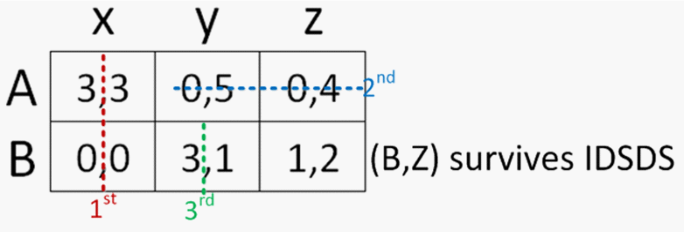
- Hence, the above matrix reduces to the following 2×1 matrix:

		Steroids	
Maurice	Steroids	2,3,3	
	No steroids	~~1,4,5~~	3^{rd} step

- Moving now to Maurice (row player), we note that "No steroids" is strictly dominated by "steroids".
- Hence, the only strategy profile surviving IDSDS is

$$(\text{Steroids},\text{Steroids},\text{Steroids})$$

Another example:



- Hence the only strategy profile surviving IDSDS is (B,Z).
- Great! But can we apply IDSDS to **all** kinds of games and obtain sharp predictions (a unique strategy profile)? No!

Imprecise equillibrium predictions

- Harrington, pp. 76-78

Player 2

Player 1	w	x	y	z
a	3,2	4,1	2,3	0,4
~~b~~	~~4,4~~	~~2,5~~	~~1,2~~	~~0,4~~
c	1,3	3,1	3,1	4,2
d	5,1	3,1	2,3	1,4

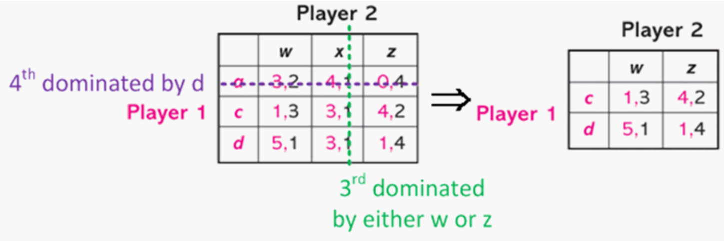
Row b: 1st (dominated by d)

Column y: 2nd (dominated by z)

- Now what? Too much information?
- Let's clarify things, by rewriting the matrix after deleting the strictly dominated strategies b and y.

Imprecise equillibrium predictions

- We can now detect that x is strictly dominated for player 2, and ...
- that a is strictly dominated for player 1, which yields the following 2×1 matrix:



- However,at this point, there are no more strictly dominated strategies for player 1 or player 2.

Imprecise equillibrium predictions

- Hence, the only prediction that we can make using IDSDS is that any of the following set of four strategy pairs surviving IDSDS can be ultimately played:
    - (c,W),(c,Z),(d,W),(d,Z).
- That's not a very precise prediction about how the game will be played!!

Battle of the Sexes game

- Another example where applying IDSDS doesn't allow us to identify a unique outcome:

		Wife	
		F	O
Husband	F	3,1	0,0
	O	1,3	1,3

- There are no strictly dominated strategies for the Husband.
- There are no strictly dominated strategies for the Wife.
- Hence, all four strategy profiles (F.F).(F,O),(O,F),(O,O), are the most precise equillibrium prediction we can provide applying IDSDS.

Battle of the Sexes game

- Is this imprecise equillibrium prediction only happening for this particular example?.. No!

Matching Pennies game

- Yet, another example in which the application of IDSDS yields imprecise equillibrium predictions:

		P_2 Head	P_2 Tails
P_1	Head	1,-1	-1,1
P_1	Tails	-1,1	1,-1

- There are no undominated strategies for Player 1 or Player 2.
  - No predictive power! (Inexistence of a single strategy part that perfectly describes how the game will be played).

Allowing for randomizations in IDSDS

- So far we applied IDSDS by checking if a player's play of a particular strategy (with 100% probability, also referred as a pure strategy) strictly dominates another pure strategy.
- What if we check if a mixed strategy (where the player randomizes among two or more strategies) strictly dominates another strategy?
  - Let's check this in the following example.

Allowing for randomizations in IDSDS

Player 2

		F	*C*	*B*
	F	0,5	2,3	2,3
Player 1	*C*	2,3	0,5	3,2
	B	5,0	3,2	2,3

- There are no strictly dominated strategies (using pure strategies) for player 1 nor for player 2.

Allowing for randomizations in IDSDS

		Player 2 F	C	B
	F	0,5	2,3	2,3
Player 1	C: (1-q)	2,3	0,5	3,2
	B: (q)	5,0	3,2	2,3

- **Mixing:** If player 1 mixes between B (with probability q) and C (with probability $1-q$) he obtains an expected utility that exceeds the utility from selecting F, regardless of what strategy player 2 chooses
  - In order to show this results, we must separately consider the case in which player 2 chooses F (left column), C (middle column), and B (right column).

Allowing for randomizations in IDSDS

			Player 2	
		F	C	B
Player 1	F	0,5	2,3	2,3
	C	2,3	0,5	3,2
	B	5,0	3,2	2,3

- **If player 2 chooses F** (the left column), then player 1's EU from mixing is $5q + 2(1-q)$, which exceeds his utility from F (zero), if

$$5q + 2(1-q) > 0 \Rightarrow 5q - 2q + 2 > 0 \Rightarrow q > -\frac{2}{3}$$

which holds by assumption given that $q \in [0, 1]$.

Allowing for randomizations in IDSDS

		Player 2 F	↓ C	B
Player 1	F	0,5	2,3	2,3
	C	2,3	0,5	3,2
	B	5,0	3,2	2,3

- **If player 2 chooses C** (the middle column), then player 1's EU from mixing is $3q + 0\,(1-q)$, which exceeds his utility from F (2), if

$$3q + 0\,(1-q) > 2 \Rightarrow 3q > 2 \Rightarrow q > \frac{2}{3}$$

- This is a necessary condition for the mixing to yield a larger EU than F. (We will keep it in mind.)

Allowing for randomizations in IDSDS

			Player 2 F	C	B
	F		0,5	2,3	2,3
Player 1	C: (1-q)		2,3	0,5	3,2
	B: (q)		5,0	3,2	2,3

- **If player 2 chooses B** (the right column), then player 1's EU from mixing is $2q + 3(1-q)$, which exceeds his utility from F (2), if

$$2q + 3(1-q) > 2 \Rightarrow 2q + 3 - 3q > 2 \Rightarrow 1 > q$$

which holds by assumption given that $q \in [0, 1]$.

Allowing for randomizations in IDSDS

- Hence, as long as player 1 mixes between B and C and assigns to B a probability $q > \frac{2}{3}$, he obtains an expected utility that exceeds the payoff he obtains from selecting F, regardless of the strategy player 2 selects.
- We can therefore delete strategy F from the matrix, since it is strictly dominated by a randomization between B and C, and thus player 1 would never use it.
- Deleting F (upper row), we obtain the following reduced matrix:

		Player 2		
		F	*C*	*B*
Player 1	*C*	2,3	0,5	3,2
	B	5,0	3,2	2,3

Allowing for randomizations in IDSDS

- Given the above reduced matrix, we can now move to player 2.
- Can we identify a strictly dominated strategy for player 2?

		Player 2		
		F	C	B
Player 1	C	2,3	0,5	3,2
	B	5,0	3,2	2,3

- Yes, F is strictly dominated by C for player 2.

Allowing for randomizations in IDSDS

- We can hence delete the column corresponding to strategy F for player 2 (since it is strictly dominated), yielding the following reduced 2×2matrix:

		Player 2	
		C	*B*
Player 1	*C*	0,5	3,2
	B	3,2	2,3

Allowing for randomizations in IDSDS

- However, for the remaining matrix does not have any further strictly dominated strategies that we can delete for player 1:

		Player 2	
		C	*B*
Player 1	*C*	0,5	3,2
	B	3,2	2,3

since

$$\underbrace{u_1(C, C)}_{0} < \underbrace{u_1(B, C)}_{3} \text{if player 2 plays } C$$

$$\text{But } \underbrace{u_1(C, B)}_{3} > \underbrace{u_1(B, B)}_{2} \text{if player 2 plays } B$$

- Hence, neither C is strictly dominated by B, nor B is strictly

Allowing for randomizations in IDSDS

- Nor for player 2. . .

		Player 2	
		C	*B*
Player 1	*C*	0,5	3,2
	B	3,2	2,3

$$u_2(C, C) \square u_2(C, B) \text{ if player 1 plays C (top row)}$$

$$\text{But } u_2(B, C) \square u_2(B, B) \text{ if player 1 plays B (bottom row)}$$

- Check signs ($>$ or $<$) as a practice.
- Hence, we cannot further eliminate strictly dominated strategies, and our (very imprecise) equilibrium prediction is $\{(C, C), (C, B), (B, C), (B, B)\}$.

Allowing for randomizations in IDSDS

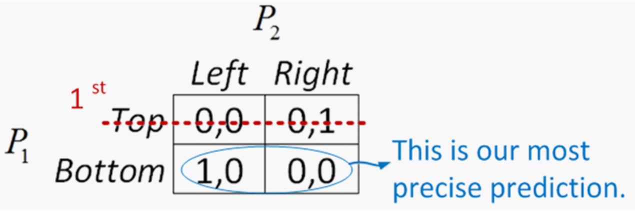
All strategy profiles
(9 in the previous example)

Strategy profiles surviving IDSDS in pure strategies
(9 as well in the previous example)

Strategy Profiles surviving IDSDS when we allow for mixed strategies.
(4 in the previous example)

Problems with dominance

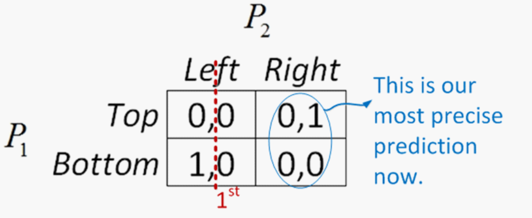
1. 1^{st} **problem**: Lack of predictive power in some games (see previous examples).
2. 2^{nd} **problem**: *Order of elimination matters*: only if we eliminate *weakly*(rather than strictly) dominated strategies.



- First, we eliminate Top as being weakly dominated by Bottom
- No further deletions for player 2 since he is indifferent between *Left* and *Right*.

Problems with dominance

- But what if we start by eliminating Left from Player 2 (it is a weakly dominated strategy for him).



- No further dominated strategies to delete since player 1 is indifferent between *Top* and *Bottom*.
- Bottom line: the set of strategies surviving IDWDS (NOT for IDSDS) depends on the order of deletion.

Problems with dominance:

3. **3^{rd} problem**: Layers of Rationality:
   - The application of IDSDS assumes the iterative thinking, sometimes requiring many layers (many steps).
   - In the prisoner's dilemma game it is reasonable to assume that my opponent will never use NC since it is strictly dominated. But we only require 2 steps of IDSDS in order to find a unique equilibrium prediction.
   - What about 3×3 matrices requiring many levels of IDSDS.
   - More importantly, what about 3×3 matrices for which we can only find strictly dominated strategies if we allow players to randomize?
     - Let's do one example (as a practice, and to confirm how cognitively demanding this process can be).

Problems with dominance

- Layers of rationality (Example of 3x3 matrix)

Player 2

		L	C : (p)	R : (1-p)
	U	5, 1	0, 4	1, 0
Player 1	M	3, 1	0, 0	1, 5
	D	3, 3	4, 4.5	2, 5

Problems with dominance

- As a practice, let's check if L is strictly dominated by a mixed strategy that puts p probability on C, and $1-p$ probability on R.
- Such mixed strategy yields a expected payoff for player 2 that exceeds his payoff from L:
  - If player 1 plays U (top row),

$$p \cdot 4 + (1-p) \cdot 0 = 4p > 1(\text{payoff from } L) \text{ if } p > \frac{1}{4}$$

  - If player 1 plays M (middle row),

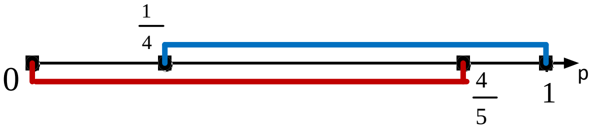
$$p \cdot 0 + (1-p) \cdot 5 = 5 - 5p > 1 \left(\text{from } L\right) \text{ if } 4 > 5p \Rightarrow \frac{4}{5} > p$$

  - If player 1 plays D (bottom row),

$$\begin{aligned} p \cdot 4.5 + (1-p) \cdot 5 &= 4.5p + 5 - 5p \\ &= 5 - 0.5p > 3 \left(\text{from } L\right) \text{ if } 2 > 0.5p \\ &\Rightarrow 4 > p \end{aligned}$$

Problems with dominance

- Hence, for all $p \in \left(\frac{1}{4}, \frac{4}{5}\right)$, the mixed strategy between C and R strictly dominates L, and we can delete L because of being strictly dominated.



Problems with dominance

- An example of the previous mixed strategy is one that assigns $p = \frac{1}{2}$, since $p = \frac{1}{2}$ indeed satisfies $p \in \left(\frac{1}{4}, \frac{4}{5}\right)$, which yields the following expected payoff for player 2:
  - If player 1 plays U,
  $$\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 0 = 2 > 1$$
  - If player 1 plays M,
  $$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 5 = \frac{5}{2} > 1$$
  - If player 1 plays D,
  $$\frac{1}{2} \cdot 4.5 + \frac{1}{2} \cdot 5 = \frac{9.5}{2} = 4.75 > 3$$
- And, hence, strategy L is strictly dominated by the mixed strategy that puts probability $p = \frac{1}{2}$to C and $1 - p = \frac{1}{2}$ to R.

Problems with dominance

- And the remaining matrix after deleting strategy L for player 2, is

		Player 2 C	Player 2 R
Player 1	U	0, 4	1, 0
Player 1	M	0, 0	1, 5
Player 1	D	4, 4.5	2, 5

Problems with dominance

- In the remaining matrix (after deleting strategy L), we can move to player 1, noting that U and M are strictly dominated by D.

			Player 2		
			C	R	
2nd Step		~~U~~	~~0, 4~~	~~1, 0~~	
	Player 1	~~M~~	~~0, 0~~	~~1, 5~~	2nd Step
		D	4, 4.5	2, 5	

Problems with dominance

- Hence, the remaining matrix is

		Player 2	
		C	R
Player 1	D	4, 4.5	2, 5

3rd Step

- Moving now to player 2, note that C is strictly dominated by R.
- Hence, (D, R) survives IDSDS, with associated payoffs $(2, 5)$.
- Can people go over 3 steps of IDSDS
    - (specifically when the first involves mixing)?

Problems with dominance

- **3^{rd} problem(cont'd)**: Layers of rationality
  - While in some games the layers of rationality might be demanding (as in the game we just analyzed)...
  - We can assume that, if the stakes are sufficiently high (millions of dollars), individuals or firms would spend as much time as necessary in order to carefully analyze these players (it took us only a few minutes!) in order to maximize their payoffs as much as possible.

Remark

- **Strict dominance equilibrium:**
  - Strategy profile $s^D = (s_1^D, s_2^D, ..., s_n^D)$ is a strict dominance equilibrium if s_i^D is a strictly dominant strategy for every player $i \in N$.
- **IDSDS:**
  - Strategy profile $s^{ED} = (s_1^{ED}, s_2^{ED}, ..., s_n^{ED})$ is iterated-dominance equilibrium if it survives IDSDS.
- Hence, if a strategy profile s is a strict dominance equilibrium, it must also survive IDSDS. (The opposite is not necessarily true, practice.)

Evaluating solution concepts

- **Strict dominance equilibrium:**
  - Existence? Not necessarily.
  - Uniqueness? Not necessarily.
  - Robustness? Yes.
- **IDSDS:**
  - Existence? Not necessarily, but better.
  - Uniqueness? Not necessarily, but better.
  - Robustness? Yes.
- **Pareto dominance:** the above equilibrium predictions are not necessarily Pareto optimal.