Bayesian Nash equilibrium

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- So far we assumed that all players knew all the relevant details in a game.
- Hence, we analyzed complete-information games.
- **Examples**:
  - Firms competing in a market observed each others' production costs,
  - A potential entrant knew the exact demand that it faces upon entry, etc.
- But, this assumption is not very sensible in several settings, where instead
  - players operate in *incomplete information* contexts.

Incomplete information:

- Situations in which one of the players (or both) knows some private information that is not observable by the other players.
- Examples:
  - Private information about marginal costs in Cournot competition,
  - Private information about market demand in Cournot competition,
  - Private information of every bidder about his/her valuation of the object for sale in an auction,

Incomplete information:

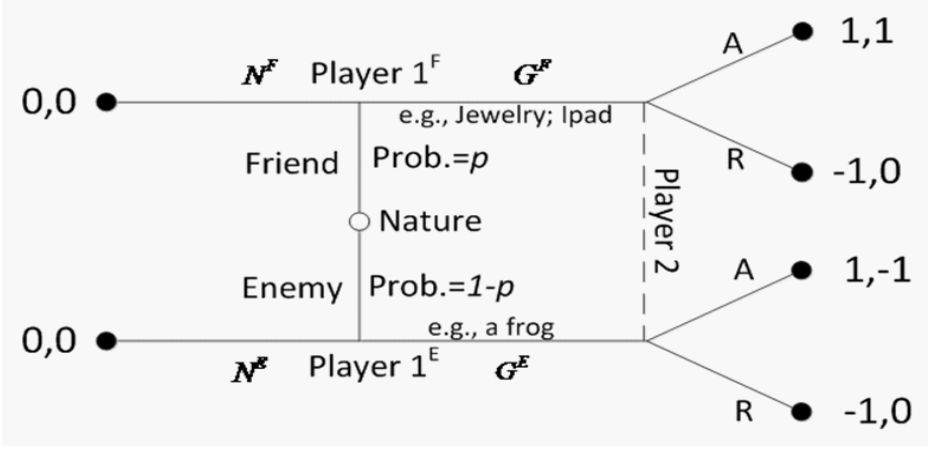
- We usually refer to this private information as "private information about player i's type, $\theta_i \in \Theta_i$"
- While uninformed players do not observe player i's type, θ_i, they know the probability (e.g., frequency) of each type in the population.
  - For instance, if $\Theta_i = \{H, L\}$, uninformed players know that $p(\theta_i = H) = p$ whereas $p(\theta_i = L) = 1 - p$, where $p \in (0, 1)$.

Reading recommendations:

- Tadelis:
  - Chapter 12.
- Osborne:
  - Chapter 9.
- Let us first:
  - See some examples of how to represent these incomplete information games using game trees.
  - We will then discuss how to solve them, i.e., finding equilibrium predictions.

Gift game

- Example #1



- Notation:

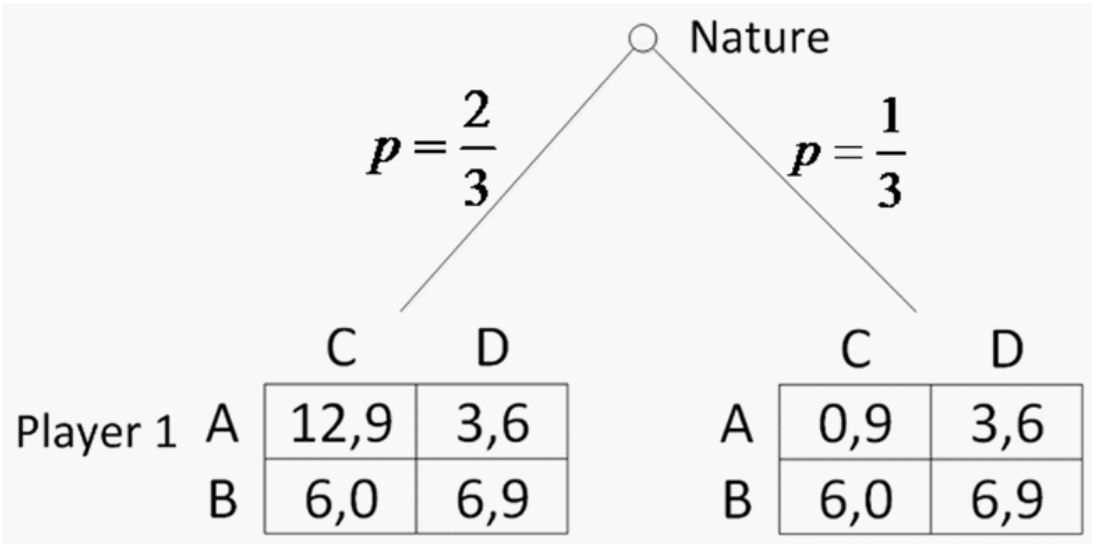
G^F : Player 1 makes a gift when being a "Friendly type";
G^E : Player 1 makes a gift when being a "Enemy type";
N^F : Player 1 does not make a gift when he is a "Friendly type";
N^E : Player 1 does not make a gift when he is a "Enemy type".

Properties of payoffs:

1. Player 1 is happy if player 2 accepts the gift:
    1. In the case of a Friendly type, he is just happy because of altruism.
    2. In the case of an Enemy type, he enjoys seeing how player 2 unwraps a box with a frog inside!
2. Both types of player 1 prefer not to make a gift (obtaining a payoff of 0), rather than making a gift that is rejected (with a payoff of -1).
3. Player 2 prefers:
    1. to accept a gift coming from a Friendly type (it is jewelry!!)
    2. to reject a gift coming from an Enemy type (it is a frog!!)

Another example

- Example #2
  - Player 1 observes whether players are interacting in the left or right matrix, which only differ in the payoff he obtains in outcome (A, C), either 12 or 0.



Another example

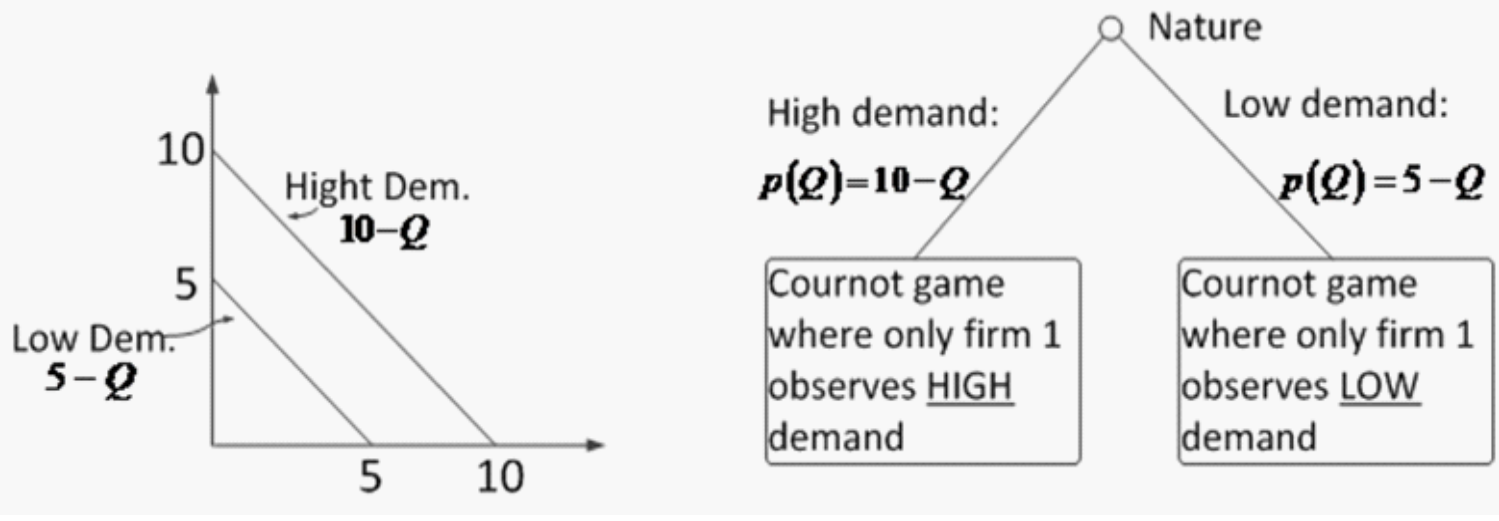
- Or more compactly...
- Player 2 is uninformed about the realization of x. Depending on whether $x = 12$ or $x = 0$, player 1 will have incentives to choose A or B, which is relevant for player 2.

		Player 2	
		C	D
Player 1	A	$x, 9$	3, 6
	B	6, 0	6, 9

$$\text{where } x = \begin{cases} 12 \text{ with probability } \frac{2}{3} \\ 0 \text{ with probability } \frac{1}{3} \end{cases}$$

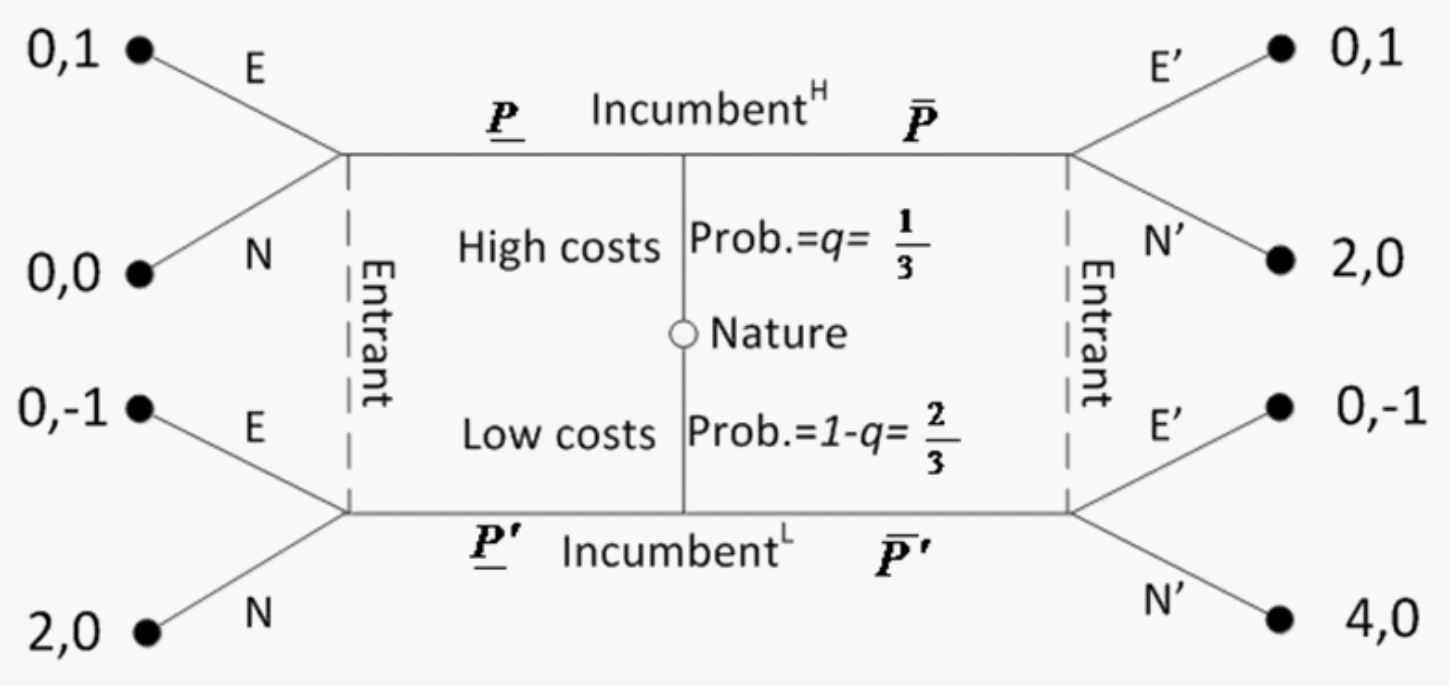
Another example:

- Example #3
  - Cournot game in which the new comer (firm 2) does not know whether demand is high or low, while the incumbent (firm 1) observes market demand after years operating in the industry.



Entry game with incomplete information:

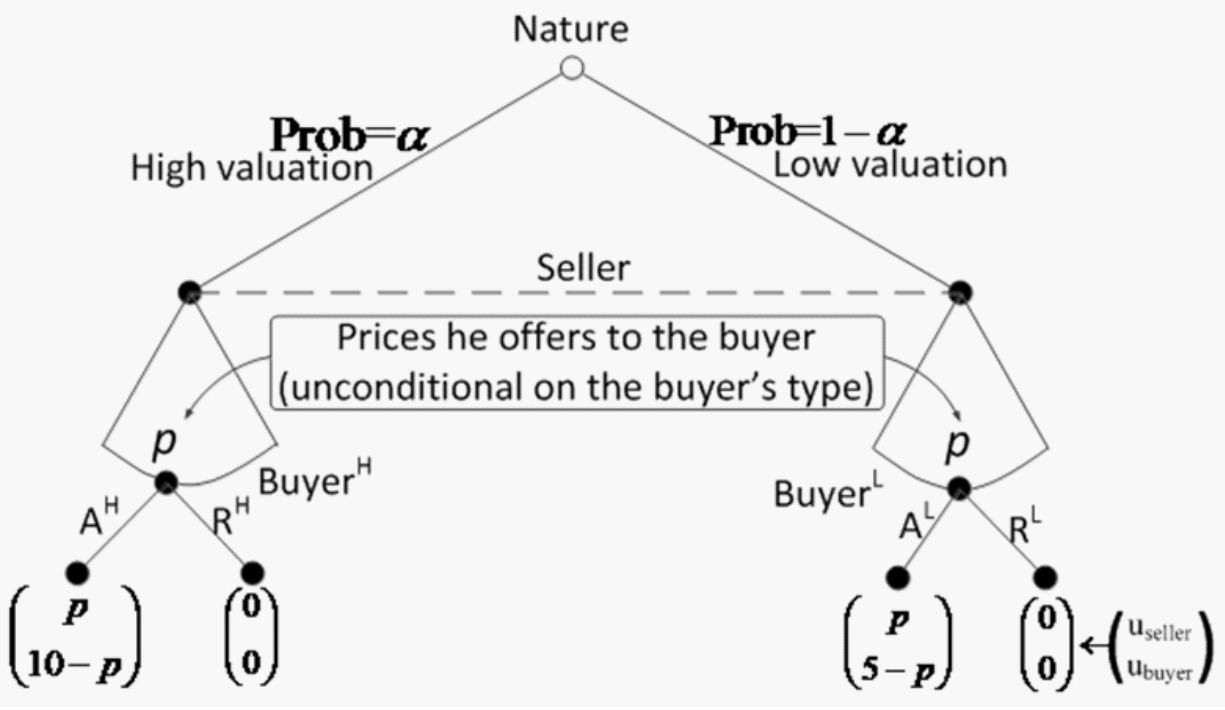
- Example #4: Entry game.



- **Notation:** E: enter, N: do not enter, $\underline{P}$: low prices, $\overline{P}$: high prices.

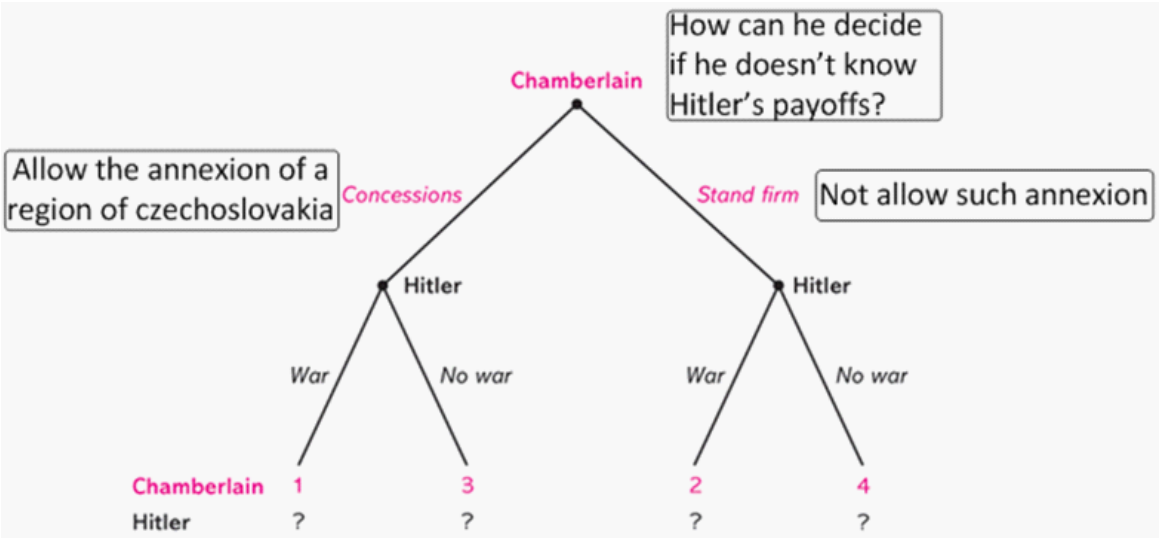
Bargaining with incomplete information (Example #5)

- Buyer has a high value from (10) or low valuation from (5) for the object (privately observed), and the seller is uninformed about such value.



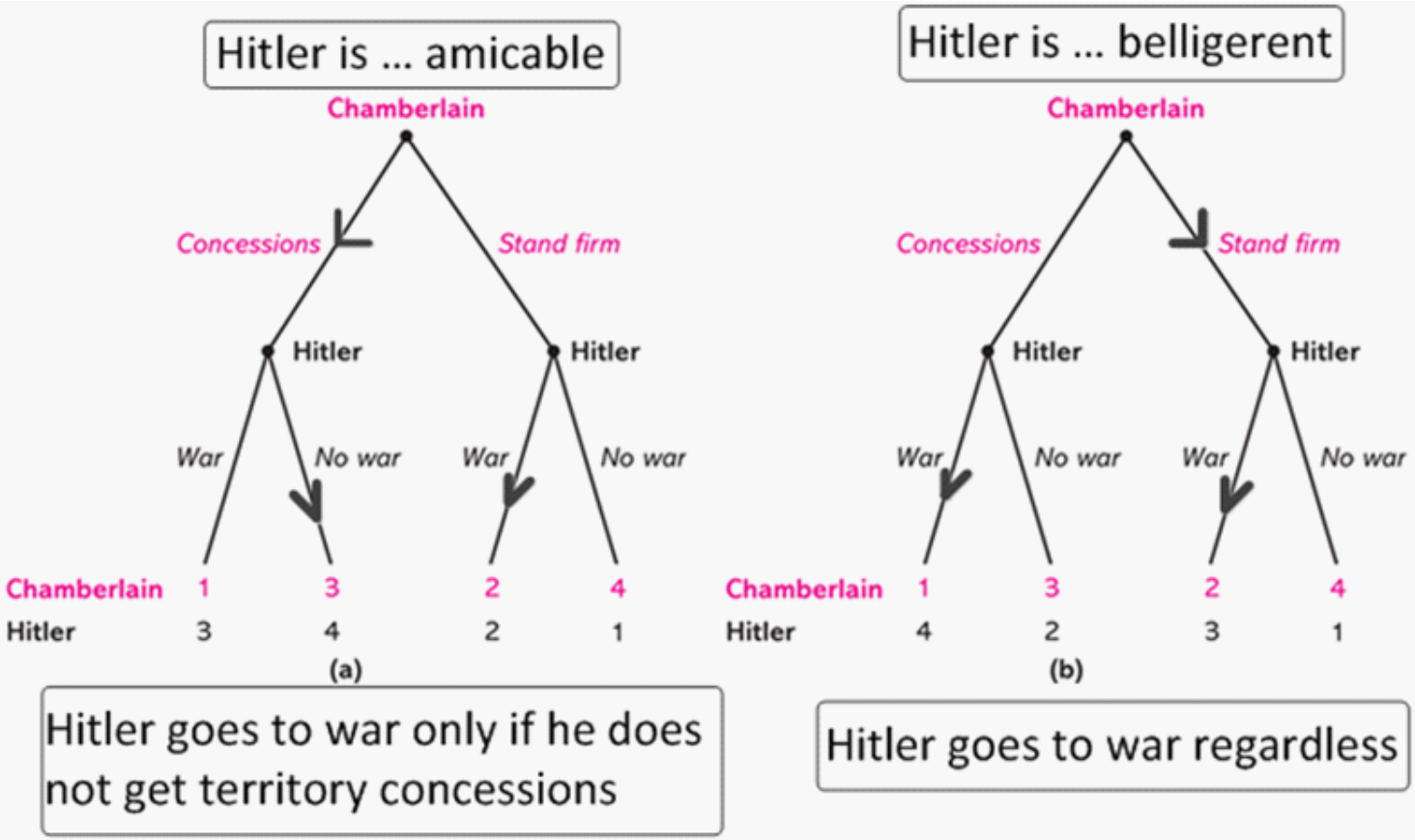
The Munich agreement (Example #6)

- It is 1938...
  - Hitler has invaded Czechoslovakia, and UK's prime minister, Chamberain, must decide whether to concede on such annexation to Germany or stand firm not allowing the occupation.
  - Chamberlain does not know Hitler's incentives as he cannot observe Hitler's payoff.

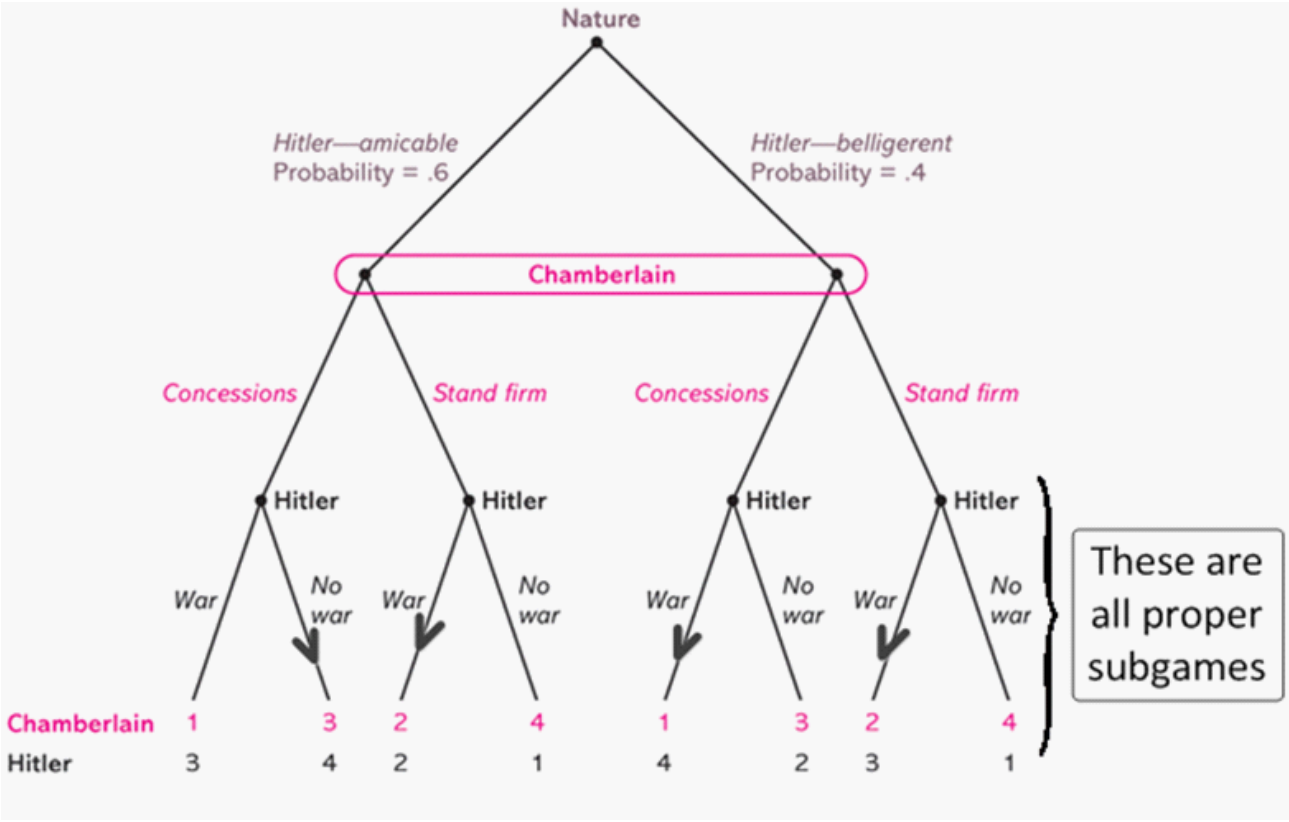


The Munich agreement

- Well, Chamberlain knows that Hitler can either be belligerent or amicable.

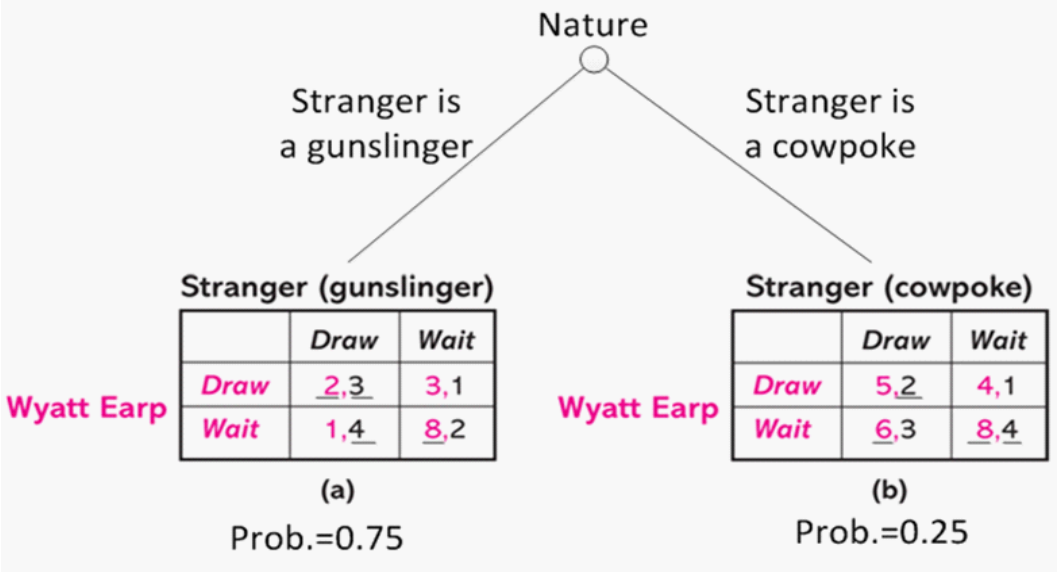


- How can we describe the above two possible games Chamberlain could face by using a single tree?
  - Simply introducing a previous move by nature which determines the "type" of Hitler: graphically, connecting with an information set the two games we described above.



Gunfight in the wild west

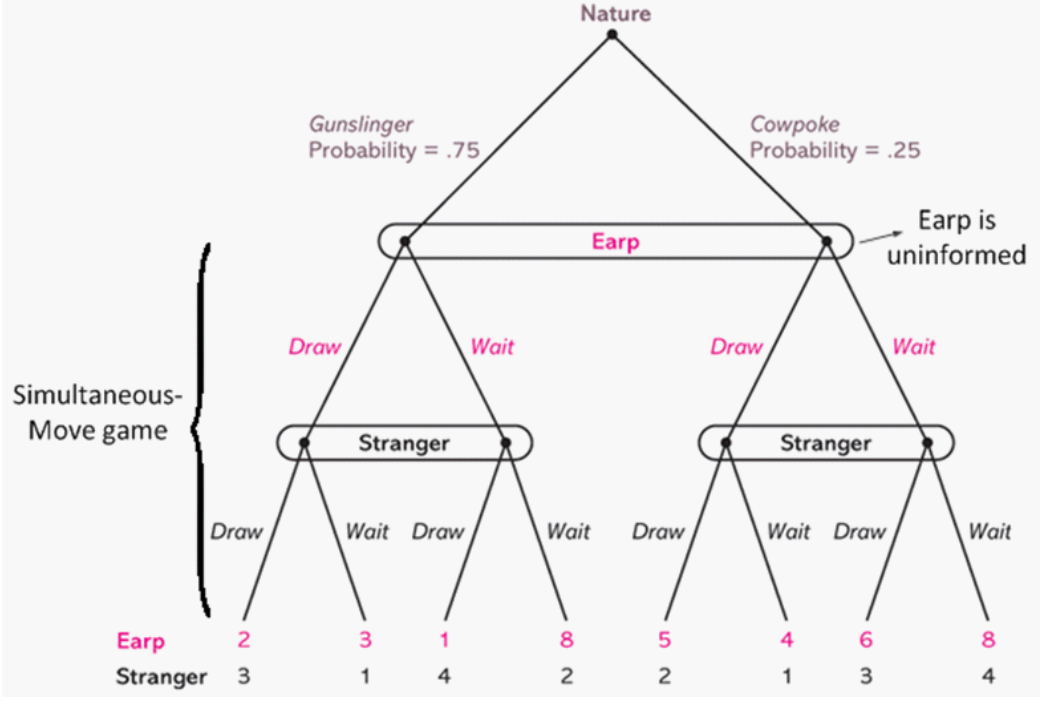
- Example #7



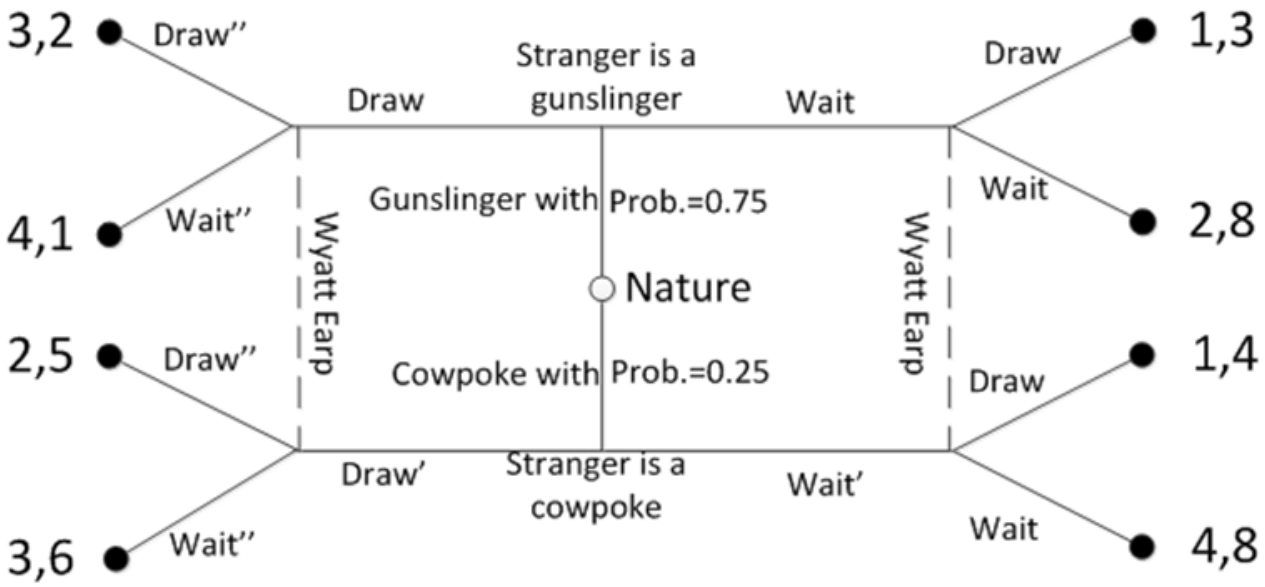
- We cannot separately analyze best responses in each payoff matrix since by doing that, we are implicitly assuming that Wyatt Earp knows the ability of the stranger (either a gunslinger or cowpoke) *before* choosing to draw or wait.
- Wyatt Earp doesn't know that!

How to describe Wyatt Earp's lack of information about the stranger's abilities?

- We can depict the game tree representation of this incomplete information game, by having nature determining the stranger's ability at the beginning of the game.



- Why don't we describe the previous incomplete information game using the following figure?



- No! This figure indicates that the stranger acts first, and Earp responds to his action,
- the previous figure illustrated that, after nature determines the stranger's ability, the game he and Earp play is simultaneous; as opposed to sequential in this figure.

Common features in all of these games:

- One player observes some piece of information
- The other player's cannot observe such element of information.
  - e.g., market demand, production costs, ability...
  - Generally about his type θ_i.
- We are now ready to describe how can we solve these games.
  - Intuitively, we want to apply the NE solution concept, but...
  - taking into account that some players maximize **expected** utility rather than simple utility, since they don't know which type they are facing (i.e., uncertainty).

Common features in all of these games:

- In addition, note that a strategy s_i for player i must now describe the actions that player i selects given that his privately observed type (e.g., ability) is θ_i.
  - Hence, we will write strategy s_i as the function $s_i(\theta_i)$.
- Similarly, the strategy of all other players, s_{-i}, must be a function of their types, i.e., $s_{-i}(\theta_{-i})$.

Common features in all of these games:

- Importantly, note that every player conditions his strategy on his own type, but not on his opponents' types, since he cannot observe their types.
  - That's why we don't write strategy s_i as $s_i(\theta_i, \theta_{-i})$.
  - If that was the case, then we would be in a complete information game, as those we analyzed during the first half of the semester.
- We can now define what we mean by equilibrium strategy profiles in games of incomplete information.

Bayesian Nash Equilibrium

- **Definition:** A strategy profile $(s_1^*(\theta_1), s_2^*(\theta_2), ..., s_n^*(\theta_n))$ is a Bayesian Nash Equilibrium of a game of incomplete information if

$$EU_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}) \geq EU_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i})$$

  for every $s_i(\theta_i) \in S_i$, every $\theta_i \in \Theta_i$, and every player i.
- In words, the expected utility that player i obtains from selecting $s_i^*(\theta_i)$ when his type is θ_i is larger than that of deviating to any other strategy $s_i(\theta_i)$. This must be true for all possible types of player i, $\theta_i \in \Theta_i$, and for all players $i \in N$ in the game.

Bayesian Nash Equilibrium

- Note an alternative way to write the previous expression, expanding the definition of expected utility:

$$\begin{aligned} & \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \times u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}) \\ \geq & \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \times u_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}) \end{aligned}$$

  for every $s_i(\theta_i) \in S_i$, every $\theta_i \in \Theta_i$, and every player i.

- Intuitively, $p\left(\theta_{-i} \middle| \theta_i\right)$ represents the probability that player i assigns, after observing that his type is θ_i, to his opponents' types being θ_{-i}.

Bayesian Nash Equilibrium

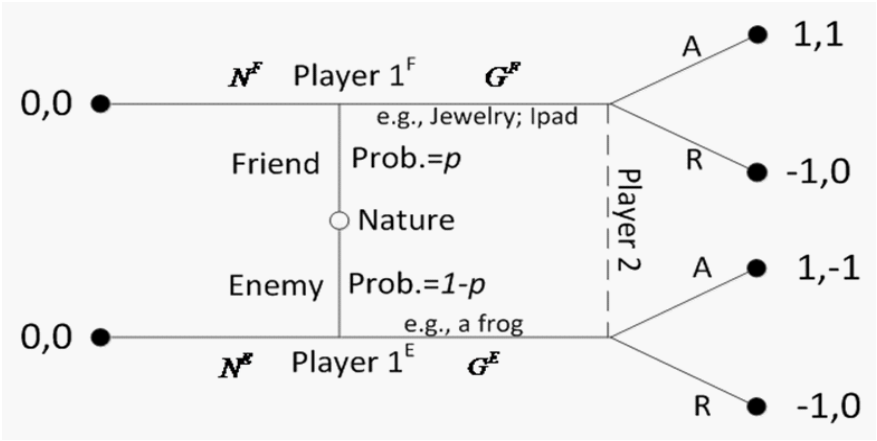
- For many of the examples we will explore $p\left(\theta_{-i} \middle| \theta_i\right) = p\left(\theta_{-i}\right)$ (e.g., $p\left(\theta_i\right) = \frac{1}{3}$), implying that the probability distribution of my type and my rivals' types are independent.
- That is, observing my type doesn't provide me with any more accurate information about my rivals' type than what I know before observing my type.

Bayesian Nash Equilibrium

- Let's apply the definition of BNE into some of the examples we described above about games of incomplete information.

Gift game

- Let's return to the game in Example #1



- Notation:

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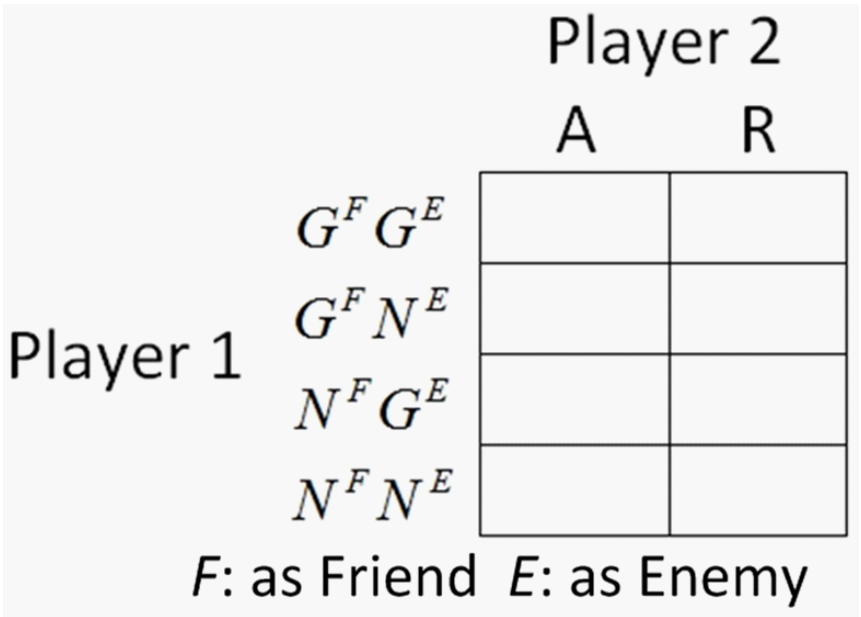
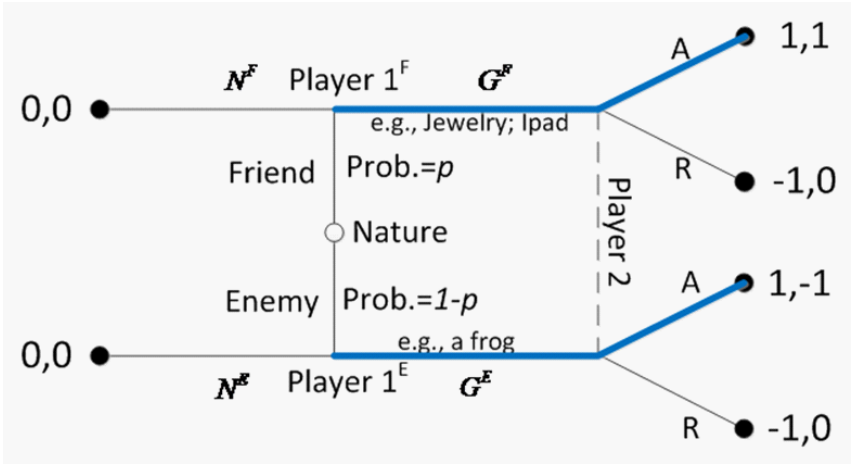
G^E : Player 1 makes a gift when being a "Enemy type";
N^F : Player 1 does not make a gift when he is a "Friendly type";
N^E : Player 1 does not make a gift when he is a "Enemy type".

"Bayesian Normal Form" representation

- Let us now transform the previous extensive-form game into its "Bayesian Normal Form" representation.
- **1st step** identify strategy spaces:
  - Player 2, $S_2 = \{A, R\}$
  - Player 1, $S_1 = \left\{ G^F G^E, G^F N^E, N^F G^E, N^F N^E \right\}$



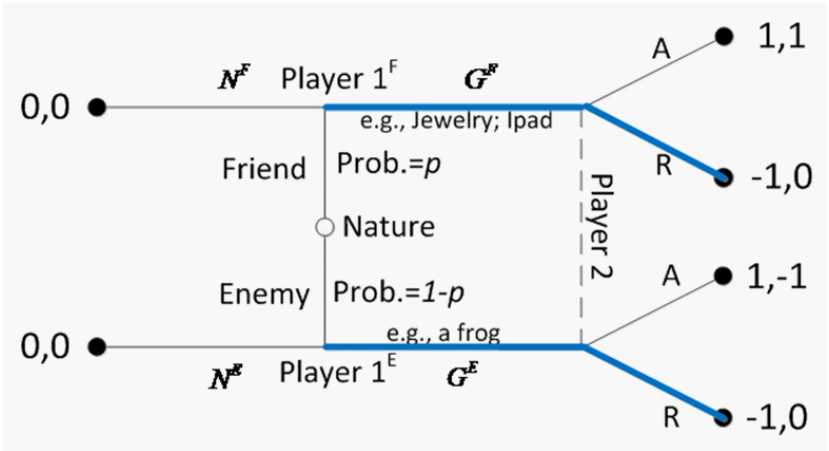
- **2nd step:** Identify the expected payoffs in each cell of the matrix.
- Strategy $\left(G^F\, G^E, A\right)$, and its associated expected payoff:



$$\begin{aligned} Eu_1 &= p \cdot 1 + (1-p) \cdot 1 = 1 \\ Eu_2 &= p \cdot 1 + (1-p) \cdot (-1) = 2p - 1 \end{aligned}$$

- Hence, the payoff pair $(1, 2p-1)$ will go in the cell of the matrix corresponding to strategy profile $\left(G^F\, G^E, A\right)$.

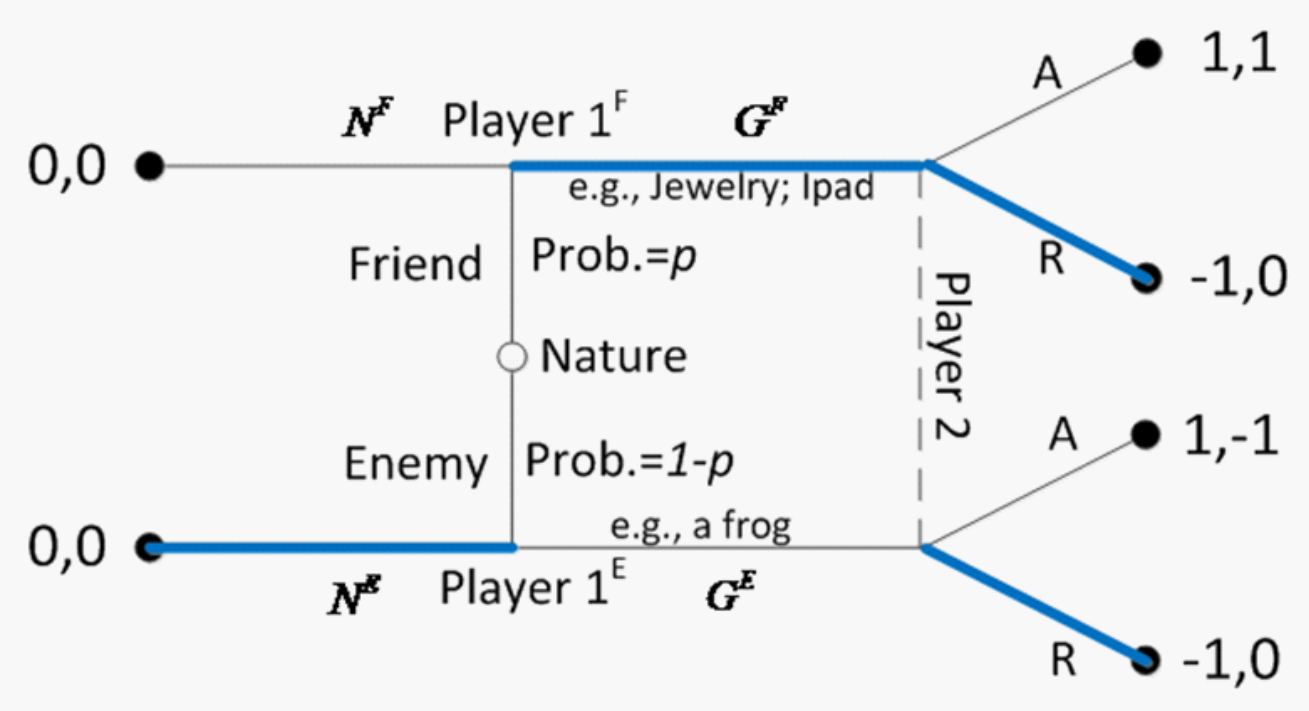
- **2nd step:** Identify the expected payoffs in each cell of the matrix.
- Strategy $\left(G^F G^E, R\right)$, and its associated expected payoff:



$$
\begin{aligned}
Eu_1 &= p \cdot (-1) + (1-p) \cdot (-1) = -1 \\
Eu_2 &= p \cdot 0 + (1-p) \cdot 0 = 0
\end{aligned}
$$

- Hence, the payoff pair $(-1, 0)$ will go in the cell of the matrix corresponding to strategy profile $\left(G^F G^E, R\right)$.

- Strategy $\left(G^F N^E, R\right)$, and its associated expected payoff:



$$\begin{aligned} Eu_1 &= p \cdot (-1) + (1-p) \cdot 0 = -p \\ Eu_2 &= p \cdot 0 + (1-p) \cdot 0 = 0 \end{aligned}$$

- Hence, expected payoff pair $(-p, 0)$

a) $\left(G^F G^E, A\right) \rightarrow (1, 2p-1)$:

$$\begin{aligned} Eu_1 &= p \cdot 1 + (1-p) \cdot 1 = 1 \\ Eu_2 &= p \cdot 1 + (1-p) \cdot (-1) = 2p-1 \end{aligned}$$

b) $\left(G^F G^E, R\right) \rightarrow (-1, 0)$:

$$\begin{aligned} Eu_1 &= p \cdot (-1) + (1-p) \cdot (-1) = -1 \\ Eu_2 &= p \cdot 0 + (1-p) \cdot 0 = 0 \end{aligned}$$

c) $\left(G^F N^E, A\right) \rightarrow$:

$$\begin{aligned} Eu_1 &= \\ Eu_2 &= \end{aligned}$$

d) $\left(G^F N^E, R\right) \rightarrow (-p, 0)$:

$$\begin{aligned} Eu_1 &= p \cdot (-1) + (1-p) \cdot 0 = -p \\ Eu_2 &= p \cdot 0 + (1-p) \cdot 0 = 0 \end{aligned}$$

- Practice:

e) $\left(N^F G^E, A\right) \rightarrow$:

$$Eu_1 =$$

$$Eu_2 =$$

f) $\left(N^F G^E, R\right) \rightarrow$:

$$Eu_1 =$$

$$Eu_2 =$$

g) $\left(N^F N^E, A\right) \rightarrow$:

$$Eu_1 =$$

$$Eu_2 =$$

h) $\left(G^F N^E, R\right) \rightarrow$:

$$Eu_1 =$$

$$Eu_2 =$$

- Inserting the expected payoffs in their corresponding cell, we obtain

		Player 2	
		A	R
Player 1	$G^F G^E$	$1, 2p-1$	$-1,\ 0$
	$G^F N^E$	$p,\ p$	$-p,\ 0$
	$N^F G^E$	$1-p,\ p-1$	$p-1, 0$
	$N^F N^E$	$0,\ 0$	$0,\ 0$

- **3rd step:** Underline best response payoffs in the matrix we built.

		Player 2	
		A	R
Player 1	$G^F G^E$	$\underline{1}, \underline{2p-1}$	$-1, \underline{0}$
	$G^F N^E$	$p, \underline{p}$	$-p, 0$
	$N^F G^E$	$1-p, p-1$	$p-1, \underline{0}$
	$N^F N^E$	$0, \underline{0}$	$\underline{0}, \underline{0}$

- If $p > \frac{1}{2}$ $(2p-1>0) \Rightarrow$ 2 B.N.Es: $\left(G^F G^E, A\right)$ and $\left(N^F N^E, R\right)$
- If $p < \frac{1}{2}$ $(2p-1<0) \Rightarrow$ only one B.N.E: $\left(N^F N^E, R\right)$

- If, for example, $p = \frac{1}{3}$ (implying that $p < \frac{1}{2}$), the above matrix becomes:

		Player 2	
		A	R
Player 1	$G^F G^E$	$\underline{1}, -\frac{1}{3}$	$-1, \underline{0}$
	$G^F N^E$	$\frac{1}{3}, \underline{\frac{1}{3}}$	$-\frac{1}{3}, 0$
	$N^F G^E$	$\frac{2}{3}, -\frac{2}{3}$	$-\frac{2}{3}, \underline{0}$
	$N^F N^E$	$0, \underline{0}$	$\underline{0}, \underline{0}$

- Only one BNE: $(N^F N^F, R)$

- **Practice:** Can you find two BNE for $p = \frac{2}{3}? > \frac{1}{2} \Rightarrow$ 2 BNEs.

Player 1	Player 2: A	Player 2: R
$G^F G^E$		
$G^F N^E$		
$N^F G^E$		
$N^F N^E$		

- Just plug $p = \frac{2}{3}$ into the matrix 2 slides ago.
- You should find 2 BNEs.

Another game with incomplete information

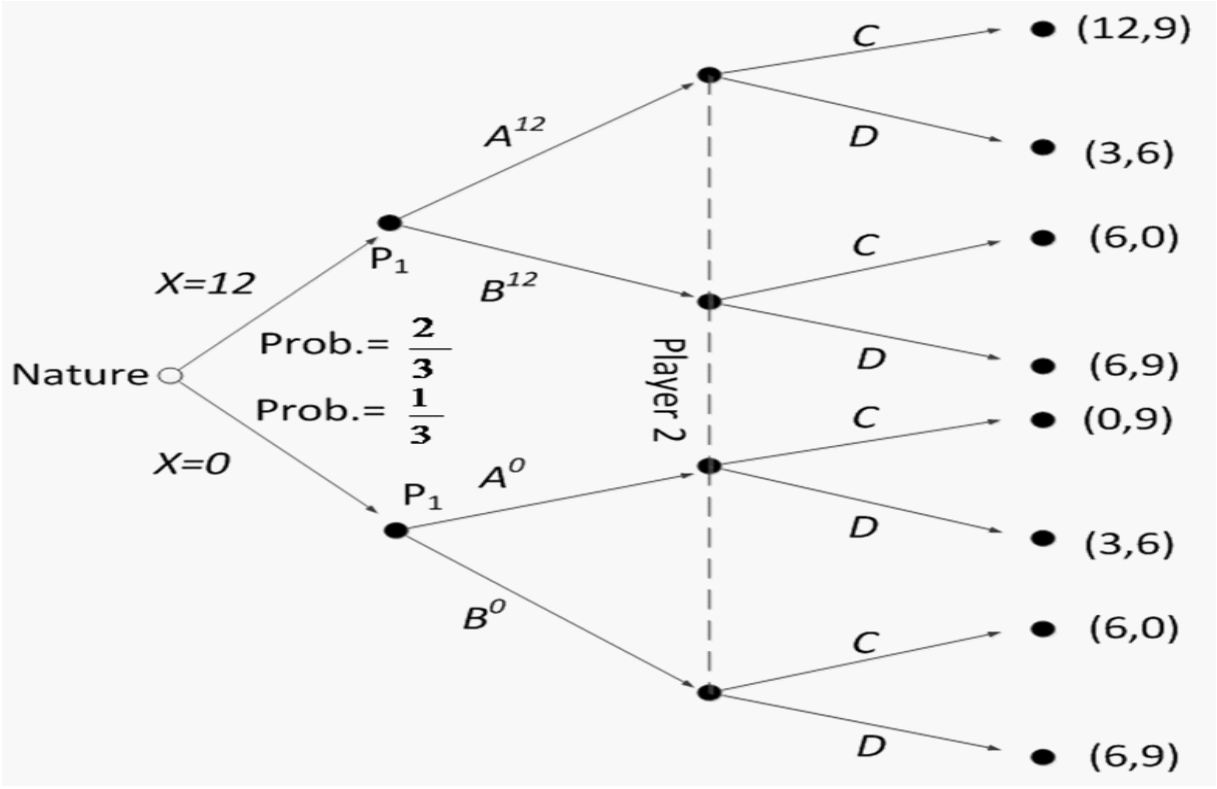
- Example #2:

		Player 2	
		C	D
Player 1	A	$x, 9$	3, 6
	B	6, 0	6, 9

$$\text{where } x = \begin{cases} 12 \text{ with probability } \frac{2}{3} \\ 0 \text{ with probability } \frac{1}{3} \end{cases}$$

- Extensive form representation$\rightarrow$figure in next slide.
- Note that player 2 here:
  - Does not observe player 1's type nor his actions $\rightarrow$ long information set.

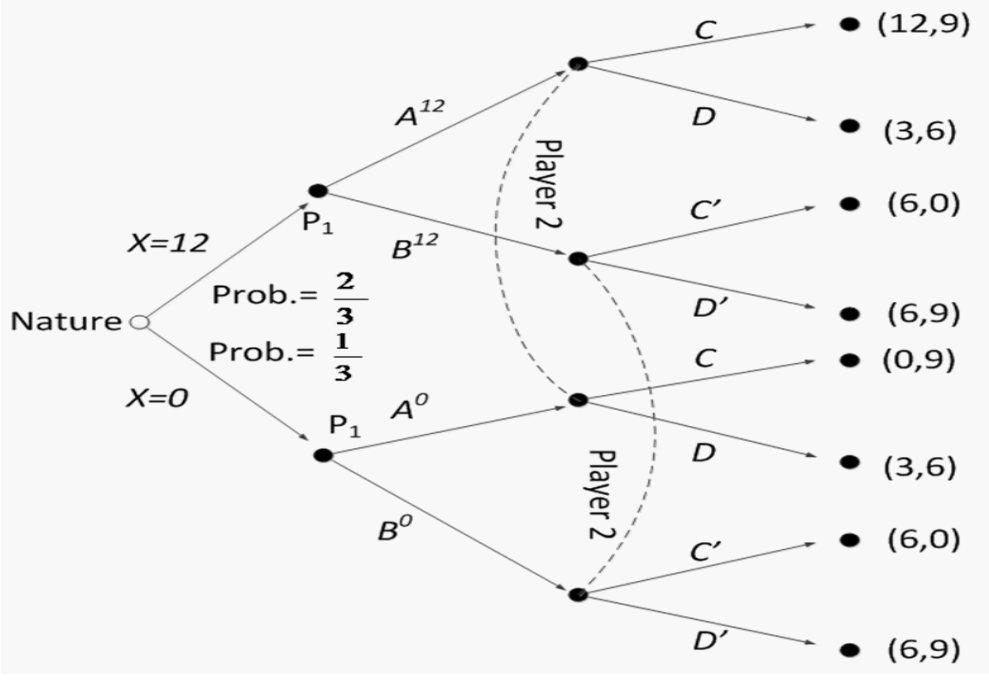
Extensive-Form Representation



- The dashed line represents that player 2 doesn't observe player 1's type nor his actions (long information set).

Extensive-Form Representation

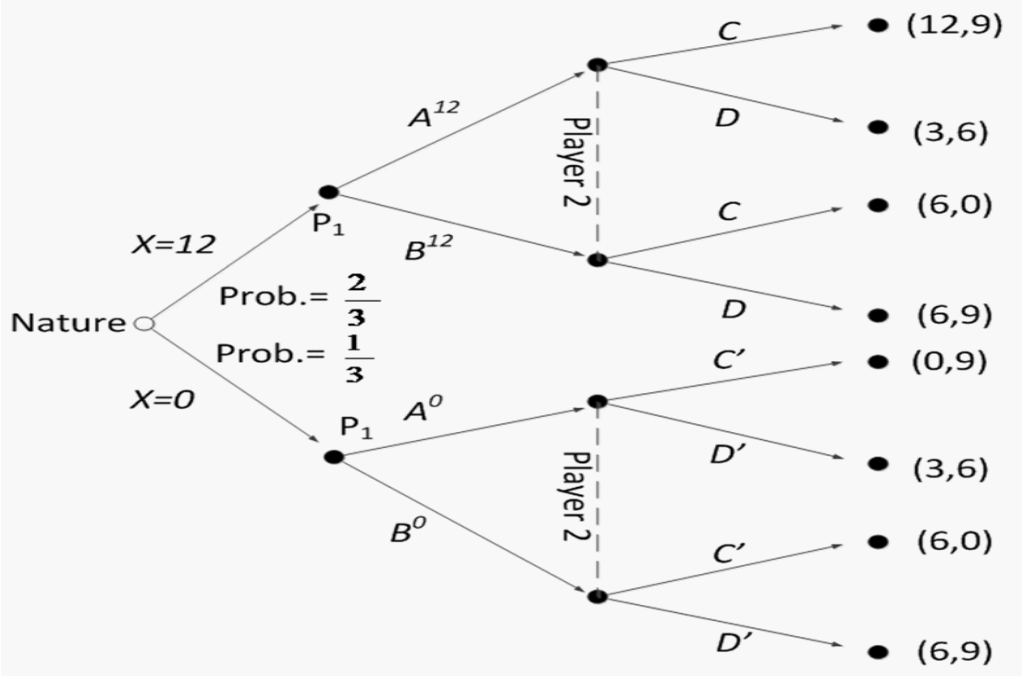
- What if player 2 observed player 1's action but not his type:



- We denote:
- C and D after observing A;
- C' D' after observing B

Extensive-Form Representation

- What if player 2 could observed player 1's type but not his action:



- We denote:
  - C and D when player 2 deals with a player 1 with $x = 12$
  - C' and D' when player 2 deals with a player 1 with $x = 0$.

- How to construct the Bayesian normal form representation of the game in which player 2 cannot observe player 1's type nor his actions depicted in the game tree two slides ago?
- **1st step:** Identify each player's strategy space.

$$S_2 = \{C, D\}$$
$$S_1 = \left\{A^{12}A^0, A^{12}B^0, B^{12}A^0, B^{12}B^0\right\}$$

- where the superscript 12 means $x = 12$, 0 means $x = 0$.

- Hence the Bayesian normal form is:

Player 1	Player 2: C	Player 2: D
$A^{12}A^{0}$		
$A^{12}B^{0}$		
$B^{12}A^{0}$		
$B^{12}B^{0}$		

- Let's find out the expected payoffs we must insert in the cells...

- **2nd step:** Find the expected payoffs arising in each strategy profile and locate them in the appropriate cell:

a) $\left(A^{12}A^{0}, C\right)$

$$\left.\begin{array}{l} Eu_1 = \frac{2}{3} \cdot 12 + \frac{1}{3} \cdot 0 = 8 \\ Eu_2 = \frac{2}{3} \cdot 9 + \frac{1}{3} \cdot 9 = 9 \end{array}\right\} \rightarrow (8, 9)$$

b) $\left(A^{12}A^{0}, D\right)$

$$\left.\begin{array}{l} Eu_1 = \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 3 = 3 \\ Eu_2 = \frac{2}{3} \cdot 6 + \frac{1}{3} \cdot 6 = 6 \end{array}\right\} \rightarrow (3, 6)$$

c) $\left(A^{12}B^{0}, C\right)$

$$\left.\begin{array}{l} Eu_1 = \frac{2}{3} \cdot 12 + \frac{1}{3} \cdot 6 = 10 \\ Eu_2 = \frac{2}{3} \cdot 9 + \frac{1}{3} \cdot 0 = 6 \end{array}\right\} \rightarrow (10, 6)$$

Practice

d) $\left(A^{12}B^{0}, D\right)$

$$\begin{aligned} Eu_1 &= \\ Eu_2 &= \end{aligned}$$

e) $\left(B^{12}A^{0}, C\right)$

$$\begin{aligned} Eu_1 &= \\ Eu_2 &= \end{aligned}$$

f) $\left(B^{12}A^{0}, D\right)$

$$\begin{aligned} Eu_1 &= \\ Eu_2 &= \end{aligned}$$

Practice

g) $\left(B^{12}B^{0}, C\right)$

$$\begin{aligned} Eu_1 &= \\ Eu_2 &= \end{aligned}$$

h) $\left(B^{12}B^{0}, D\right)$

$$\begin{aligned} Eu_1 &= \\ Eu_2 &= \end{aligned}$$

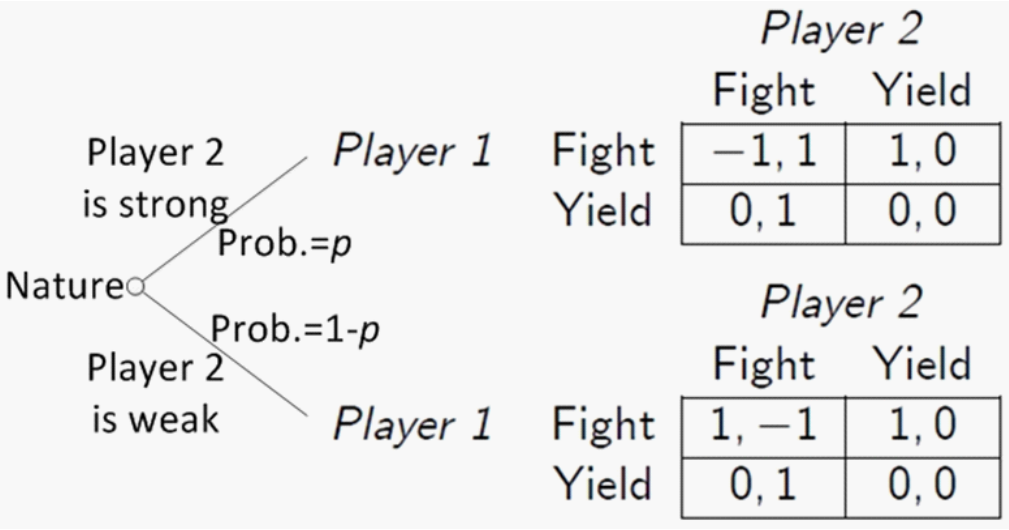
- **3rd step:** Inserting the expected payoffs in the cells of the matrix, we are ready to find the B.N.E. of the game by underlining best response payoffs:

		Player 2	
		C	D
Player 1	$A^{12}A^{0}$	8 , $\underline{9}$	3 , 6
	$A^{12}B^{0}$	$\underline{10}$, 6	4 , $\underline{7}$
	$B^{12}A^{0}$	4 , 3	5 , $\underline{8}$
	$B^{12}B^{0}$	6 , 0	$\underline{6}$, $\underline{9}$

- Hence, the Unique B.N.E. is $\left(B^{12}B^{0}, D\right)$

Two players in a dispute

- Two people are in a dispute. P2 knows her own type, either Strong or Weak, but P1 does know P2's type.



Player 2

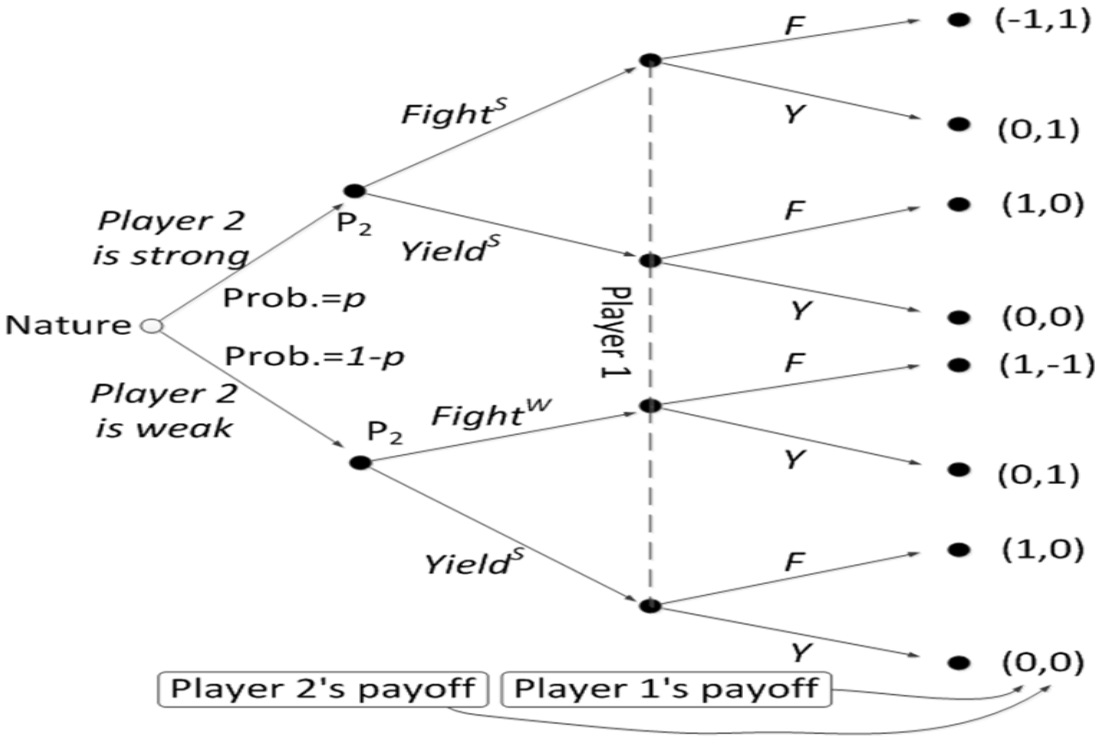
	Fight	Yield
Fight	$-1, 1$	$1, 0$
Yield	$0, 1$	$0, 0$

Player 2

	Fight	Yield
Fight	$1, -1$	$1, 0$
Yield	$0, 1$	$0, 0$

- Intuitively, P1 is in good shape in (Fight, Fight) if P2 is weak, but in bad shape otherwise.
- Game tree of this incomplete information game?$\longrightarrow$

Extensive Form Representation



- S: strong; W: weak;
- Only difference in payoffs occurs if both players fight.
- Let's next construct the Bayesian normal form representation of the game, in order to find the BNEs of this game.

Bayesian Normal Form Representation

- **1st step:** Identify players' strategy spaces.

$S_1 = \{F, Y\}$
$S_2 = \{F^S F^W, F^S Y^W, Y^S F^W, Y^S Y^W\}$
which entails the following Bayesian normal form.

		Player 2			
		$F^S F^W$	$F^S Y^W$	$Y^S F^W$	$Y^S Y^W$
Player 1	F				
	Y				

Bayesian Normal Form Representation

- **2nd step:** Let's start finding the expected payoffs to insert in the cells...

1) $\left(F, F^S F^W\right)$

$$\left.\begin{array}{l} Eu_1 = p \cdot (-1) + (1-p) \cdot 1 = 1 - 2p \\ Eu_2 = p \cdot 1 + (1-p) \cdot (-1) = 2p - 1 \end{array}\right\} \rightarrow (1 - 2p, 2p - 1)$$

Finding expected payoffs (Cont'd)

2) $(F, F^S Y^W)$

$$\left.\begin{aligned} Eu_1 = p \cdot (-1) + (1-p) \cdot 1 = 1 - 2p \\ Eu_2 = p \cdot 1 + (1-p) \cdot 0 = p \end{aligned}\right\} \rightarrow (1-2p, p)$$

3) $(F, Y^S F^W)$

$$\left.\begin{aligned} Eu_1 = p \cdot 1 + (1-p) \cdot 1 = 1 - 2p \\ Eu_2 = p \cdot 1 + (1-p) \cdot (-1) = p - 1 \end{aligned}\right\} \rightarrow (1-2p, p-1)$$

4) $(F, Y^S Y^W)$

$$\left.\begin{aligned} Eu_1 = p \cdot 1 + (1-p) \cdot 1 = 1 \\ Eu_2 = p \cdot 0 + (1-p) \cdot 0 = 0 \end{aligned}\right\} \rightarrow (1, 0)$$

5) $(Y, F^S F^W)$

$$\left.\begin{aligned} Eu_1 = p \cdot 0 + (1-p) \cdot 0 = 0 \\ Eu_2 = p \cdot 1 + (1-p) \cdot 1 = 1 \end{aligned}\right\} \rightarrow (0, 1)$$

Finding expected payoffs (Cont'd)

6) $(Y, F^S Y^W)$

$$\left.\begin{aligned} Eu_1 &= p \cdot 0 + (1-p) \cdot 0 = 0 \\ Eu_2 &= p \cdot 1 + (1-p) \cdot 0 = p \end{aligned}\right\} \rightarrow (0, p)$$

7) $(Y, Y^S F^W)$

$$\left.\begin{aligned} Eu_1 &= p \cdot 0 + (1-p) \cdot 0 = 0 \\ Eu_2 &= p \cdot 0 + (1-p) \cdot 1 = 1-p \end{aligned}\right\} \rightarrow (0, 1-p)$$

8) $(Y, Y^S Y^W)$

$$\left.\begin{aligned} Eu_1 &= p \cdot 0 + (1-p) \cdot 0 = 0 \\ Eu_2 &= p \cdot 0 + (1-p) \cdot 0 = 0 \end{aligned}\right\} \rightarrow (0, 0)$$

- Inserting these 8 expected payoff pairs in the matrix, we obtain:

		Player 2			
		$F^S F^W$	$F^S Y^W$	$Y^S F^W$	$Y^S Y^W$
Player 1	F	$1-2p, 2p-1$	$1-2p, p$	$1-2p, p-1$	$1, 0$
	Y	$0, 1$	$0, p$	$0, 1-p$	$0, 0$

- **3rd step:** Underline best response payoffs for each player.

		Player 2			
		$F^S F^W$	$F^S Y^W$	$Y^S F^W$	$Y^S Y^W$
Player 1	F	1–2p,2p–1	1–2p,p	1–2p,p–1	1,0
	Y	0,1	0,p	0,1–p	0,0

- Comparing for player 1 his payoff $1-2p$ against 0, we find that $1-2p \geq 0$ if $p \leq \frac{1}{2}$; otherwise $1-2p < 0$.
- In addition, for player 2 $2p-1 < p$ since $2p-p < 1 \Leftrightarrow p < 1$,which holds by definition, i.e., $p \in [0,1]$ and $p > p-1$ since $p \in [0,1]$.
- We can hence divide our analysis into two cases: case 1, where $p > \frac{1}{2}$; case 2, where $p \leq \frac{1}{2}$ $\rightarrow$next

- **Case 1:** $p \leq \frac{1}{2}$

Player 1	Player 2: $F^S F^W$	$F^S Y^W$	$Y^S F^W$	$Y^S Y^W$
F	$\underline{1-2p}, 2p-1$	$\underline{1-2p}, \underline{p}$	$\underline{1-2p}, p-1$	$\underline{1}, 0$
Y	$0, \underline{1}$	$0, p$	$0, 1-p$	$0, 0$

- $1-2p \geq 0$ since in this case $p \leq \frac{1}{2}$. $\rightarrow$that's why we underlined $1-2p$ (and not 0) in the first 3 columns.
- Hence, we found only one B.N.E. when $p \leq \frac{1}{2}$: $\left(F, F^S Y^W\right)$.

- **Case 2:** $p > \frac{1}{2}$

Player 1	Player 2: $F^S F^W$	$F^S Y^W$	$Y^S F^W$	$Y^S Y^W$
F	$1-2p, 2p-1$	$1-2p, \underline{p}$	$1-2p, p-1$	$\underline{1}, 0$
Y	$\underline{0}, \underline{1}$	$\underline{0}, p$	$\underline{0}, 1-p$	$0, 0$

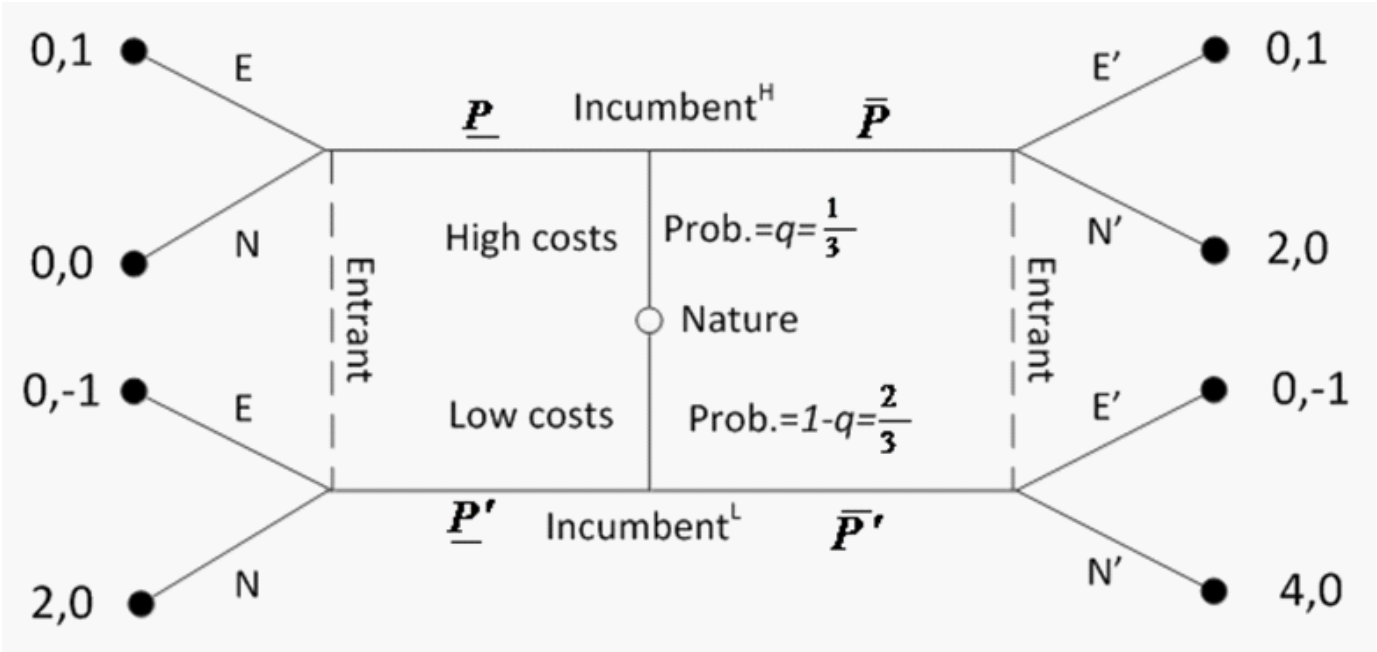
- $1 - 2p < 0$ since in this case $p > \frac{1}{2}$. $\rightarrow$that's why we underlined 0 in the first 3 columns.
- We have now found one (but different) B.N.E. when $p > \frac{1}{2} : \left(Y, F^S Y^W\right)$.

- Intuitively, when P1 knows that P2 is likely strong $\left(p > \frac{1}{2}\right)$, he yields in the BNE $\left(Y, F^S Y^W\right)$; whereas when he is most probably weak $\left(p \leq \frac{1}{2}\right)$, he fights in the BNE $\left(F, F^S Y^w\right)$.
- However, P2 behaves in the same way regardless of the precise value of p; he fights when strong but yields when weak, i.e., $F^S Y^s$, in both BNEs.

Remark

- Unlike in our search of mixed strategy equilibria, the probability p is now *not endogenously determined* by each player.
  - In a msNE each player could alter the frequency of his randomizations.
  - In contrast, it is now an exogenous variable (given to us) in the exercise.
- Hence,
  - if I give you the previous exercise with $p \leq \frac{1}{2}$(e.g., $p = \frac{1}{3}$), you will find that the unique BNE is $\left(F, F^S Y^W\right)$, and
  - if I give you the previous exercise with $p > \frac{1}{2}$ $\left(\text{e.g., } p = \frac{3}{4}\right)$ you will find that the unique BNE is $\left(Y, F^S F^W\right)$.

Entry game with incomplete information (Exercise #4)



- Notation: $\underline{P}$: low prices, $\overline{P}$: high prices, E: enter after low prices, N: do not enter after low prices, E': enter after high prices, N': do not enter after high prices.
- Verbal explanation on next slide.

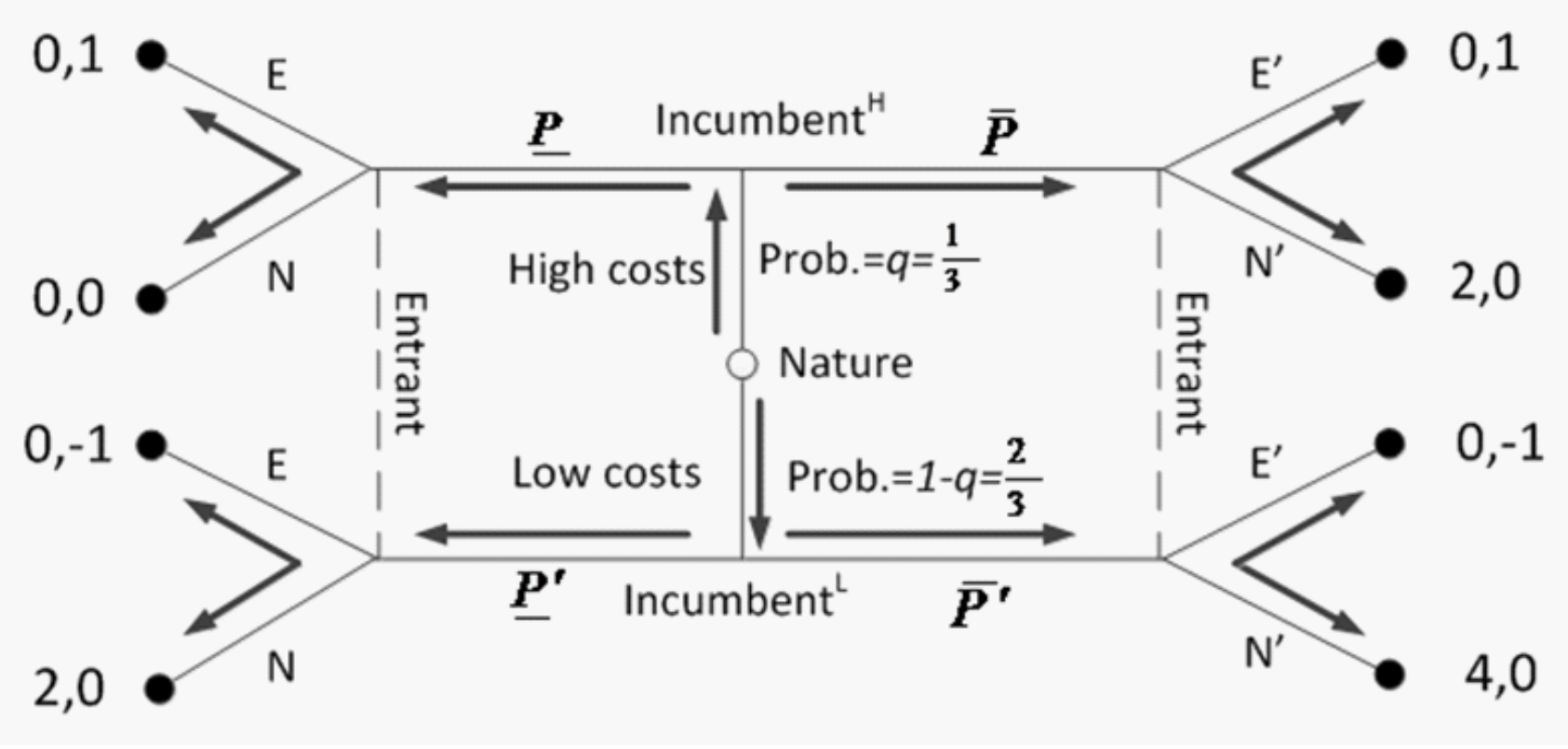
Time structure of the game:

The following sequential-move game with incomplete information is played between an incumbent and a potential entrant.

1. First, nature determines whether the incumbent experiences high or low costs, with probability q and $1-q$, e.g., $\frac{1}{3}$ and $\frac{2}{3}$, respectively.
2. Second, the incumbent, observing his cost structure (something that is not observed by the entrant), decides to set either a high price ($\overline{p}$) or a low price ($\underline{p}$).
3. Finally, observing the price that the incumbent sets (either high $\overline{p}$ or low $\underline{p}$), but without observing the incumbent's type, the entrant decides to enter or not enter the market.

Note that we use different notation, depending on the incumbent's type $\left(\overline{p} \text{ and } \underline{p}\right)$ and depending on the price observed by the entrant before deciding to enter $\left(E \text{ or } N,\ E' \text{ or } N'\right)$.

Entry game with incomplete information:



- You can think about its time structure in this way (starting from nature of the center of the game tree).

Let us now find the BNE of this game

- In order to do that, we first need to build the Bayesian Normal Form matrix.
- **1st step:** Identify the strategy spaces for each player.

$$S_{inc} = \left\{ \overline{PP}', \overline{P}\underline{P}', \underline{P}\overline{P}', \underline{PP}' \right\} \text{ 4 strategies}$$

$$S_{ent} = \left\{ EE', EN', NE', NN' \right\} \text{ 4 strategies}$$

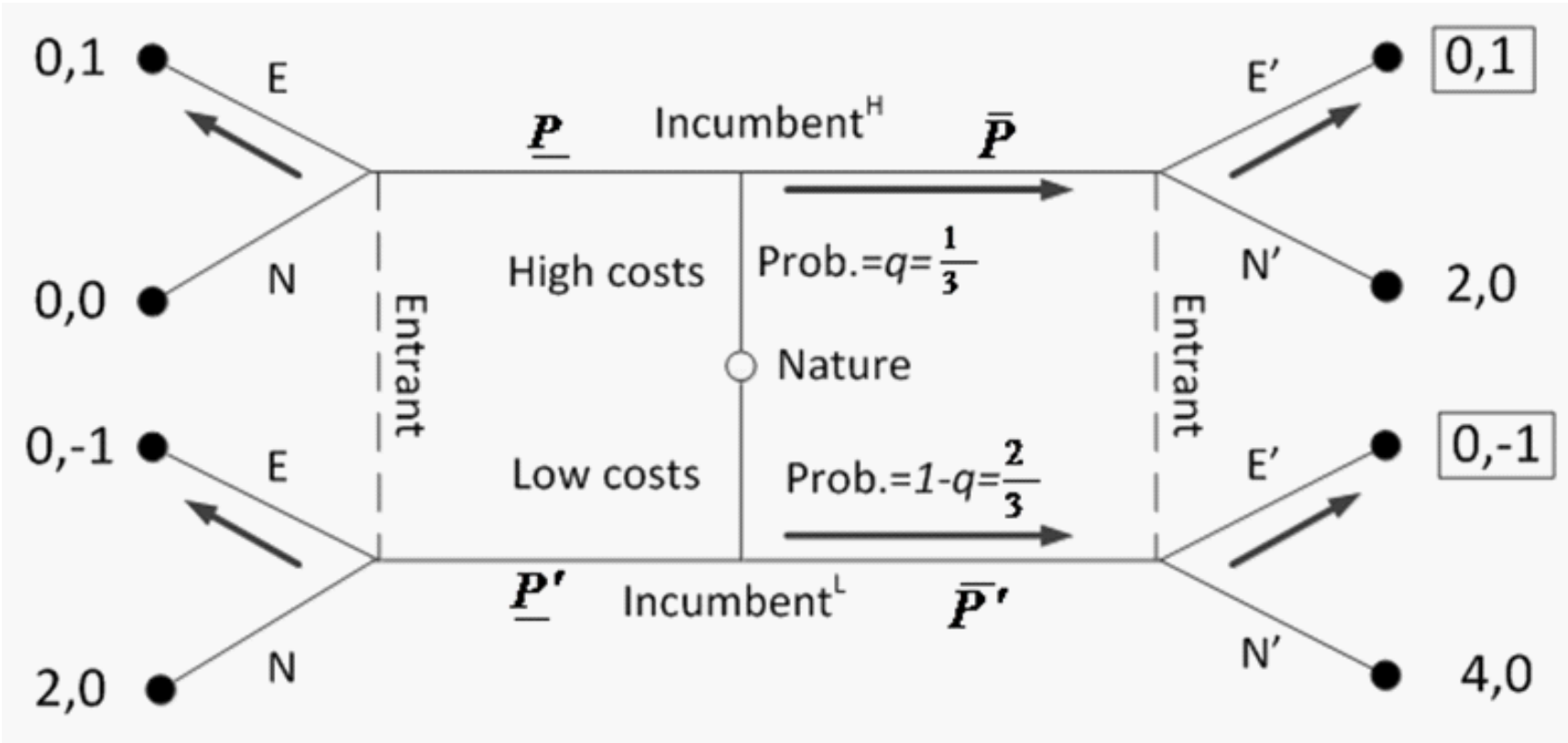
- We hence need to build a 4×4 Bayesian normal from matrix such as the following:

		Entrant			
		EE'	EN'	NE'	NN'
Incumbent	$\overline{P}\overline{P}'$	1	2	3	4
	$\overline{P}\underline{P}'$	5	6	7	8
	$\underline{P}\overline{P}'$	9	10	11	12
	$\underline{P}\underline{P}'$	13	14	15	16

- **2nd step:** We will need to find the expected payoff pairs in each of the $4 \times 4 = 16$ cells.

Entry game with incomplete information:

1. Strategy profile $\left(\overline{PP}'EE'\right)$



Let's fill the cells!

- First Row (where the incumbent chooses $\overline{PP}'$):
  1) $\overline{PP}'EE'$:

$$\left.\begin{array}{r} \text{Inc.} \to 0 \cdot q + 0 \cdot (1-q) = 0 \\ \text{Ent.} \to 1 \cdot q + (-1) \cdot (1-q) = 2q-1 \end{array}\right\} \to (0, 2q-1)$$

  2) $\overline{PP}'EN'$:

$$\left.\begin{array}{r} \text{Inc.} \to 2 \cdot q + 4 \cdot (1-q) = 4-2q \\ \text{Ent.} \to 0 \cdot q + 0 \cdot (1-q) = 0 \end{array}\right\} \to (4-2q, 0)$$

  3) $\overline{PP}'NE'$:

$$\left.\begin{array}{r} \text{Inc.} \to 0 \cdot q + 0 \cdot (1-q) = 0 \\ \text{Ent.} \to 1 \cdot q + (-1) \cdot (1-q) = 2q-1 \end{array}\right\} \to (0, 2q-1)$$

  4) $\overline{PP}'NN'$:

$$\left.\begin{array}{r} \text{Inc.} \to 2 \cdot q + 4 \cdot (1-q) = 4-2q \\ \text{Ent.} \to 0 \cdot q + 0 \cdot (1-q) = 0 \end{array}\right\} \to (4-2q, 0)$$

- Second Row (where the incumbent chooses $\overline{P}\underline{P'}$):
  5) $\overline{P}\underline{P'}EE'$:

$$\left.\begin{array}{c}\text{Inc.} \to 0 \cdot q + 0 \cdot (1-q) = 0 \\ \text{Ent.} \to 1 \cdot q + (-1) \cdot (1-q) = 2q-1\end{array}\right\} \to (0, 2q-1)$$

  6) $\overline{P}\underline{P'}EN'$:

$$\left.\begin{array}{c}\text{Inc.} \to 2 \cdot q + 0 \cdot (1-q) = 2q \\ \text{Ent.} \to 0 \cdot q + (-1) \cdot (1-q) = q-1\end{array}\right\} \to (2q, q-1)$$

  7) $\overline{P}\underline{P'}NE'$:

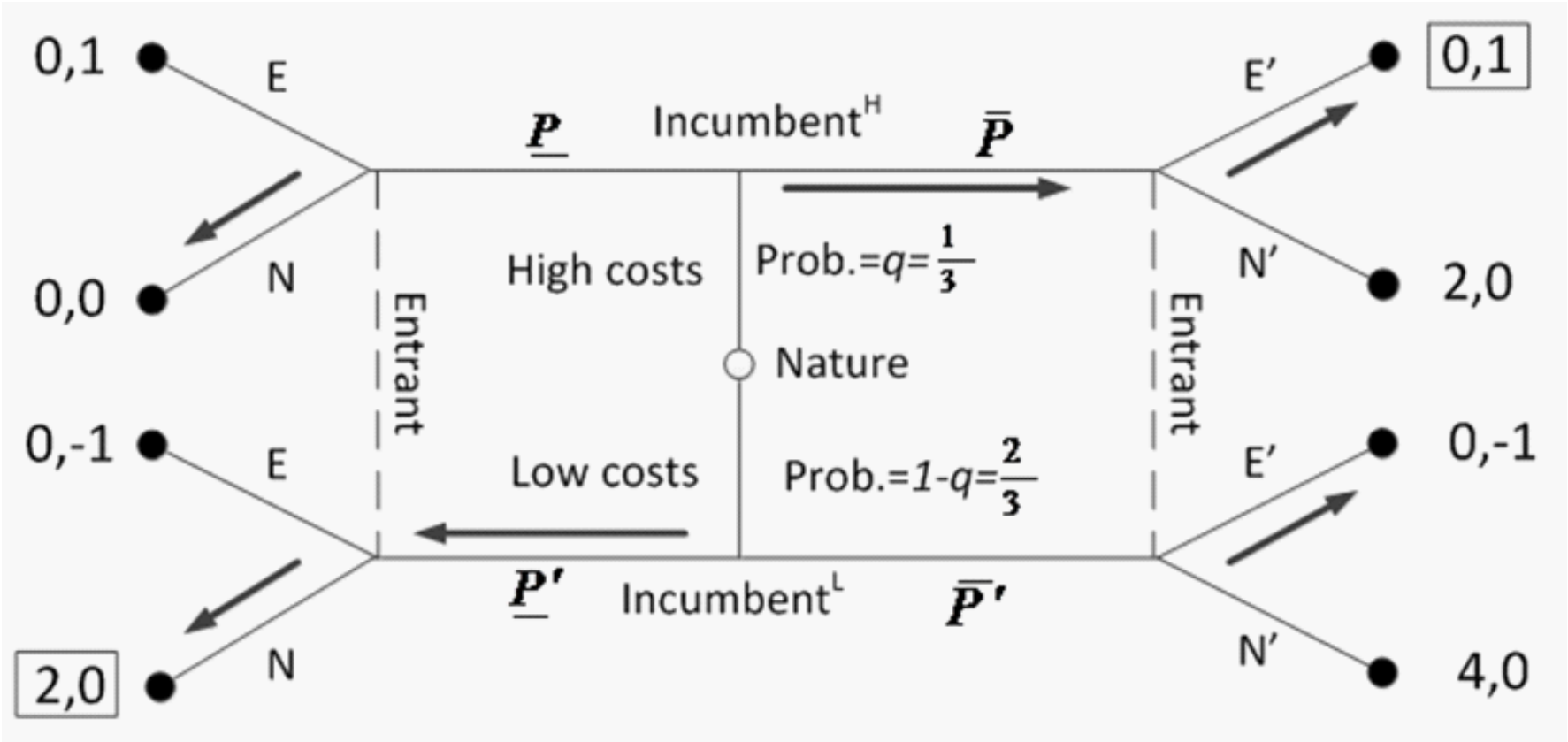
$$\left.\begin{array}{c}\text{Inc.} \to 0 \cdot q + 2 \cdot (1-q) = 2-2q \\ \text{Ent.} \to 1 \cdot q + 0 \cdot (1-q) = 1-q\end{array}\right\} \to (2-2q, 1-q)$$

  8) $\overline{P}\underline{P'}NN'$:

$$\left.\begin{array}{c}\text{Inc.} \to 2 \cdot q + 2 \cdot (1-q) = 2 \\ \text{Ent.} \to 0 \cdot q + 0 \cdot (1-q) = 0\end{array}\right\} \to (2, 0)$$

Entry game with incomplete information:

7. Strategy profile $(\overline{P}\underline{P}'NE')$



- Third Row (where the incumbent chooses $\underline{P}\overline{P}'$):

  9) $\underline{P}\overline{P}'EE'$:

$$\left.\begin{array}{c}\text{Inc.} \rightarrow 0 \cdot q + 0 \cdot (1-q) = 0 \\ \text{Ent.} \rightarrow 1 \cdot q + (-1) \cdot (1-q) = 2q-1\end{array}\right\} \rightarrow (0, 2q-1)$$

  10) $\underline{P}\overline{P}'EN'$:

$$\left.\begin{array}{c}\text{Inc.} \rightarrow 0 \cdot q + 4 \cdot (1-q) = 4-4q \\ \text{Ent.} \rightarrow 1 \cdot q + 0 \cdot (1-q) = q\end{array}\right\} \rightarrow (4-4q, q)$$

  11) $\underline{P}\overline{P}'NE'$:

$$\left.\begin{array}{c}\text{Inc.} \rightarrow 0 \cdot q + 0 \cdot (1-q) = 0 \\ \text{Ent.} \rightarrow 0 \cdot q + (-1) \cdot (1-q) = q-1\end{array}\right\} \rightarrow (0, q-1)$$

  12) $\underline{P}\overline{P}'NN'$:

$$\left.\begin{array}{c}\text{Inc.} \rightarrow 0 \cdot q + 4 \cdot (1-q) = 4-4q \\ \text{Ent.} \rightarrow 0 \cdot q + 0 \cdot (1-q) = 0\end{array}\right\} \rightarrow (4-4q, 0)$$

- Four Row (where the incumbent chooses $\underline{PP'}$):

13) $\underline{PP'}EE'$:

$$\left.\begin{array}{r} \text{Inc.} \rightarrow 0 \cdot q + 0 \cdot (1-q) = 0 \\ \text{Ent.} \rightarrow 1 \cdot q + (-1) \cdot (1-q) = 2q-1 \end{array}\right\} \rightarrow (0, 2q-1)$$

14) $\underline{PP'}EN'$:

$$\left.\begin{array}{r} \text{Inc.} \rightarrow 0 \cdot q + 0 \cdot (1-q) = 0 \\ \text{Ent.} \rightarrow 1 \cdot q + (-1) \cdot (1-q) = 2q-1 \end{array}\right\} \rightarrow (0, 2q-1)$$

15) $\underline{PP'}NE'$:

$$\left.\begin{array}{r} \text{Inc.} \rightarrow 0 \cdot q + 2 \cdot (1-q) = 2-2q \\ \text{Ent.} \rightarrow 0 \cdot q + 0 \cdot (1-q) = 0 \end{array}\right\} \rightarrow (2-2q, 0)$$

16) $\underline{PP'}NN'$:

$$\left.\begin{array}{r} \text{Inc.} \rightarrow 0 \cdot q + 2 \cdot (1-q) = 2-2q \\ \text{Ent.} \rightarrow 0 \cdot q + 0 \cdot (1-q) = 0 \end{array}\right\} \rightarrow (2-2q, 0)$$

Inserting these expected payoff pairs yields:

		Entrant			
		EE'	EN'	NE'	NN'
Incumbent	$\overline{P}\overline{P}'$	$0, 2q-1$	$4-2q, 0$	$0, 2q-1$	$4-2q, 0$
	$\overline{P}\underline{P}'$	$0, 2q-1$	$2q, q-1$	$2-2q, q$	$2, \ 0$
	$\underline{P}\overline{P}'$	$0, 2q-1$	$4-4q, q$	$0, q-1$	$4-4q, 0$
	$\underline{P}\underline{P}'$	$0, 2q-1$	$0, 2q-1$	$2-2q, 0$	$2-2q, 0$

- Before starting our underlining, let's carefully compare the incumbent's and entrant's expected payoffs$\longrightarrow$

- Comparing the **Incumbent's** expected payoffs:
  - under EE', the incumbent's payoff is 0 regardless of the strategy he chooses (i.e., for all rows). under EN', $4 - 2q > 2q$ since $4 > 4q$ for any $q < 1$ and $4 - 2q > 4 - 4q$, which simplifies to $4q > 2q \Rightarrow 4 > 2$ and $4 - 2q > 0 \rightarrow 4 > 2q \rightarrow 2 > q$
  - under NE', $2 - 2q > 0$ since $2 > 2q$ for any $q < 1$
  - under NN', $4 - 2q > 2$ since $2 > 2q$ for any $q < 1$
    - and $4 - 2q > 4 - 4q \Rightarrow 4q > 2q \Rightarrow 4 > 2$
    - and $4 - 2q > 2 - 2q$ since $4 > 2$

- Comparing the **Entrant's** expected payoffs:
  - under $\overline{PP}'$, $2q-1>0$ if $q>\frac{1}{2}$ (otherwise, $2q-1<0$)
  - under $\overline{P}\underline{P}'$, $q>2q-1$ since $1>q$ and we have $2q-1>q-1$ since $q>0$.
    - Hence $q>2q-1>q-1$
  - under $\underline{P}\overline{P}'$, $q>2q-1>q-1$ (as above).
  - under $\underline{PP}'$, $2q-1>0$ only if $q>\frac{1}{2}$ (otherwise, $2q-1<0$).

For clarity...

- We can separate our analysis into two cases
  - When $q > \frac{1}{2}$ (see the matrix in the next slide).
  - When $q < \frac{1}{2}$ (see the matrix two slides from now).
- Note that these cases emerged from our comparison of the entrant's payoff alone, since the payoffs of the incumbent could be unambiguously ranked without the need to introduce any condition on q.
- In the following matrix, this implies that the payoffs underlined in blue (for the Incumbent.) are independent on the precise value of q, while the payoffs underlined in red (for the entrant) depend on q.

- **Case 1:** $q > \frac{1}{2} \rightarrow$ so that $2q - 1 > 0$

		Entrant			
		EE'	EN'	NE'	NN'
Incumbent	$\overline{P}\overline{P}'$	$0, 2q-1$	$4-2q, 0$	$0, 2q-1$	$4-2q, 0$
	$\overline{P}\underline{P}'$	$0, 2q-1$	$2q, q-1$	$2-2q, q$	$2, 0$
	$\underline{P}\overline{P}'$	$0, 2q-1$	$4-4q, q$	$0, q-1$	$4-4q, 0$
	$\underline{P}\underline{P}'$	$0, 2q-1$	$0, 2q-1$	$2-2q, 0$	$2-2q, 0$

- 3 BNEs: $\left(\overline{PP}', EE'\right)$, $\left(\overline{P}\underline{P}', NE'\right)$ and $\left(\underline{PP}', EE'\right)$

- **Case 2:** $q < \frac{1}{2} \rightarrow$ so that $2q - 1 < 0$

		Entrant			
		EE'	EN'	NE'	NN'
Incumbent	$\overline{P}\overline{P}'$	0,2q−1	4−2q,0	0,2q−1	4−2q,0
	$\overline{P}\underline{P}'$	0,2q−1	2q,q−1	2−2q,q	2, 0
	$\underline{P}\overline{P}'$	0,2q−1	4−4q,q	0,q−1	4−4q,0
	$\underline{P}\underline{P}'$	0,2q−1	0,2q−1	2−2q,0	2−2q,0

- 4 BNEs: $\left(\overline{PP}', EN'\right)$, $\left(\overline{P}\underline{P}', NE'\right)$, $\left(\underline{PP}', NE'\right)$ and $\left(\overline{\overline{PP}}', NN'\right)$

- **Practice:** let's assume that $q = \frac{1}{3}$. Then, the Bayesian Normal Form matrix becomes:

		Entrant			
		EE'	EN'	NE'	NN'
Incumbent	$\overline{P}\overline{P}'$	$0, -\frac{1}{3}$	$\frac{10}{3}, 0$	$0, -\frac{1}{3}$	$\frac{10}{3}, 0$
	$\overline{P}\underline{P}'$	$0, -\frac{1}{3}$	$\frac{2}{3}, -\frac{2}{3}$	$\frac{4}{3}, \frac{1}{3}$	$2, 0$
	$\underline{P}\overline{P}'$	$0, -\frac{1}{3}$	$\frac{8}{3}, \frac{1}{3}$	$0, -\frac{2}{3}$	$\frac{8}{3}, 0$
	$\underline{P}\underline{P}'$	$0, -\frac{1}{3}$	$0, -\frac{1}{3}$	$\frac{4}{3}, 0$	$\frac{4}{3}, 0$

- The payoff comparison is now faster, as we only compare numbers.
- 4 BNEs $\rightarrow$ the same set of BNEs as when $q < \frac{1}{2}$.

Alternative methodology

- There is an alternative way to approach these exercise. . .
  - Which is especially useful in exercises that are difficult to represent graphically.
  - *Example*:
    - Cournot games with incomplete information,
    - Bargaining games with incomplete information,
    - and, generally, games with a continuum of strategies available to each player.
- The methodology is relatively simple:
  - Focus on the informed player first, determining what he would do for each of his possible types, e.g., when he is strong and then when he is weak.
  - Then move on to the uninformed player.

Alternative methodology

- Before applying this alternative methodology in Cournot or bargaining games...
- Let's redo the "Two players in a dispute" exercise, using this method.
- For simplicity, let us focus on the case that $p = \frac{1}{3}$.
  - We want to show that we can obtain the same BNE as with the previous methodology (Constructing the Bayesian Normal Form matrix).
  - In particular, recall that the BNE we found constructing the Bayesian Normal form matrix was $\left(F, F^S Y^W\right)$

Two players in a dispute

- Two people are in a dispute. $P2$ knows her own type, either Strong or Weak, but $P1$ does know P2's type.

Player 2 strong with probability P

Player 1	Player 2: Fight α	Player 2: Yield $1-\alpha$
Fight β	-1, 1	1, 0
Yield $1-\beta$	0, 1	0, 0

Player 2 weak with probability $1-P$

Player 1	Player 2: Fight γ	Player 2: Yield $1-\gamma$
Fight β	1, -1	1, 0
Yield $1-\beta$	0, 1	0, 0

- **Notation:** β is prob. of fighting for the uninformed P1, $\alpha\,(\gamma)$ is the prob. of fighting for P2 when he is strong (weak, respectively).

Two players in a dispute

- **1st step:** Privately informed player (player 2):
  - If player 2 is **strong**, fighting is strictly dominant (yielding is strictly dominated for him when being strong).
    - You can delete that column from the first matrix.

Two players in a dispute

- Privately informed player (player 2):
  - If player 2 is **weak**, there are no strictly dominated actions.
    - Hence (looking at the lower matrix, corresponding to the weak P_2) we must compare his expected utility of fighting and yielding.

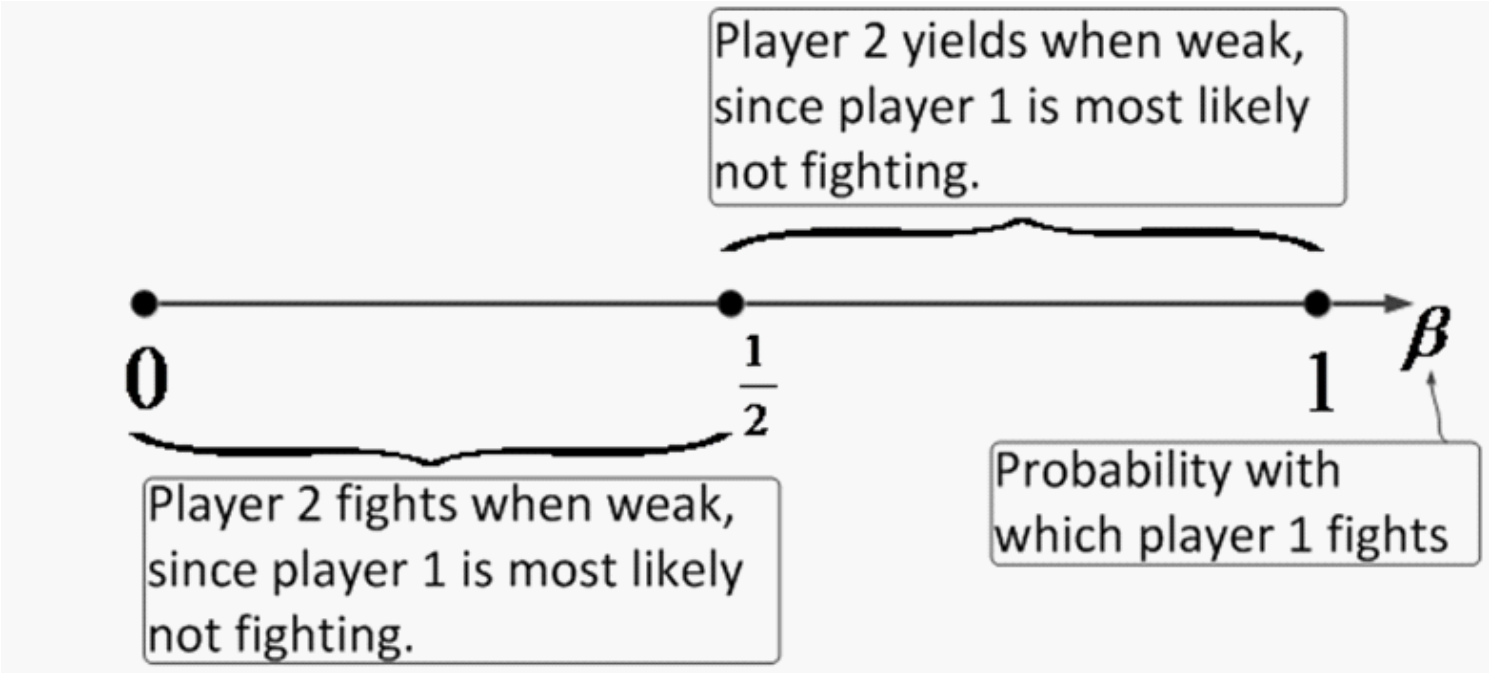
$$\begin{aligned} EU_2\left(F\,|Weak\right) &= -1\cdot\beta+1\cdot(1-\beta)=1-2\beta \\ EU_2\left(Y\,|Weak\right) &= 0\cdot\beta+0\cdot(1-\beta)=0 \end{aligned}$$

where β is the probability that player 1 plays Fight, and $1-\beta$ is the probability that he plays Yield. (See figure in previous slide)

-
  - Therefore, $EU_2\left(F\,|Weak\right) \geq EU_2\left(Y\,|Weak\right)$ if $1-2\beta \geq 0$, which is true only if $\beta \leq \frac{1}{2}$.

Two players in a dispute

- Thus, when $\beta \leq \frac{1}{2}$ player 2 fights, and when $\beta > \frac{1}{2}$ player 2 yields



Two players in a dispute

- **2nd step:** Uninformed player (player 1):
  - On the other hand, player 1 plays fight or yield unconditional on player 2's type, since he is uninformed about P2's type. Indeed, his expected utility of fighting is

$$EU_1(F) = \underbrace{p \cdot (-1)}_{\text{if P2 is strong, P2 fights}}$$

$$+\underbrace{(1-p) \cdot \left[\overbrace{\gamma \cdot 1}^{\text{if P2 fights when weak}} + \overbrace{(1-\gamma) \cdot 1}^{\text{if P2 yields when weak}}\right]}_{\text{if P2 is weak}}$$

$$= 1 - 2p$$

and since $p = \frac{1}{3}$, $1 - 2p$ becomes $1 - 2 \times \frac{1}{3} = \frac{1}{3}$.

Two players in a dispute

- And P1's expected utility of yielding is:

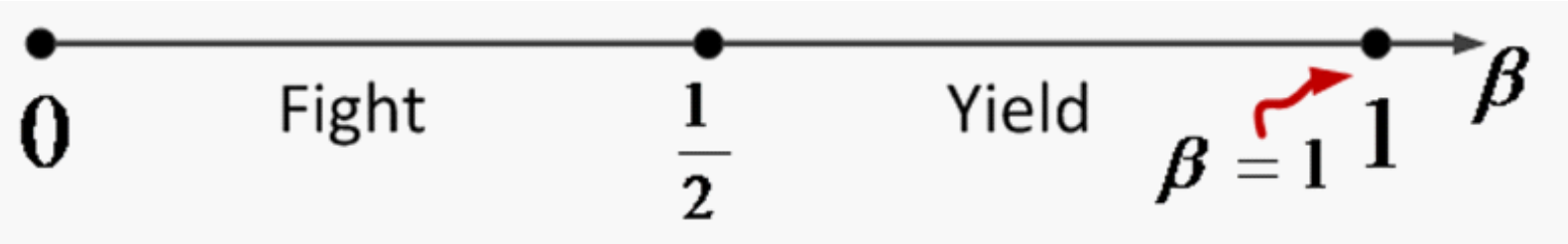
$$EU_1(Y) = \underbrace{p \cdot (0)}_{\text{if P2 is strong, P2 fights}} + \underbrace{(1-p) \cdot \left[\overbrace{\gamma \cdot 0}^{\text{if P2 fights when weak}} + \overbrace{(1-\gamma) \cdot 0}^{\text{if P2 yields when weak}}\right]}_{\text{if P2 is weak}}$$

$$= 0$$

  - Therefore, $EU_1(F) > EU_1(Y)$, since $\frac{1}{3} > 0$, which implies that player 1 fights.
  - Hence, since β represents the prob. with which player 1 fights, we have that $\beta = 1$.

Two players in a dispute

- We just determined that $\beta = 1$.
- Therefore, β is definitely **larger** than $\frac{1}{2}$, leading player 2 to **Yield** when he is weak. Recall that P2's decision rule when weak was as depicted in the next figure: yield if and only if $\beta > \frac{1}{2}$.



Two players in a dispute

- We are now ready to summarize the BNE of this game, for the particular case in which $p = \frac{1}{3}$,

$$\left\{ \underbrace{\text{Fight}}_{\text{player 1}}, \underbrace{(\text{Fight if Strong, Yield if Weak})}_{\text{player 2}} \right\}$$

- This BNE coincides with that under $p \leq \frac{1}{2} : \left\{ F, F^S Y^W \right\}$ we found using the other method.

Two players in a dispute

- **Practice for you:** Let's redo the previous exercise, but with $p = \frac{2}{3}$.
- Nothing changes in this slide...
- Two people are in a dispute: $P2$ knows her own type, either Strong or Weak, but $P1$ does know P2's type.

Player 2 strong with probability P

Player 1	Player 2: Fight α	Player 2: Yield $1-\alpha$
Fight β	-1, 1	1, 0
Yield $1-\beta$	0, 1	0, 0

Player 2 weak with probability $1-P$

Player 1	Player 2: Fight γ	Player 2: Yield $1-\gamma$
Fight β	1, -1	1, 0
Yield $1-\beta$	0, 1	0, 0

Two players in a dispute

- **1st step:** Privately informed player (player 2): (nothing changes in this slide either)
  - If player 2 is **strong**, fighting is strictly dominant (yielding is strictly dominated for him when being strong).
    - You can delete that column from the first matrix.

Two players in a dispute

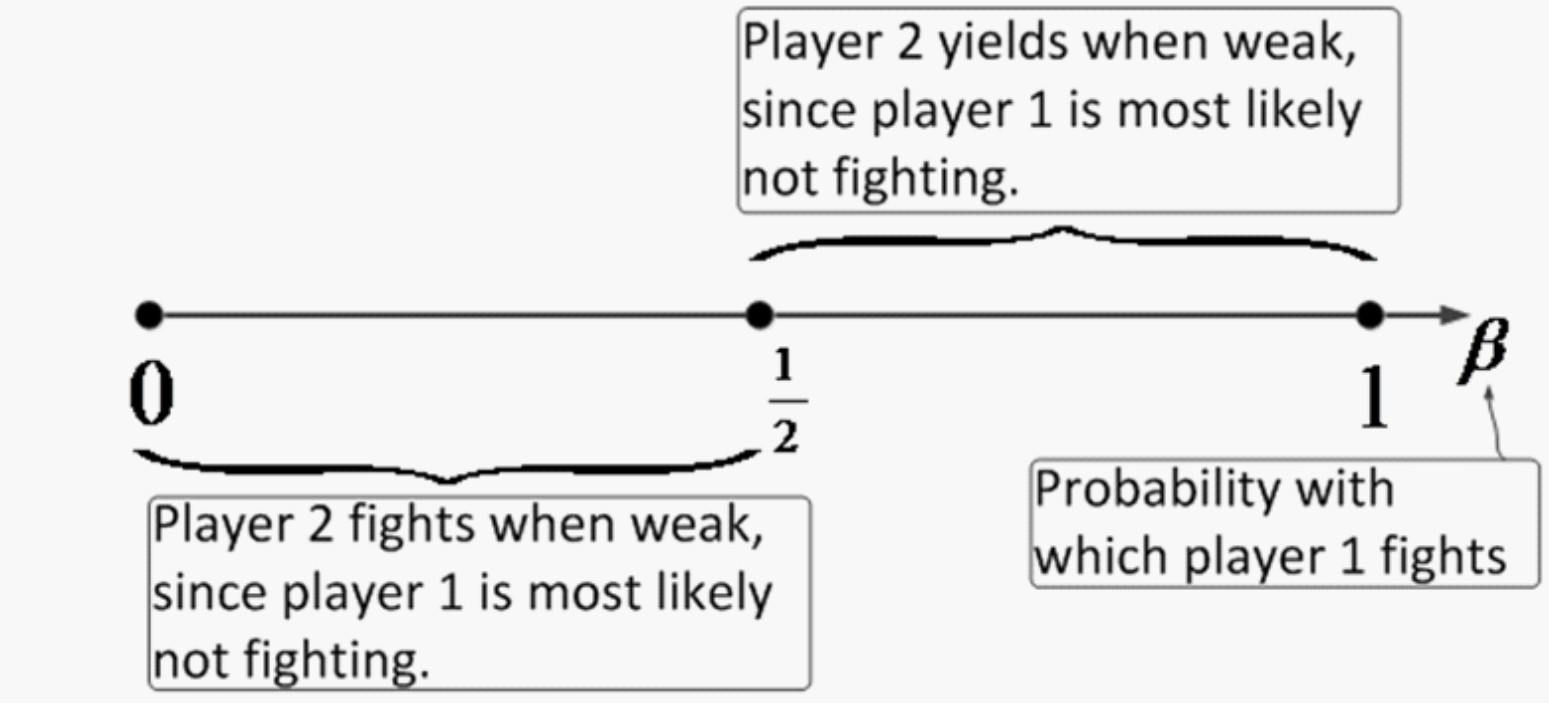
- Privately informed player (player 2): (nothing charges in this slide either)
  - If player 2 is **weak**, there are no strictly dominated actions.
    - Hence (looking at the lower matrix, corresponding to the weak P_2):

$$\begin{aligned} EU_2(F \,|\, Weak) &= -1 \cdot \beta + 1 \cdot (1-\beta) = 1 - 2\beta \\ EU_2(Y \,|\, Weak) &= 0 \cdot \beta + 0 \cdot (1-\beta) = 0 \end{aligned}$$

  - where β is the probability that player 1 plays Fight, and $1-\beta$ is the probability that he plays Yield. (See figure in previous slide)
- Therefore, $EU_2(F \,|\, Weak) \geq EU_2(Y \,|\, Weak)$ if $1 - 2\beta \geq 0$, which is true only if $\beta \leq \frac{1}{2}$.

Two players in a dispute

- Thus, when $\beta \leq \frac{1}{2}$ player 2 fights, and when $\beta > \frac{1}{2}$ player 2 yields.



Two players in a dispute

- **2nd step:** Uninformed player (player 1): (Here is when things start to change)
  - On the other hand, player 1 plays fight or yield unconditional on player 2's type. Indeed, P1's expected utility of fighting is

$$
\begin{aligned}
EU_1(F) &= \underbrace{p \cdot (-1)}_{\text{if P2 is strong, P2 fights}} \\
&\quad + \underbrace{(1-p) \cdot \Big[\overbrace{\gamma \cdot 1}^{\text{if P2 fights when weak}} + \overbrace{(1-\gamma) \cdot 1}^{\text{if P2 yields when weak}}\Big]}_{\text{if P2 is weak}} \\
&= 1 - 2p
\end{aligned}
$$

and since $p = \frac{2}{3}$, $1 - 2p$ becomes $1 - 2 \times \frac{2}{3} = -\frac{1}{3}$.

Two players in a dispute

- And P1's expected utility of yielding is

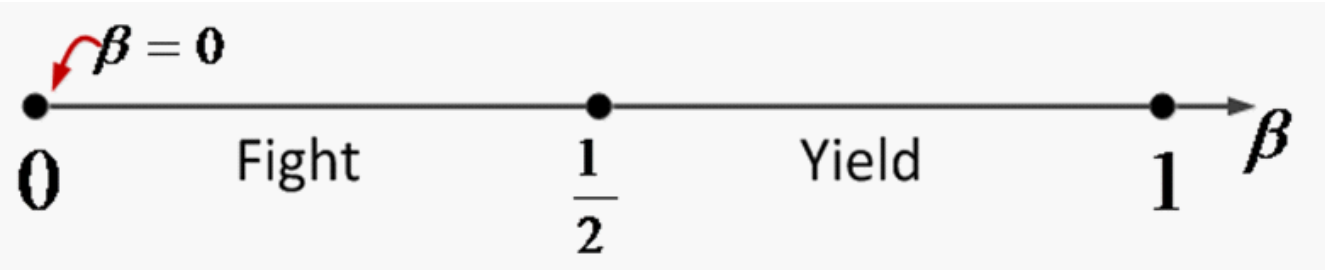
$$EU_1(Y) = \underbrace{p \cdot (0)}_{\text{if P2 is strong, P2 fights}} + \underbrace{(1-p) \cdot \left[\overbrace{\gamma \cdot 0}^{\text{if P2 fights when weak}} + \overbrace{(1-\gamma) \cdot 0}^{\text{if P2 yields when weak}} \right]}_{\text{if P2 is weak}}$$

$$= 0$$

  - Therefore, $EU_1(F) < EU_1(Y)$, i.e., $-\frac{1}{3} < 0$, which implies that player 1 fights.
  - Hence, since β represents the prob. with which player 1 fights, $EU_1(F) < EU_1(Y)$ entails $\beta = 0$.

Two players in a dispute

- And things keep changing. . .
  - Since $\beta = 0$, β is definitely **smaller** than $\frac{1}{2}$, leading player 2 to **Fight** when he is weak, as illustrated in P2's decision rule when weak in the following line.



Two players in a dispute

- We are now ready to summarize the BNE of this game, for the particular case of $p = \frac{2}{3}$,

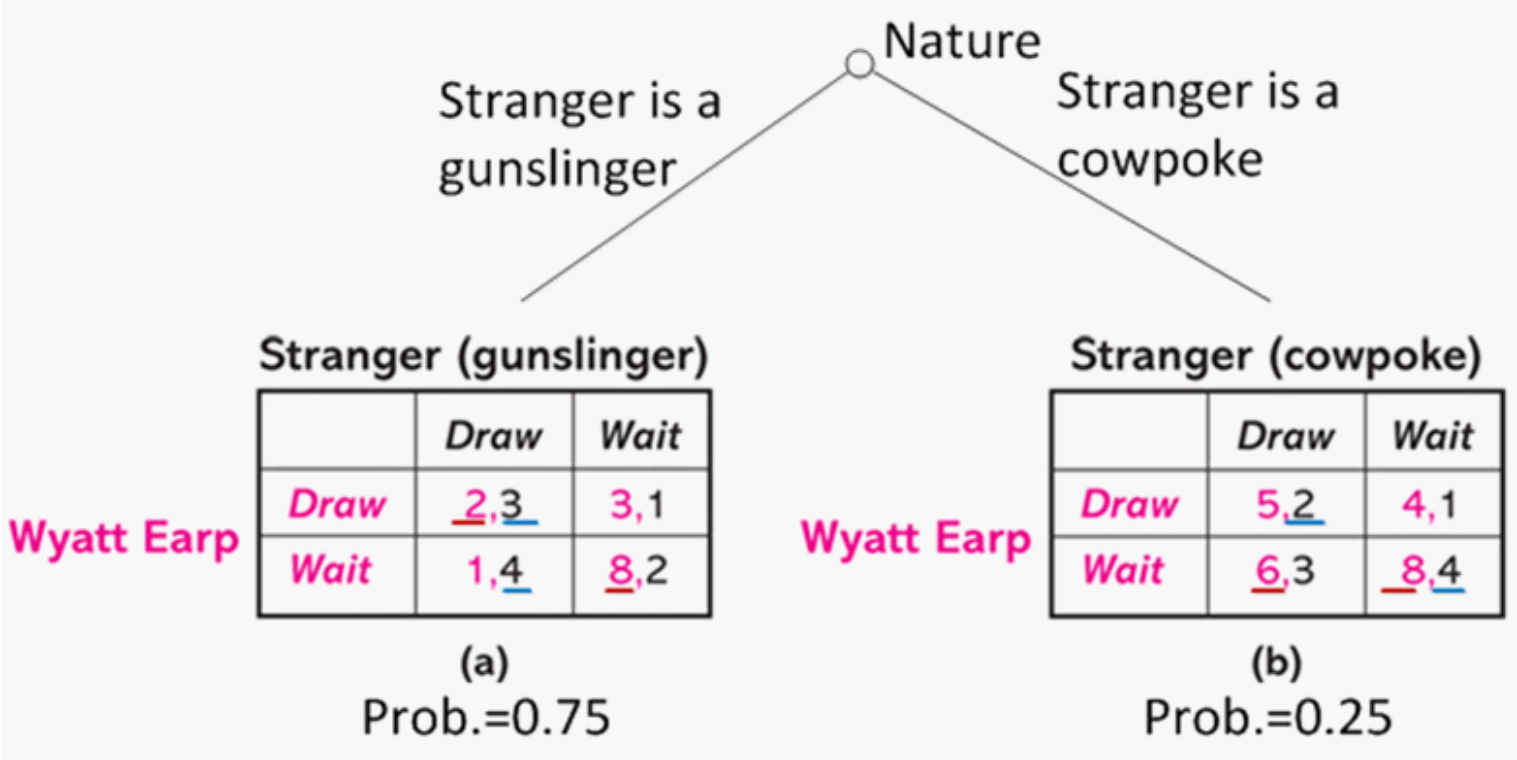
$$\left\{ \underbrace{\text{Yield}}_{\text{player 1}} , \underbrace{(\text{Fight if Strong, Fight if Weak})}_{\text{player 2}} \right\}$$

which coincides with the BNE we found for all $p > \frac{1}{2} : \left\{ Y, F^S Y^W \right\}$.

Two players in a dispute

- Summarizing, the set of BNEs is...
  - $\left\{F, F^S Y^W\right\}$ when $p \leq \frac{1}{2}$
  - $\left\{Y, F^S Y^W\right\}$ when $p > \frac{1}{2}$
- Importantly, we could find them using either of the two methodologies:
  - Constructing the Bayesian normal form representation of the game with a matrix (as we did in our last class); or
  - Focusing on the informed player first, and then moving to the uniformed player (as we did today).

Gunfight in the wild west



Description of the payoffs:

If Wyatt Earp knew for sure that the Stranger is a gunslinger (left matrix):

1. Earp doesn't have a dominant strategy (he would Draw if the stranger Draws, but Wait if the stranger Waits).
2. The gunslinger, in contrast, has a dominant strategy: Draw.

If Wyatt Earp knew for sure that the Stranger is a cowpoke (right hand matrix):

1. Now, Earp has a dominant strategy: Wait.
2. In contrast, the cowpoke would draw only if he thinks Earp is planning to do so. In particular, he Draws if Earp Draws, but Waits if Earp Waits.

Description of the payoffs:

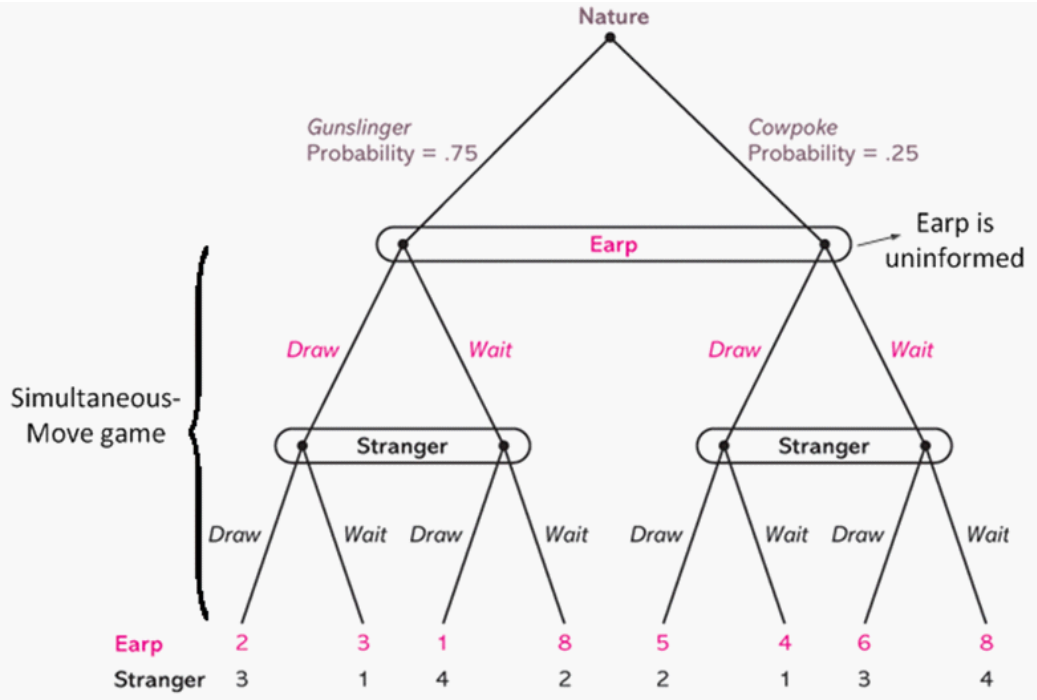
- This is a common feature in games of incomplete information:
  - The uninformed player (Wyatt Earp) does not have a strictly dominant strategy which would allow him to choose the same action...
  - regardless of the informed player's type (gunslinger/cowpoke).
- Otherwise, he wouldn't care what type of player he is facing. He would simply choose his dominant strategy, e.g., shoot!
  - That is, uncertainty would be irrelevant.
- Hence, the lack of a dominant strategy for the uninformed player makes the analysis interesting.

Description of the payoffs:

- Later on, we will study games of incomplete information where the privately informed player acts first and the uniformed player responds.
  - In that context, we will see that the uniformed player's lack of a strictly dominant strategy allows the informed player to use his actions to signal his own type...
  - either revealing or concealing his type to the uniformed player...
  - Ultimately affecting the uninformed player's response.
- Example from the gunfight in the wild west:
  - Did the stranger order a "whisky on the rocks" for breakfast at the local saloon, or
  - is he drinking a glass of milk?

How to describe Wyatt Earp's lack of information about the stranger's ability?

- Nature determines the stranger's type (gunslinger or cowpoke), but Earp doesn't observe that.

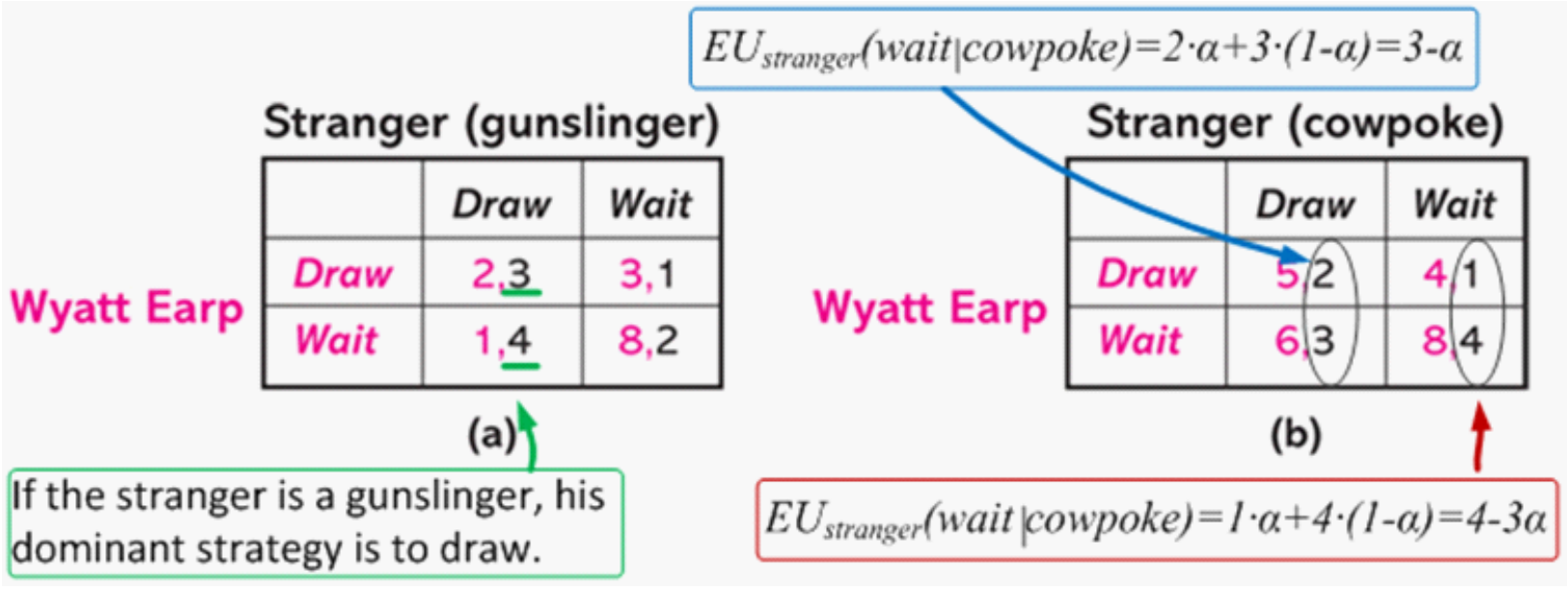


- Analog to the "two players in a dispute" game.

Let's apply the previous methodology!

1. Let us hence focus on the informed player first, separately analyzing his optimal strategy:
    1. When he is a gunslinger, and
    2. When he is a cowpoke.
2. After examining the informed player (stranger) we can move on to the optimal strategy for Wyatt Earp (uninformed player).
    1. Note that Wyatt Earp's strategy will be *unconditional on types*, since he cannot observe the stranger's type.

1st step: stranger (informed player)



Stranger:

- **If Gunslinger:** he selects to Draw (since Draw is his dominant strategy).
- **If Cowpoke:** in this case the stranger doesn't have a dominant strategy. Hence, he needs to compare his expected payoff from drawing and waiting.

$$EU_{Stranger}(Draw|Cowpoke) = \underbrace{2\alpha}_{\text{if Earp Draws}} + \underbrace{3(1-\alpha)}_{\text{if Earp Waits}} = 3-\alpha$$

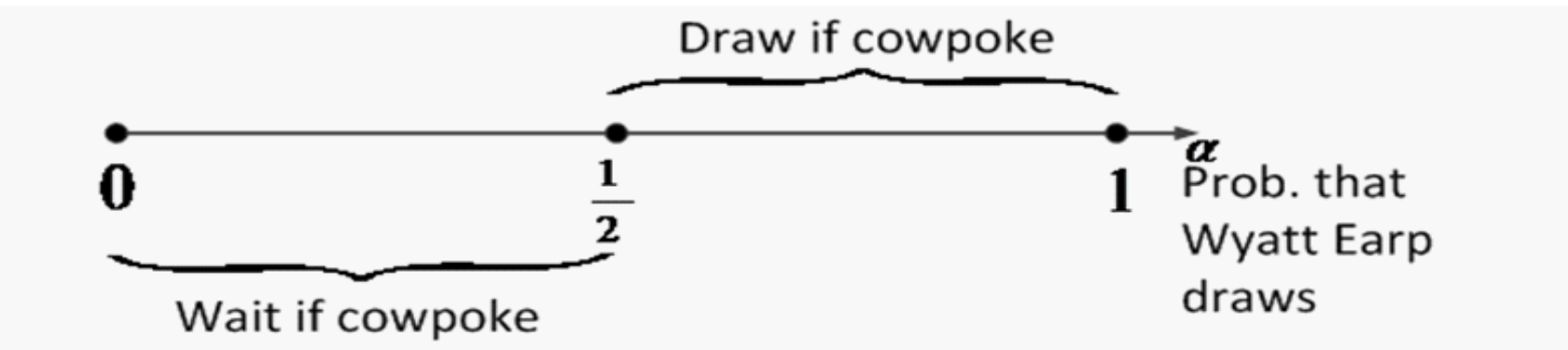
$$EU_{Stranger}(Wait|Cowpoke) = \underbrace{1\alpha}_{\text{if Earp Draws}} + \underbrace{4(1-\alpha)}_{\text{if Earp Waits}} = 4-3\alpha$$

- where α denotes the probability with which Earp draws.
- Hence, the Cowpoke decides to Draw if:

$$3-\alpha \geq 4-3\alpha \implies \alpha \geq \frac{1}{2} \rightarrow \text{next figure}$$

Cutoff strategy for the stranger:

- When the stranger is a gunslinger he draws, but when he is a cowpoke the following figure summarizes the decision rule we just found:



- Let us now turn to the uninformed player (Wyatt Earp)$\longrightarrow$

Uninformed player - first case:

IF $\alpha \geq \frac{1}{2}$

- The Stranger Draws as a Cowpoke since $\alpha \geq \frac{1}{2}$.
- Then, the expected payoffs for the uninformed player (Earp) are

$$EU_{\text{Earp}}(Draw) = \underbrace{0.75 \times 2}_{\text{if gunslinger}} + \underbrace{0.25 \times 5}_{\text{if cowpoke}} = 2.75$$

$$EU_{\text{Earp}}(Wait) = \underbrace{0.75 \times 1}_{\text{if gunslinger}} + \underbrace{0.25 \times 6}_{\text{if cowpoke}} = 2.25$$

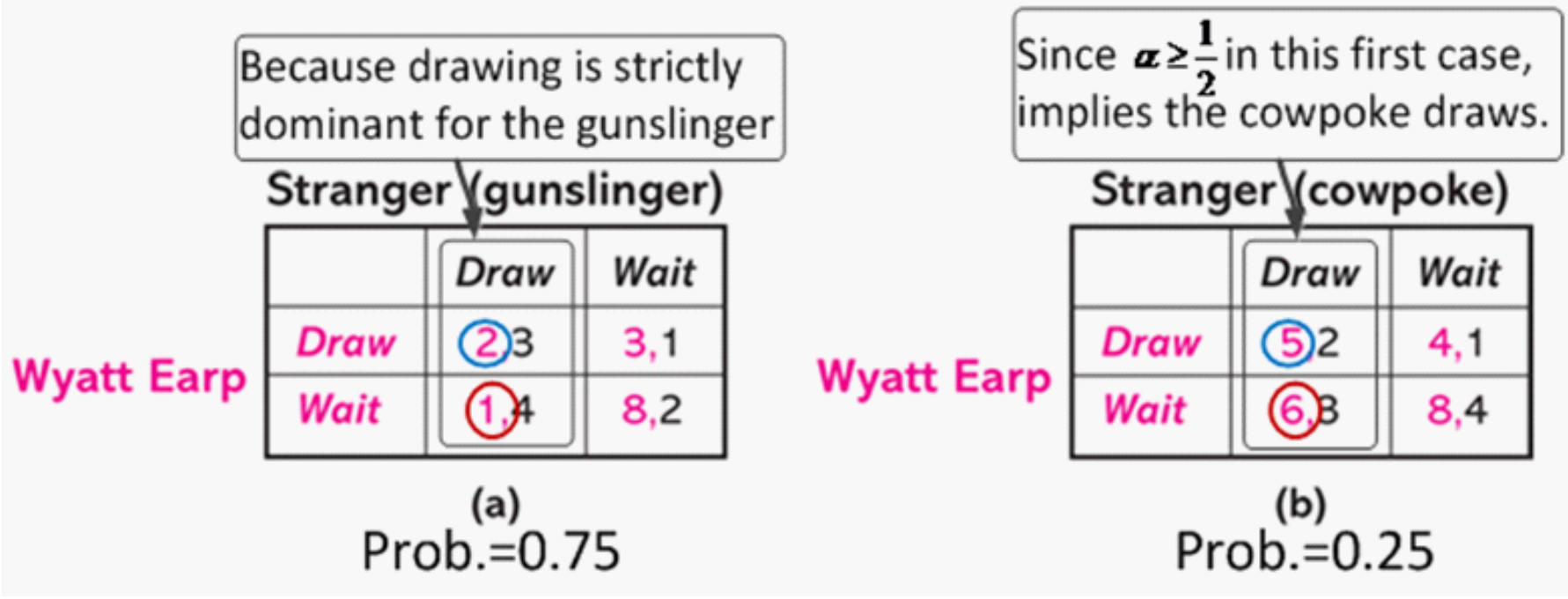
$\rightarrow$figure of these payoffs in next slide

- Hence, if $\alpha \geq \frac{1}{2}$ Earp chooses to Draw since $2.75 > 2.25$.
- The BNE of this game in the case that $\alpha \geq \frac{1}{2}$ is

$$\underbrace{\text{Draw}}_{\text{Earp}}, \underbrace{(\text{Draw,Draw})}_{\text{Stranger}}$$

Uninformed player - first case:

Case 1: $\alpha \geq \frac{1}{2}$



Uninformed player - second case:

IF $\alpha < \frac{1}{2}$

- The Stranger Waits as a Cowpoke since $\alpha < \frac{1}{2}$.
- Then, the expected payoffs for the uninformed player (Earp) are

$$EU_{\text{Earp}}(Draw) = \underbrace{0.75 \times 2}_{\text{if gunslinger}} + \underbrace{0.25 \times 4}_{\text{if cowpoke}} = 2.5$$

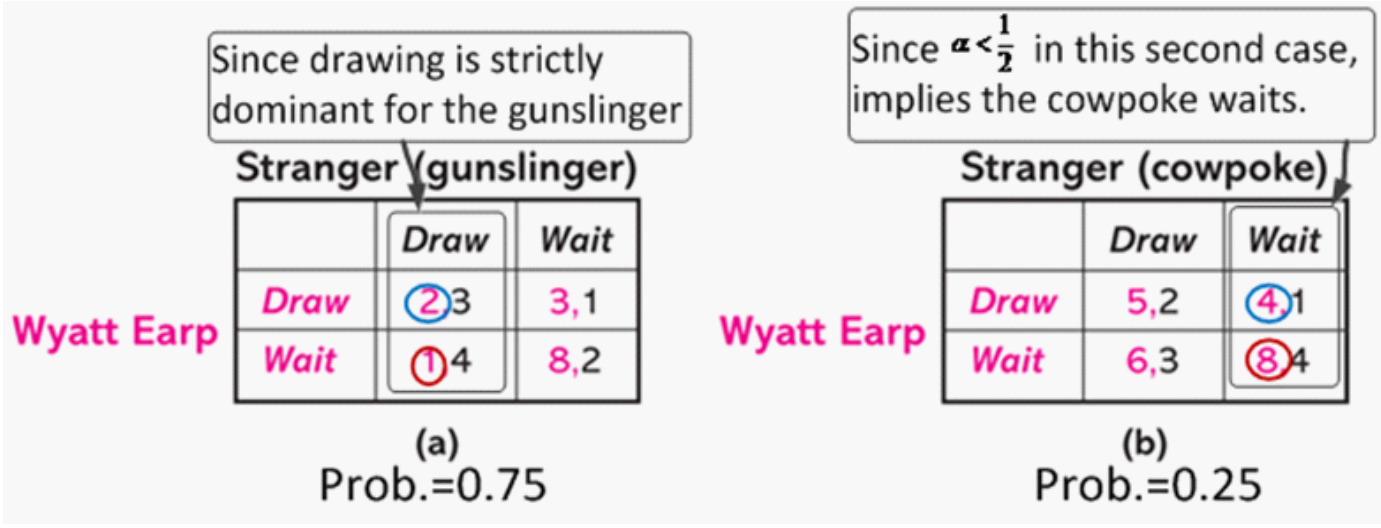
$$EU_{\text{Earp}}(Wait) = \underbrace{0.75 \times 1}_{\text{if gunslinger}} + \underbrace{0.25 \times 8}_{\text{if cowpoke}} = 2.75$$

  $\rightarrow$figure of these payoffs in next slide
- Hence, if $\alpha < \frac{1}{2}$ Earp chooses to Wait since $2.5 < 2.75$.
- The BNE of this game in the case that $\alpha < \frac{1}{2}$ is

$$\underbrace{\text{Wait}}_{\text{Earp}}, \underbrace{(\text{Draw,Wait})}_{\text{Stranger}}$$

Uninformed player - second case:

Case 2: $\alpha < \frac{1}{2}$

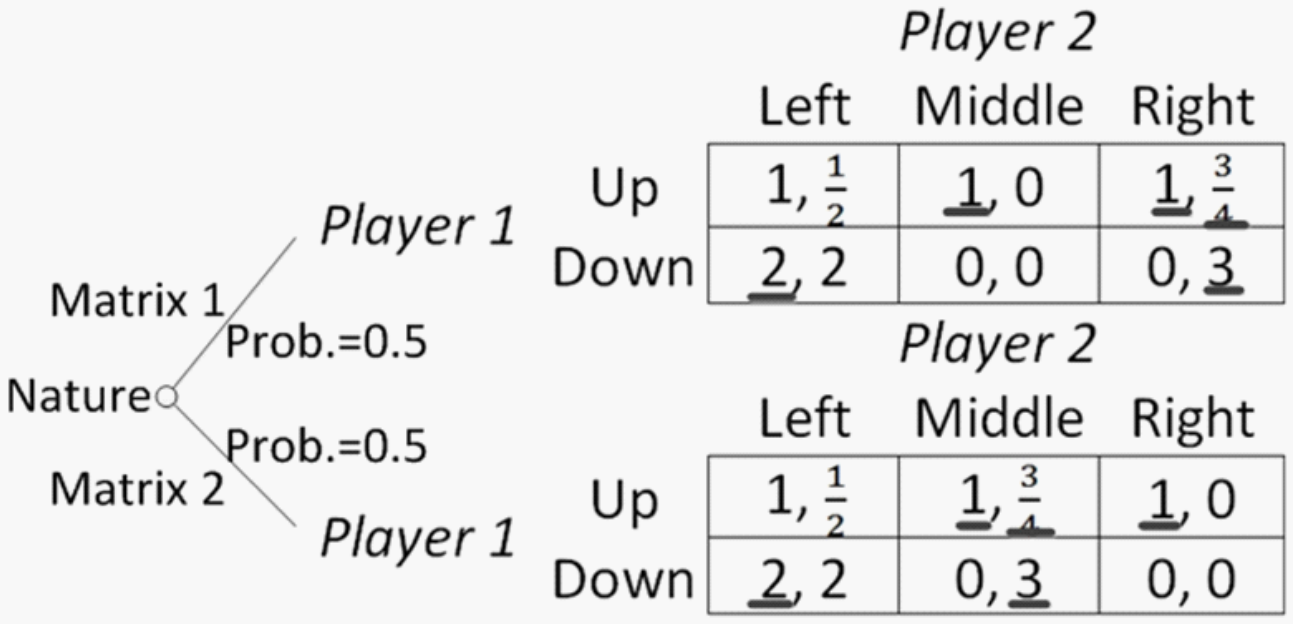


More information may hurt!

- In some contexts, the uninformed player might prefer to remain as he is (uninformed)
  - thus playing the BNE of the incomplete information game, than...
- becoming perfectly informed about all relevant information (e.g., the other player's type)
  - in which case he would be playing the standard NE of the complete information game.
- In order to show that, let us consider a game where player 2 is uninformed about which particular payoff matrix he plays...
  - while player 1 is privately informed about it.

More information may hurt!

Two players play the following game, where player 1 is privately informed about the particular payoff matrix they play.



Complete information...

1. For practice, let us first find the set of psNE of these two matrices if both players were perfectly informed:
   1. (U,R) for matrix 1, with associated equilibrium payoffs of $\left(1, \frac{3}{4}\right)$, and
   2. (U,M) for matrix 2, with the same associated equilibrium payoffs of $\left(1, \frac{3}{4}\right)$.
2. Therefore, player 2 would obtain a payoff of $\frac{3}{4}$, both:
   1. if he was perfectly informed of playing matrix 1, and
   2. if he was perfectly informed of playing matrix 2.

Complete information...

1. But, of course, player 2 is **uninformed** about which particular matrix he plays.
    1. Let us next find the BNE of the incomplete information game, and
    2. the associated *expected payoff* for the uninformed player 2.

- Recall that our goal is to check that the expected payoff for the uninformed player 2 in the BNE is lower than $\frac{3}{4}$.

Incomplete information:

1. Let us now find the set of BNEs.
2. We start with the **informed** player (player 1),
    1. who knows whether he is playing the upper, or lower matrix.
    2. Let's analyze the informed player separately in each of two matrices.

Informed player (P1) - Upper matrix

1. If he plays the **upper matrix**:
   1. His expected payoff of choosing Up (in the first row) is...

$$EU_1(Up) = \underbrace{1p}_{\text{if P2 chooses L}} + \underbrace{1q}_{\text{if P2 chooses M}} + \underbrace{1(1-p-q)}_{\text{if P2 chooses R}} = 1$$

      1. where p denotes the probability that P2 chooses L,
      2. q the probability that P2 chooses M, and
      3. $1-p-q$ the probability that P2 selects R (for a reference, see the annotated matrices in the next slide)

   2. And his expected payoff from choosing Down (in the second row) is ...

$$EU_1(Down) = \underbrace{2p}_{\text{if P2 chooses L}} + \underbrace{0q}_{\text{if P2 chooses M}} + \underbrace{0(1-p-q)}_{\text{if P2 chooses R}} = 2p$$

Informed player (P1) - Upper matrix

Informed player (P1) - Upper matrix

- Hence, when playing the upper matrix, the informed P1 chooses Up if and only if

$$EU_1(Up) > EU_1(Down) \Leftrightarrow 1 > 2p \Leftrightarrow \frac{1}{2} > p$$

Information player (P1)- Lower matrix

1. Similarly when he plays the **lower matrix:**
   1. His expected payoff of choosing Up (in the first row) is...

$$EU_1(Up) = \underbrace{1p}_{\text{if P2 chooses L}} + \underbrace{1q}_{\text{if P2 chooses M}} + \underbrace{1(1-p-q)}_{\text{if P2 chooses R}} = 1$$

   2. And his expected payoff from choosing Down (in the second row) is ...

$$EU_1(Down) = \underbrace{2p}_{\text{if P2 chooses L}} + \underbrace{0q}_{\text{if P2 chooses M}} + \underbrace{0(1-p-q)}_{\text{if P2 chooses R}} = 2p$$

(For a reference, see the Up and Down row of the lower matrix in the next slide.)

Information player (P1)- Lower matrix

Information player (P1)- Lower matrix

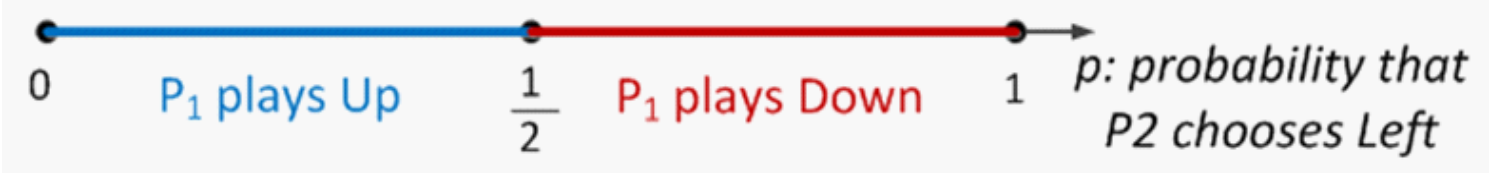
- Therefore, when playing in the lower matrix, the informed P1 chooses Up if and only if

$$EU_1(Up) > EU_1(Down) \Leftrightarrow 1 > 2p \Leftrightarrow \frac{1}{2} > p$$

- which coincides with the same decision rule that P1 uses when playing in the upper matrix.
- This happens because P1's payoffs are symmetric across matrices.

Informed player (P1)

- Summarizing, the informed player 1's decision rule can be depicted as follows



Uninformed player (P2)

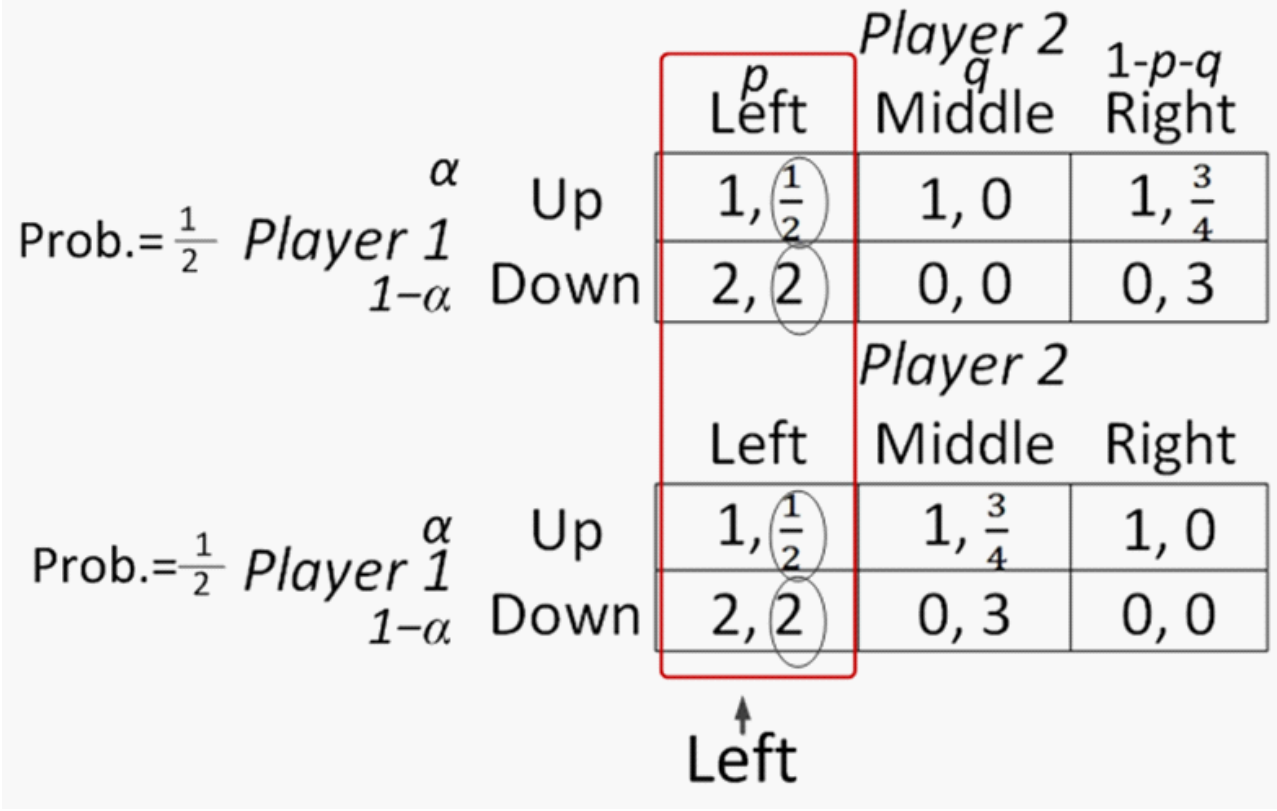
1. Regarding the **uninformed player** (player 2), he doesn't know if player 1 is playing Up or Down, so he assigns a probability α to player 1 playing Up,

$$
\begin{aligned}
EU_2(Left) &= \overbrace{\frac{1}{2}\left[\underbrace{\frac{1}{2}\alpha}_{\text{if } P_1 \text{ plays Up}} + \underbrace{2(1-\alpha)}_{\text{if } P_1 \text{ plays Down}}\right]}^{\text{if upper matrix}} \\
&\quad + \overbrace{\frac{1}{2}\left[\frac{1}{2}\alpha + 2(1-\alpha)\right]}^{\text{if lower matrix}} \\
&= 2 - \frac{3}{2}\alpha
\end{aligned}
$$

(for a visual reference of these expected payoffs, $\rightarrow$ next slide)

Uninformed player (P2) - Left Column

If P2 chooses in the left column...

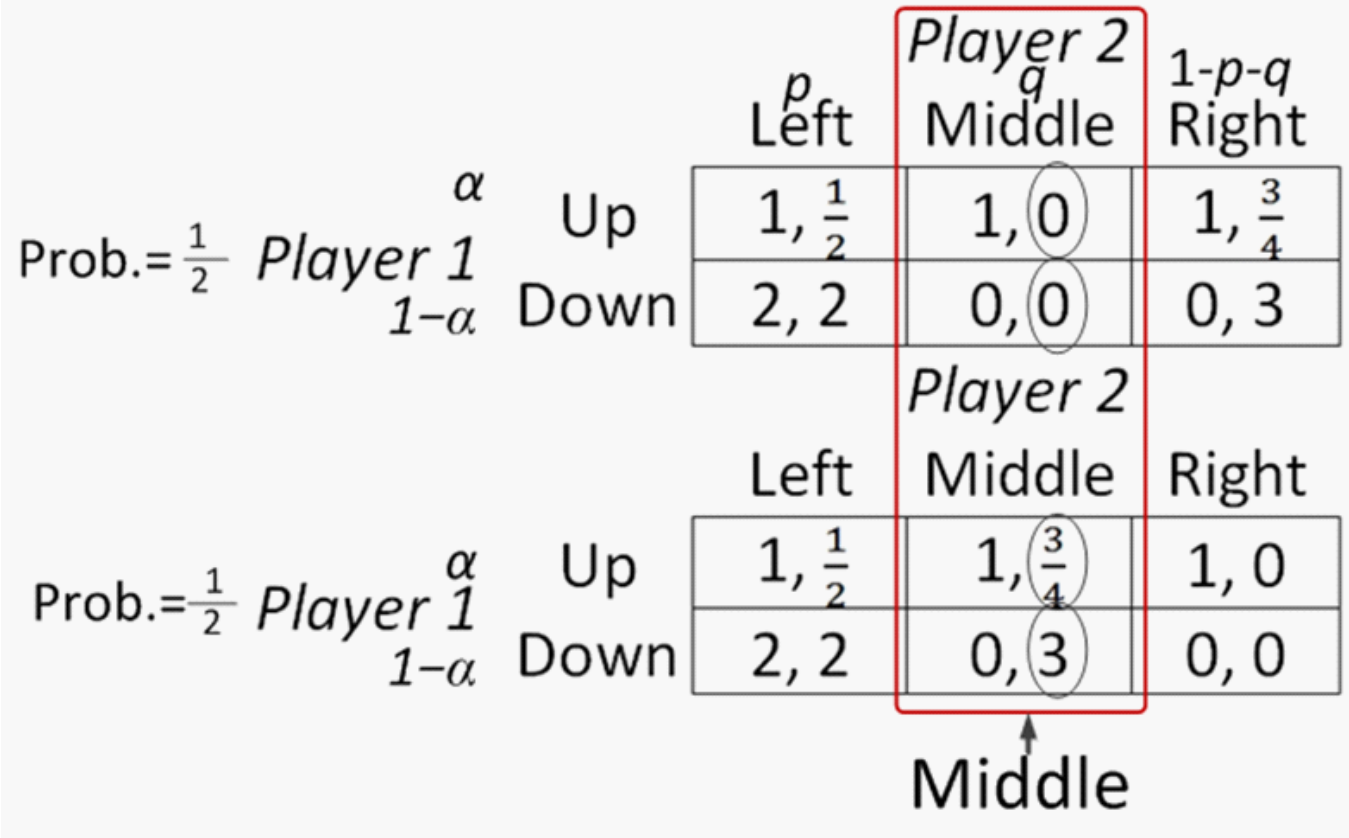


Uninformed player (P2) - Middle Column

$$
\begin{aligned}
EU_2\left(Middle\right) &= \frac{1}{2}\overbrace{\left[0\alpha + 0\left(1-\alpha\right)\right]}^{\text{if upper matrix}} + \frac{1}{2}\overbrace{\left[\frac{3}{4}\alpha + 3\left(1-\alpha\right)\right]}^{\text{if lower matrix}} \\
&= \frac{3}{2} - \frac{9}{8}\alpha
\end{aligned}
$$

Uninformed player (P2) - Middle Column

If P2 chooses in the Middle column...



Uninformed player (P2) - Right Column

$$
\begin{aligned}
EU_2\left(Right\right) &= \frac{1}{2}\overbrace{\left[\frac{3}{4}\alpha + 3\left(1-\alpha\right)\right]}^{\text{if upper matrix}} + \frac{1}{2}\overbrace{\left[0\alpha + 0\left(1-\alpha\right)\right]}^{\text{if lower matrix}} \\
&= \frac{3}{2} - \frac{9}{8}\alpha
\end{aligned}
$$

Uninformed player (P2) - Right Column

If P2 chooses in the Right column...

Uninformed player (P2)

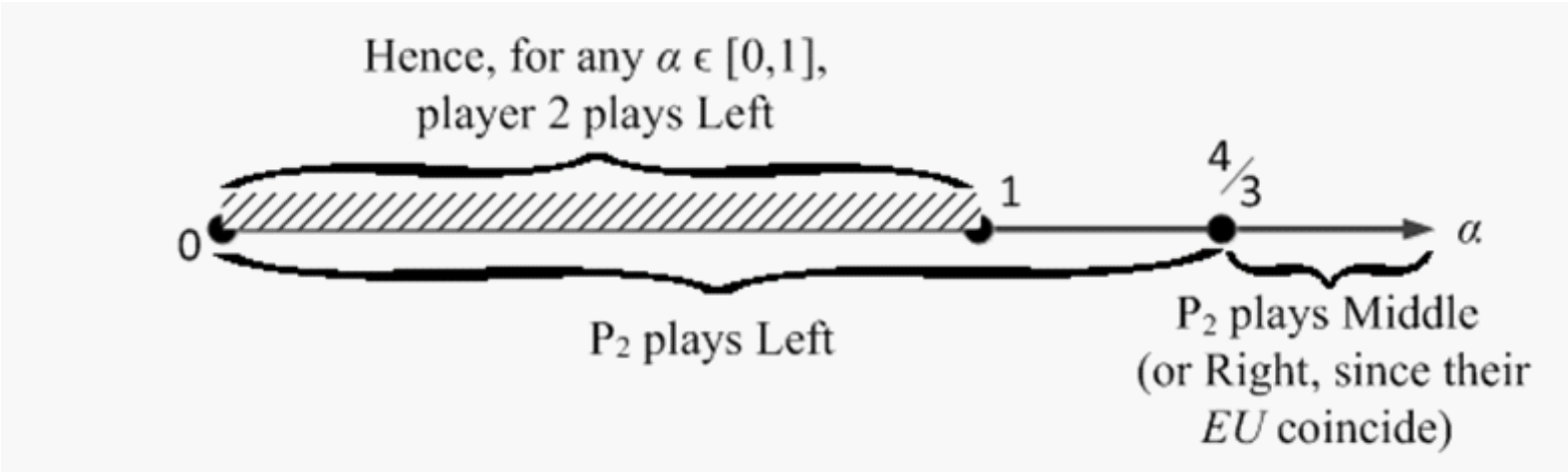
- Hence, player 2 plays Left instead of Middle, if

$$\begin{aligned} EU_2\left(\text{Left}\right) &\geq EU_2\left(\text{Middle}\right) \\ 2-\frac{3}{2}\alpha &\geq \frac{3}{2}-\frac{9}{8}\alpha \Leftrightarrow \alpha \leq \frac{4}{3} \end{aligned}$$

- [Note that the expected payoff from Middle and Right coincide, i.e., $EU_2\left(\text{Middle}\right) = EU_2\left(\text{Right}\right)$, implying that checking $EU_2\left(\text{Left}\right) \geq EU_2\left(\text{Middle}\right)$ is enough.]

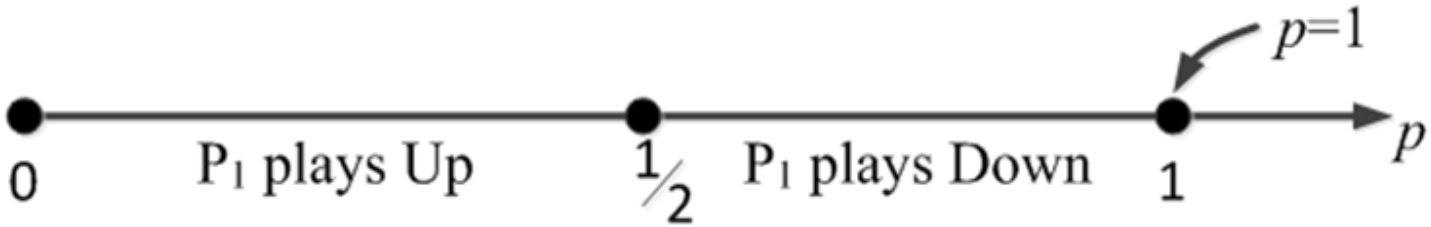
Uninformed player (P2)

- However, condition $\alpha \leq \frac{4}{3}$ holds for all probabilities $\alpha \in [0, 1]$.
  - Hence, player 2 chooses Left.



Uninformed player (P2)

- Therefore, the value of p (which denotes the probability that player 2 chooses Left) must be p=1.
- And p=1, in turn, implies that player 1...



plays Down.

- Therefore, the BNE can be summarized as follows:

$$\left\{ \underbrace{(\text{Down if matrix 1, Down if matrix 2})}_{\text{player 1}}, \quad \underbrace{\text{Left}}_{\text{player 2}} \right\}$$

Payoff comparison:

- Therefore, in the BNE the expected payoff for the uninformed player 2 is...

$$\frac{1}{2} \times 2 + \frac{1}{2} \times 2 = 2$$

- since he obtains \$2 both when the upper and lower matrices are played in the BNE:

$$\{(\text{Down if matrix 1, Down if matrix 2}),\ \text{Left}\}$$

Payoff comparison:

- Indeed, the uninformed player 2's payoff is \$2 (circled payoffs in both matrices), entailing a expected payoff of \$2 as well.

Left

		Player 2		
		Left	Middle	Right
Player 1	Up	$1, \frac{1}{2}$	$1, 0$	$1, \frac{3}{4}$
→	Down	$2, (2)$	$0, 0$	$0, 3$

		Player 2		
		Left	Middle	Right
Player 1	Up	$1, \frac{1}{2}$	$1, \frac{3}{4}$	$1, 0$
→	Down	$2, (2)$	$0, 3$	$0, 0$

Payoff comparison:

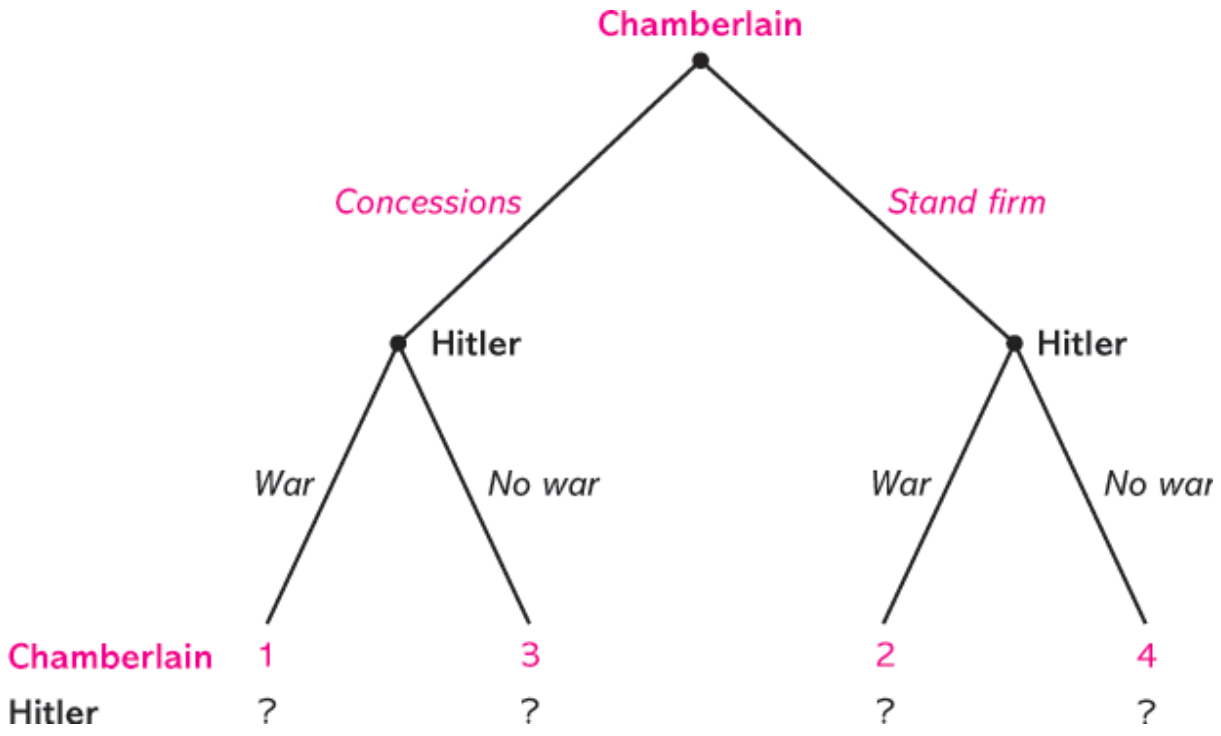
- What was player 2's payoff if he was perfectly informed about the matrix being played?
  - $\frac{3}{4}$ if he was perfectly informed of playing matrix 1 (less than in the BNE), or
  - $\frac{3}{4}$ if he was perfectly informed of playing matrix 2 (less than in the BNE).
- In contrast, in the BNE the expected payoff for the uninformed player 2 is \$2.
- Hence, more information definitely hurts the uninformed player 2!!

The Munich agreement

- Let us now turn to the Munich agreement (Harrington, Ch. 10)

The Munich agreement

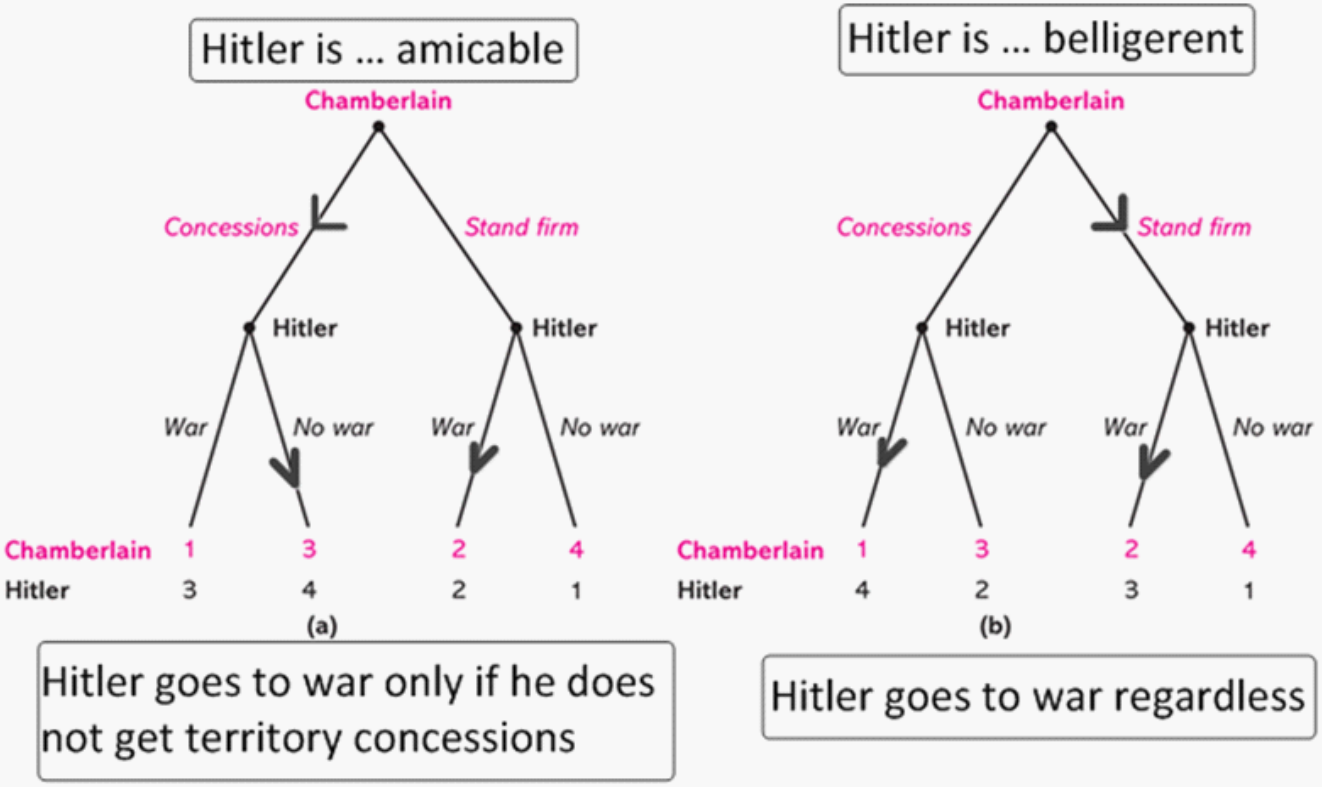
- Chamberlain does not know which are Hitler's payoffs at each contingency (i.e., each terminal node)



- How can Chamberlain decide if he does not observe Hitler's payoff?

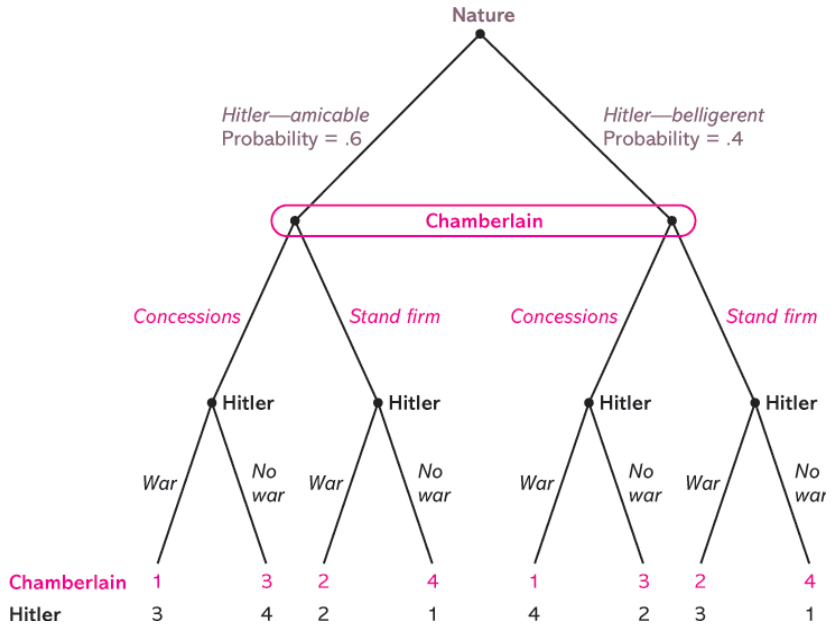
The Munich agreement

- Well, Chamberlain knows that Hitler is either belligerent or amicable.



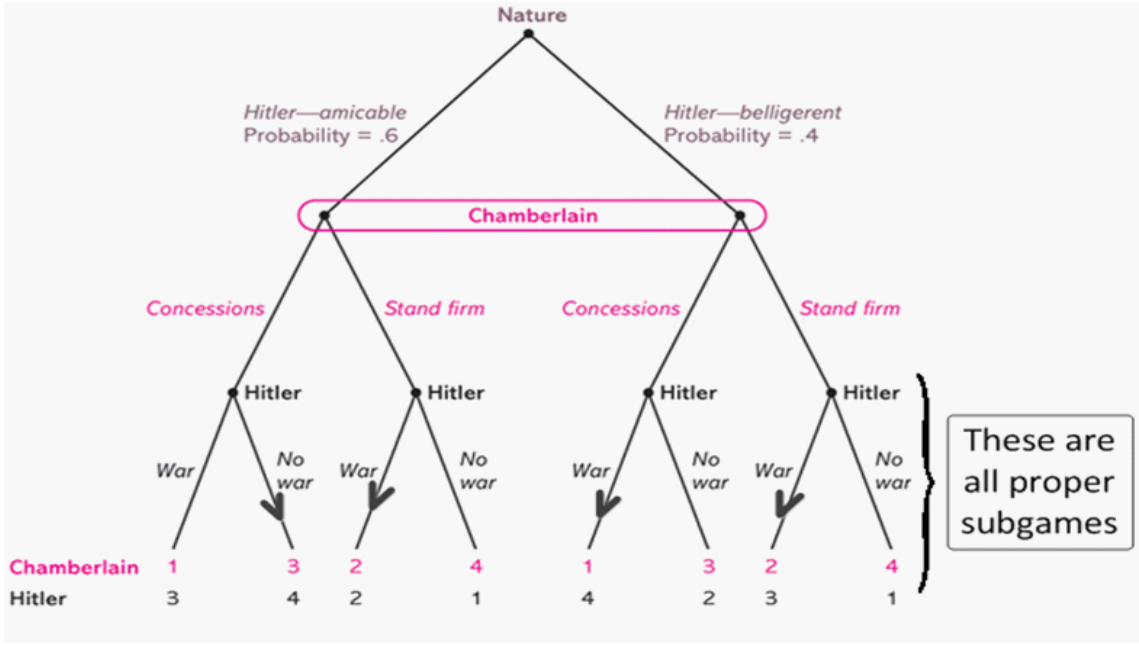
The Munich agreement

- How can we describe the above two possible games Chamberlain could face by using a single tree?
  - Simply introducing a previous move by nature which determines the "type" of Hitler.
  - Graphically, we connect both games with an information set to represent Chamberlain's uncertainty.



The Munich agreement

- In addition, Hitler's actions at the end of the game can be anticipated since these subgames are all proper.



- Hence, up to these subgames we can use backward induction (see arrows in the branches)

The Munich agreement - Hitler

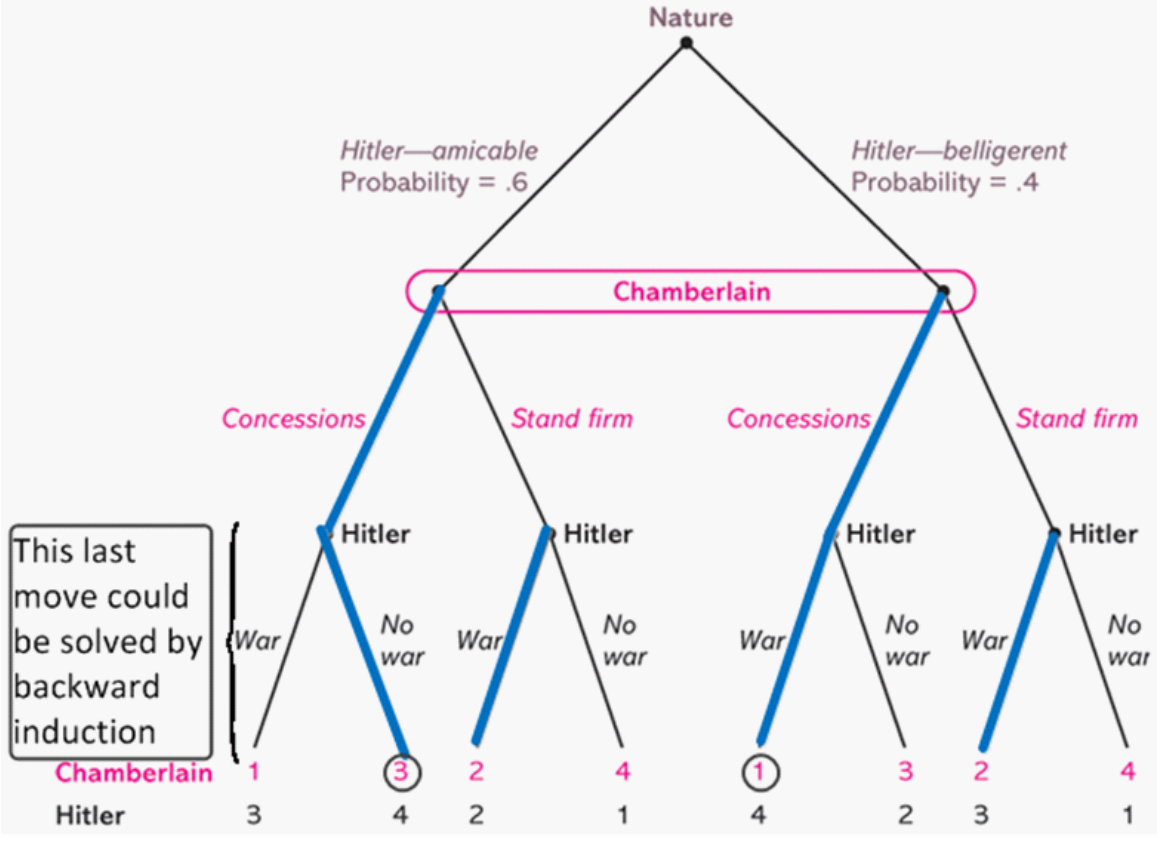
- Let's start analyzing the informed player (Hilter in this game).
- Since he is the last mover in the game, the study of his optimal actions can be done applying backward induction (see arrows in the previous game tree), as follows:
- When he is **amicable** (left side of tree), he responds choosing:
  - No war after Chamberlain gives him concessions.
  - War after Chamberlain stands firm.
- When he is **belligerent** (right side of tree), he responds choosing:
  - War after Chamberlain gives him concessions; and
  - War after Chamberlain stands firm.

The Munich agreement - Chamberlain

- Let's now move to the uninformed player (Chamberlain)
  - Note that he must choose Concessions/Stand firm unconditional on Hitler's type...
  - since Chamberlain doesn't observe Hiltler's type.
- Let's separately find Chamberlain's EU from selecting
  - Concessions (next slide).
  - Stand firm (two slides ahead)

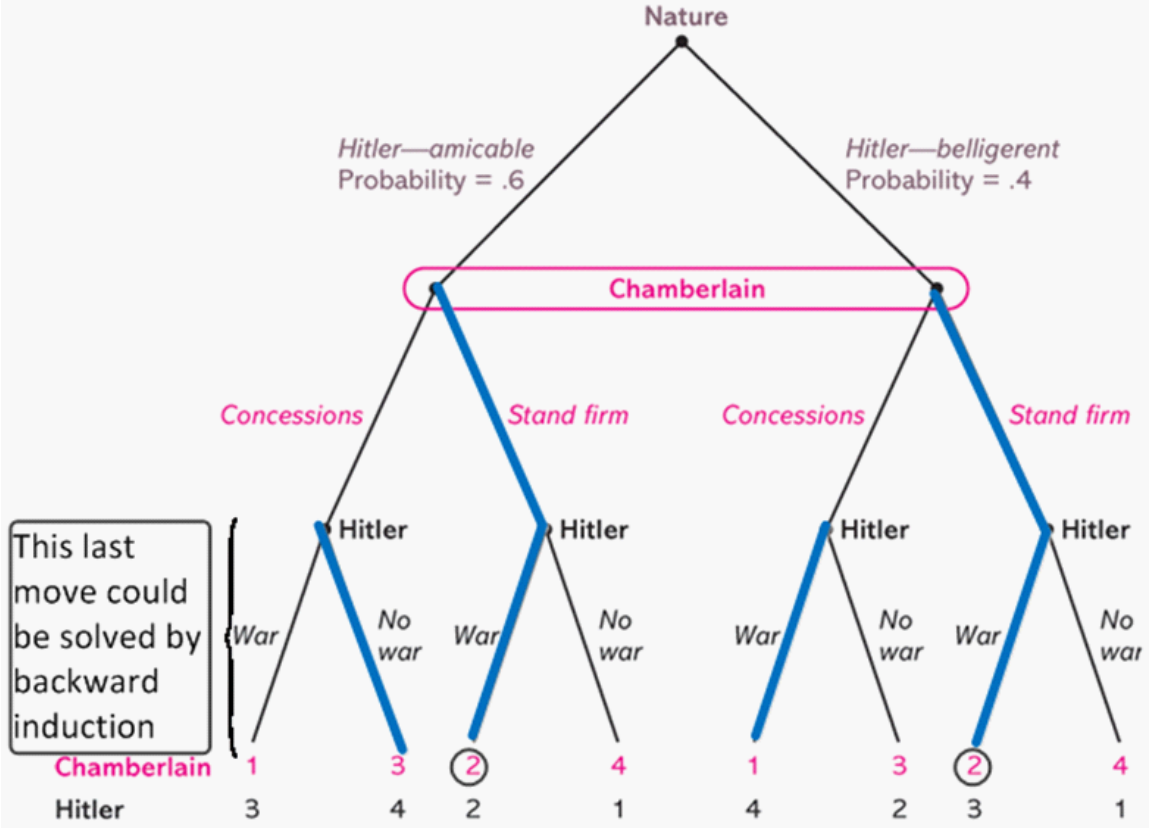
The Munich agreement - Chamberlain

- If Chamberlain chooses **Concessions**:
  - Expected payoff $= 0.6 \times 3 + 0.4 \times 1 = 2.2$



The Munich agreement - Chamberlain

- If Chamberlain chooses to **Stand firm**:
  - Expected payoff $= 0.6 \times 2 + 0.4 \times 2 = 2$



The Munich agreement - Chamberlain

- How to find out Chamberlain's best strategy?
  - If he chooses concessions:

$$\underbrace{0.6 \times 3}_{\text{if Hitler is amicable}} + \underbrace{0.4 \times 1}_{\text{if Hitler is belligerent}} = 2.2$$

  - If he chooses to stand firm:

$$\underbrace{0.6 \times 2}_{\text{if Hitler is amicable}} + \underbrace{0.4 \times 2}_{\text{if Hitler is belligerent}} = 2$$

  - Hence, Chamberlain chooses to give concessions.

The Munich agreement - Summary

- Therefore, we can summarize the BNE as
  - **Chamberlain:** gives Concessions (at the only point in which he is called on to move i.e., at the beginning of the game);
  - **Hitler:**
    - When he is amicable: NW after concessions, W after stand firm.
    - When he is belligerent: W after concessions, W after stand firm.

Cournot with incomplete information

- Thus far we considered incomplete information games in which players chose among a set of **discrete** strategies.
  - War/No war, Draw/Wait, A/B/C, etc.
- What if players have a **continuous** action space at their disposal, e.g., as in a Cournot game whereby firms can choose any output level q in $[0, \infty)$?
- Next two examples:
  - Incomplete information in market demand, and
  - Incomplete information in the cost structure.

Incomplete information about firms' costs

- Let us consider an oligopoly game where two firms compete in quantities.
- Market demand is given by the expression $p = 1 - q_1 - q_2$, and firms have incomplete information about their marginal costs.
- In particular, firm 2 privately knows whether its marginal costs are low (MC_2=0), or high (MC_2=$\frac{1}{4}$), as follows:

$$MC_2 = \begin{cases} 0 \text{ with probability } 1/2 \\ 1/4 \text{ with probability } 1/2 \end{cases}$$

Incomplete information about firms' costs

- On the other hand, firm 1 does not know firm 2's cost structure.
- Firm 1's marginal costs are $MC_1 = 0$, and this information is common knowledge among both firms (firm 2 also knows it).
- Let us find the Bayesian Nash equilibrium of this oligopoly game, specifying how much every firm produces.

Incomplete information about firms' costs

- **Firm 2.** First, let us focus on Firm 2, the informed player in this game, as we usually do when solving for the BNE of games of incomplete information.
- When firm 2 has *low costs* (L superscript), its profits are

$$Profits_2^L = (1 - q_1 - q_2^L)q_2^L = q_2^L - q_1 q_2^L \left(q_2^L\right)^2$$

- Differentiating with respect to q_2^L, we can obtain firm 2's best response function when experiencing low costs, $BRF_2^L(q_1)$.

$$1 - q_1 - 2q_2^L = 0 \Longrightarrow q_2^L(q_1) = \frac{1}{2} - \frac{q_1}{2}$$

- On the other hand, when firm 2 has *high costs* ($MC = \frac{1}{4}$), its profits are

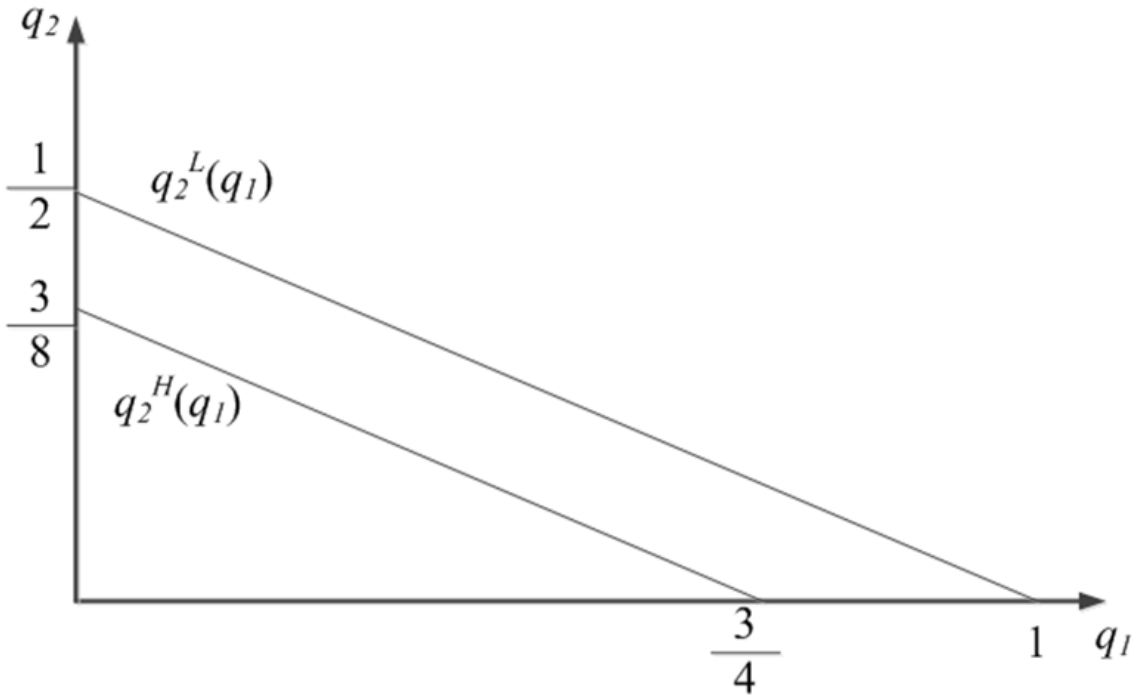
$$Profits_2^H = (1 - q_1 - q_2^H)q_2^H - \frac{1}{4}q_2^H = q_2^H - q_1 q_2^H \left(q_2^H\right)^2 - \frac{1}{4}q_2^H$$

- Differentiating with respect to q_2^H, we obtain firm 2's best response function when experiencing high costs, $BRF_2^H(q_1)$.

$$1 - q_1 - 2q_2^H - \frac{1}{4} = 0 \Longrightarrow q_2^H(q_1) = \frac{\frac{3}{4} - q_1}{2} = \frac{3}{8} - \frac{q_1}{2}$$

Incomplete information about firms' costs

- Intuitively, for a given producion of its rival (firm 1), q_1, firm 2 produces a larger output level when its costs are low than when they are high, $q_2^L(q_1) > q_2^H(q_1)$, as depicted in the figure.



Incomplete information about firms' costs

- **Firm 1.** Let us now analyze Firm 1 (the uninformed player in this game).
- First note that its profits must be expressed in expected terms, since firm 1 does not know whether firm 2 has low or high costs.

$$Profits_1 = \underbrace{\frac{1}{2}(1 - q_1 - q_2^L)q_1}_{\text{if firm 2 has low costs}} + \underbrace{\frac{1}{2}(1 - q_1 - q_2^H)q_1}_{\text{if firm 2 has high costs}}$$

Incomplete information about firms' costs

we can rewrite the profits of firm 1 as follows

$$Profits_1 = \left(\frac{1}{2} - \frac{q_1}{2} - \frac{q_2^L}{2} + \frac{1}{2} - \frac{q_1}{2} - \frac{q_2^H}{2} \right) q_1$$

And rearranging

$$Profits_1 = \left(1 - q_1 - \frac{q_2^L}{2} - \frac{q_2^H}{2} \right) q_1 = q_1 - (q_1)^2 - \frac{q_2^L}{2} q_1 - \frac{q_2^H}{2} q_1$$

Information about firms' costs

- Differentiating with respect to q_1, we obtain firm 1's best response function, $BRF_1(q_2^L, q_2^H)$.
- Note that we do not have to differentiate for the case of low and high costs, since firm 1 does not observe such information). In particular,

$$1 - 2q_1 - \frac{q_2^L}{2} - \frac{q_2^H}{2} = 0 \Longrightarrow q_1\left(q_2^L, q_2^H\right) = \frac{1}{2} - \frac{q_2^L}{2} - \frac{q_2^H}{2}$$

Incomplete information about firms' costs

- After finding the best response functions for both types of Firm 2, and for the unique type of Firm 1, we are ready to plug the first two BRFs into the latter.
- Specifically,

$$q_1 = \frac{1}{2} - \frac{\frac{1-q_1}{2}}{2} - \frac{\frac{3}{8} - \frac{q_1}{2}}{2}$$

And solving for q_1, we find $q_1 = \frac{3}{8}$.

Incomplete information about firms' costs

- With this information, i.e., $q_1 = \frac{3}{8}$, it is easy to find the particular level of production for firm 2 when experiencing *low* marginal costs,

$$q_2^L(q_1) = \frac{1 - q_1}{2} = \frac{1 - \frac{3}{8}}{2} = \frac{5}{16}$$

Incomplete information about firms' costs

- As well as the level of production for firm 2 when experiencing *high* marginal costs,

$$q_2^H(q_1) = \frac{3}{8} - \frac{\frac{3}{8}}{2} = \frac{3}{16}$$

- Therefore, the Bayesian Nash equilibrium of this oligopoly game with incomplete information about firm 2's marginal costs prescribes the following production levels

$$\left(q_1, q_2^L, q_2^H\right) = \left(\frac{3}{8}, \frac{5}{16}, \frac{3}{16}\right)$$

Incomplete information about market demand

- Let us consider an oligopoly game where two firms compete in quantities. Both firms have the same marginal costs, $MC = \$1$, but they are now asymmetrically informed about the actual state of market demand.

Incomplete information about market demand

- In particular, Firm 2 does not know what is the actual state of demand, but knows that it is distributed with the following probability distribution

$$p(Q) = \begin{cases} 10 - Q \text{ with probability } 1/2 \\ 5 - Q \text{ with probability } 1/2 \end{cases}$$

- On the other hand, firm 1 knows the actual state of market demand, and firm 2 knows that firm 1 knows this information (i.e., it is common knowledge among the players).

- **Firm 1.** First, let us focus on Firm 1, the informed player in this game, as we usually do when solving for the BNE of games of incomplete information.
- When firm 1 observes *a high* demand market its profits are

$$\begin{aligned} Profits_1^H &= (10 - Q)q_1^H - 1q_1^H \\ &= (10 - q_1^H - q_2)q_1^H - q_1^H \\ &= 10q_1^H - \left(q_1^H\right)^2 - q_2 q_1^H - 1q_1^H \end{aligned}$$

- Differentiating with respect to q_1^H, we can obtain firm 1's best response function when experiencing high demand, $BRF_1^H(q_2)$.

$$10 - 2q_1^H - q_2 - 1 = 0 \Longrightarrow q_1^H(q_2) = 4.5 - \frac{q_2}{2}$$

Incomplete information about market demand

- On the other hand, when firm 1 observes a *low demand* its profits are

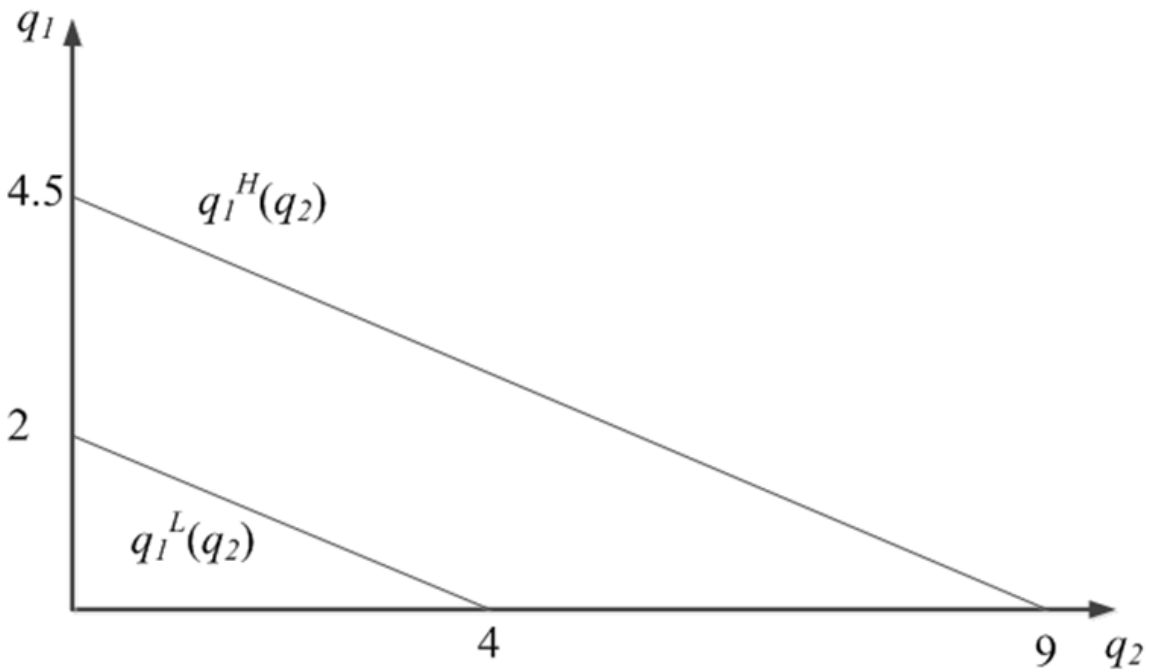
$$Profits_1^L = (5 - q_1^L - q_2)q_1^L - 1q_1^L = 5q_1^L - \left(q_1^L\right)^2 - q_2 q_1^L - 1q_1^L$$

- Differentiating with respect to q_1^L, we can obtain firm 1's best response function when experiencing low demand, $BRF_1^L(q_2)$.

$$5 - 2q_1^L - q_2 - 1 = 0 \Longrightarrow q_1^L(q_2) = 2 - \frac{q_2}{2}$$

Incomplete information about market demand

- Intuitively, for a given output level of its rival (firm 2), q_2, firm 1 produces more when facing a high than a low demand, $q_1^H(q_2) > q_1^L(q_2)$, as depicted in the figure below.



Incomplete information about market demand

- **Firm 2.** Let us now analyze Firm 2 (the *uninformed* player in this game).
- First, note that its profits must be expressed in expected terms, since firm 2 does not know whether market demand is high or low.

$$\begin{aligned} Profits_2 = & \frac{1}{2}\underbrace{\left[(10 - q_1^H - q_2)q_2 - 1q_2\right]}_{\text{demand is high}} \\ & + \frac{1}{2}\underbrace{\left[(5 - q_1^L - q_2)q_2 - 1q_2\right]}_{\text{demand is low}} \end{aligned}$$

Incomplete information about market demand

The profits of firm 2 can be rewritten as follows

$$\begin{aligned} Profits_2 &= \frac{1}{2}\left[10q_2 - q_1^H q_2 - (q_2)^2 - q_2\right] \\ &\quad + \frac{1}{2}\left[5q_2 - q_1^L q_2 - (q_2)^2 - q_2\right] \end{aligned}$$

Incomplete information about market demand

- Differentiating with respect to q_2, we obtain firm 2's best response function, $BRF_2(q_1^L, q_1^H)$.
- Note that we do not have to differentiate for the case of low and high demand, since firm 2 does not observe such information). In particular,

$$\frac{1}{2}\left[10 - q_1^H - 2q_2 - 1\right] + \frac{1}{2}\left[5 - q_1^L - 2q_2 - 1\right] = 0$$

Incomplete information about market demand

Rearranging,

$$13 - q_1^H - 4q_2 - q_1^L = 0$$

And solving for q_2, we find $\text{BRF}_2\left(q_1^L, q_1^H\right)$

$$q_2\left(q_1^L, q_1^H\right) = \frac{13 - q_1^L - q_1^H}{4} = 3.25 - 0.25\left(q_1^L + q_1^H\right)$$

Incomplete information about market demand

- After finding the best response functions for both types of Firm 1, and for the unique type of Firm 2, we are ready to plug the first two BRFs into the latter.
- Specifically,

$$q_2 = 3.25 - 0.25\left(\underbrace{\left[2 - \frac{q_2}{2}\right]}_{q_1^L} + \underbrace{\left[4.5 - \frac{q_2}{2}\right]}_{q_1^H}\right)$$

- And solving for q_2, we find $q_2 = 2.167$.

Incomplete information about market demand

- With this information, i.e., $q_2 = 2.167$, it is easy to find the particular level of production for firm 1 when experiencing *low* market demand,

$$q_1^L(q_2) = 2 - \frac{q_2}{2} = 2 - \frac{2.167}{2} = 0.916$$

Incomplete information about market demand

- As well as the level of production for firm 1 when experiencing *high* market demand,

$$q_1^H(q_2) = 4.5 - \frac{q_2}{2} = 4.5 - \frac{2.167}{2} = 3.4167$$

- Therefore, the Bayesian Nash equilibrium (BNE) of this oligopoly game with incomplete information about market demand prescribes the following production levels

$$\left(q_1^H, q_1^L, q_2\right) = (3.416, 0.916, 2.167)$$

Bargaining with incomplete information

- One buyer and one seller. The seller's valuation for an object is zero, and wants to sell it. The buyer's valuation, v, is

$$v = \begin{cases} \$10 \text{ (high) with probability } \alpha \\ \$2 \text{ (low) with probability } 1-\alpha \end{cases}$$

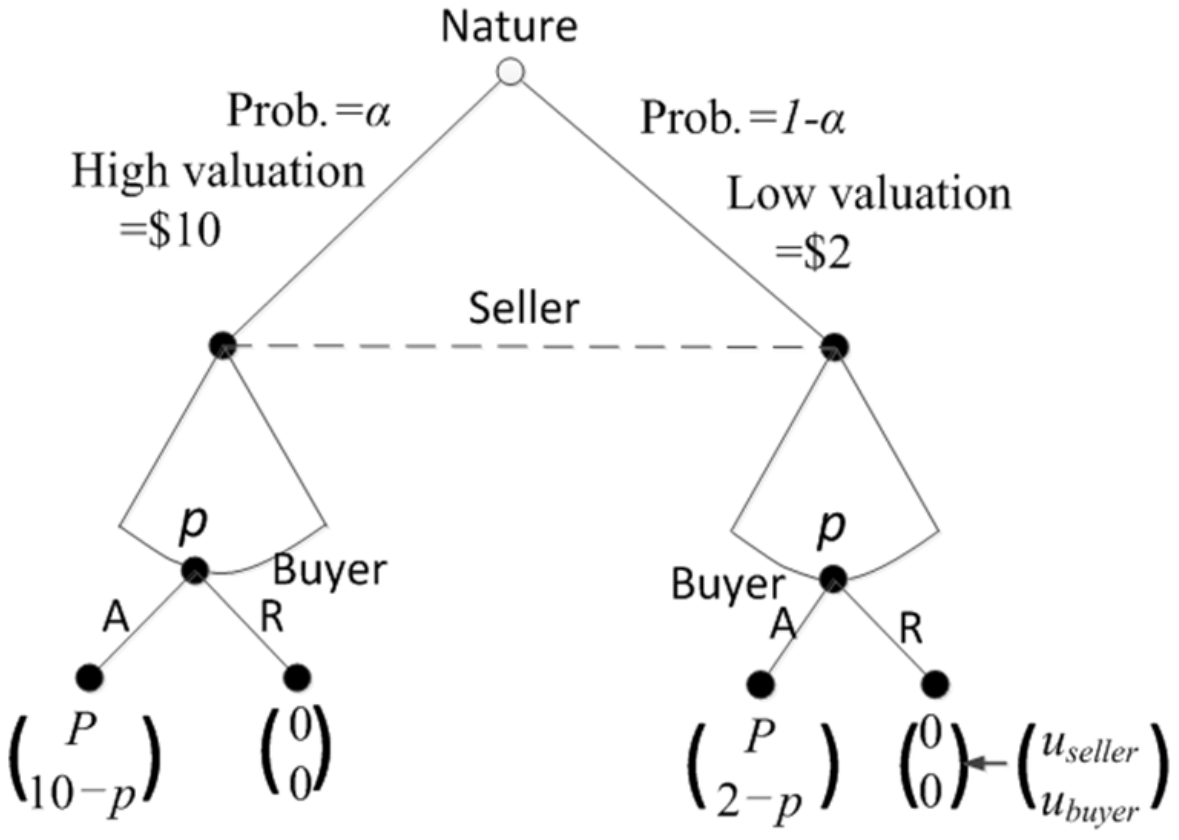
- Note that buyer's valuation v is just a normalization: it could be that
  - Buyer's value for the object is $v_{\text{buyer}} > 0$, and that of seller is $v_{\text{seller}} > 0$.
  - But we normalize both values by subtracting v_{seller}, as follows

$$v_{\text{buyer}} - v_{\text{seller}} \overset{\text{definition}}{\equiv} v$$

$$v_{\text{seller}} - v_{\text{seller}} = 0$$

- (Graphical representation of the game)

Bargaining with incomplete information

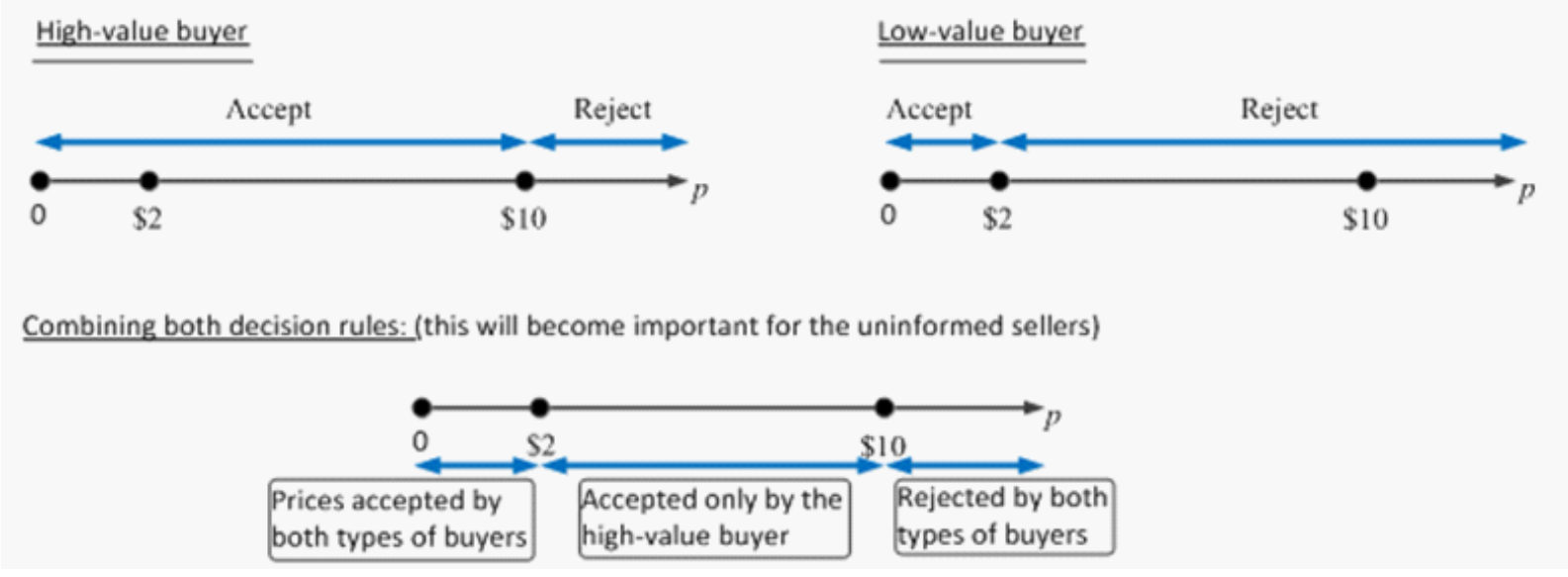


Bargaining with incomplete information

Informed player (Buyer): As usual, we start from the agent who is privately about his/her type (here the buyer is informed about her own valuation for the object).

- If her valuation is *High*, the buyer accepts any price p, such that
$$10 - p \geq 0 \Leftrightarrow p \leq 10.$$
- If her valuation is *Low*, the buyer accepts any price p, such that
$$2 - p \geq 0 \Leftrightarrow p \leq 2.$$
- Figure summarizing these acceptance rules in next slide

Bargaining with incomplete information



Bargaining with incomplete information

- **Uninformed player (Seller):** Now, regarding the seller, he sets a price p=\$10 if he knew that buyer is High, and a price of p=\$2 if he knew that he is Low.
- But he only knows the probability of High and Low. Hence, he sets a price of p=\$10 if and only if

$$EU_{\text{seller}}\left(p = \$10\right) \geq EU_{\text{seller}}\left(p = \$2\right)$$

$$\Leftrightarrow 10\alpha + 0\left(1-\alpha\right) \geq 2\alpha + 2\left(1-\alpha\right)$$

- since for a price of p= \$10 only the High-value buyer buys the good (which occurs with a probability α), whereas...
- both types of buyer purchase the good when the price is only p=\$2.

Bargaining with incomplete information

- **Uniformed player (Seller):** Solving for α in the expected utility comparison...

$$EU_{\text{seller}}\left(p = \$10\right) \geq EU_{\text{seller}}\left(p = \$2\right)$$

$$\Leftrightarrow 10\alpha + 0\left(1-\alpha\right) \geq \underbrace{2\alpha + 2\left(1-\alpha\right)}_{2}$$

we obtain

$$10\alpha + 0\left(1-\alpha\right) \geq 2 \Leftrightarrow \alpha \geq \frac{1}{5}$$

Bargaining with incomplete information

Natural questions at this point:

1. Why not set p=\$8? Or generally, why not set a price between \$2 and \$10?
    1. Low-value buyers won't be willing to buy the good.
    2. High-value buyers will be able to buy, but the seller doesn't extract as much surplus as by setting a price of p=\$10.
2. Why not set p>\$10?
    1. No customers of either types are willing to buy the good!
3. Why not set p<\$2?
    1. Both types of customers are attracted, but the seller could be making more profits by simply setting p=\$2.

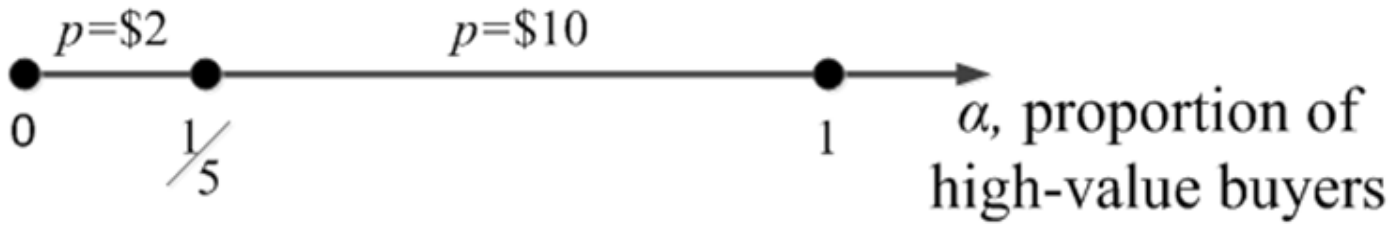
Bargaining with incomplete information

Summarizing... We have two BNE:

1. 1st BNE: if $\alpha \geq \frac{1}{5}$, (High-value buyers are very likely)
   1. the seller sets a price of $p = \$10$, and
   2. the buyer accepts any price $p \leq \$10$ if his valuation is High, and $p \leq \$2$ if his valuation is Low.
2. 2nd BNE: if $\alpha < \frac{1}{5}$, (High value buyers are unlikely)
   1. the seller sets a price of $p = \$2$, and
   2. the buyer accepts any price $p \leq \$10$ if his valuation is High, and $p \leq \$2$ if his valuation is Low.

Bargaining with incomplete information

Summarizing, the seller sets. . .



- *Comment*: The seller might get zero profits by setting $p = \$10$.This could happen if, for instance, $\alpha = \frac{3}{5}$ so the seller sets $p = \$10$, but the buyer happens to be one of the few low-value buyers who won't accept such a price.
- Nonetheless, **in expectation**, it is optimal for the seller to set $p = \$10$ when it is relatively likely that the buyer's valuation is high, i.e., $\alpha \geq \frac{1}{5}$.