EconS 301 – Intermediate Microeconomics
Review Session #5

Exercise 1
You might think that when a production function has a diminishing marginal rate of technical substitution of labor for capital, it cannot have increasing marginal products of capital and labor. Show that this is not true, using the production function $Q = K^2L^2$, with the corresponding marginal products $MP_k = 2KL^2$ and $MP_L = 2K^2L$.

Answer
Immediately we can see that the marginal products for capital and labor are increasing, since they are both positive. Now we simply need to find the $MRTS_{L,K}$. Recall that the $MRTS_{L,K}$ is simply the slope of the isoquant, and is equal to the ratio of marginal products. It is analogous to the MRS in consumer theory. So we have,

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{2K^2L}{2KL^2} = \frac{K}{L}.$$

Note that labor is in the denominator, thus as labor increases, the $MRTS_{L,K}$ is decreasing or diminishing. That is, as we move along the isoquant, increasing L, the slope is getting "less steep". Graphically, isoquants that exhibit diminishing marginal rates of technical substitution are convex to the origin (bowed toward the origin).

Thus, we have shown a production function with increasing marginal products of labor and capital can have a diminishing marginal rate of technical substitution.

Exercise 2
A firm produces quantity Q of breakfast cereal using labor L and material M with the production function $Q = 50\left(ML\right)^{\frac{1}{2}} + M + L$. The marginal product functions for this production function are

$$MP_L = 25\left(\frac{M}{L}\right)^{\frac{1}{2}} + 1$$

$$MP_M = 25\left(\frac{L}{M}\right)^{\frac{1}{2}} + 1$$

a) Are the returns to scale increasing, constant, or decreasing for this production function?

Answer
To determine the nature of returns to scale, increase all inputs by some factor λ and determine if output goes up by a factor more than, less than, or equal to λ.

$$
\begin{aligned}
Q_\lambda &= 50\left(\lambda M \lambda L\right)^{\frac{1}{2}} + \lambda M + \lambda L \\
&= 50\lambda^{\frac{1}{2}}\lambda^{\frac{1}{2}}\left(ML\right)^{\frac{1}{2}} + \lambda M + \lambda L \\
&= 50\lambda\left(ML\right)^{\frac{1}{2}} + \lambda M + \lambda L \\
&= \lambda\left[50\left(ML\right)^{\frac{1}{2}} + M + L\right] \\
&= \lambda Q
\end{aligned}
$$

Thus, by increasing all inputs by a factor λ, output goes up by a factor of λ. Since output goes up by the same factor as the inputs, this production function exhibits constant returns to scale.

b) Is the marginal product of labor ever diminishing for this production function? If so, when? Is it ever negative, and if so, when?

Answer

Recall $MP_L = 25\left(\frac{M}{L}\right)^{\frac{1}{2}} + 1$. Suppose M > 0. Holding M constant, increasing L will decrease the MP_L. The marginal product of labor is decreasing for all levels of labor. The MP_L, however, will never be negative since both components of the equation will always be greater or equal to zero. In fact, for this production function, $MP_L \geq 1$.

Exercise 3

A firm's production function is $Q = 5L^{\frac{2}{3}}K^{\frac{1}{3}}$ with $MP_K = \left(\frac{5}{3}\right)L^{\frac{2}{3}}K^{-\frac{2}{3}}$ and $MP_L = \left(\frac{10}{3}\right)L^{-\frac{1}{3}}K^{\frac{1}{3}}$.

a) Does this production function exhibit constant, increasing, or decreasing returns to scale?

Answer

$$
\begin{aligned}
Q_\lambda &= 5\left(\lambda L\right)^{\frac{2}{3}}\left(\lambda K\right)^{\frac{1}{3}} \\
&= 5\lambda^{\frac{2}{3}}\lambda^{\frac{1}{3}}L^{\frac{2}{3}}K^{\frac{1}{3}} \\
&= 5\lambda L^{\frac{2}{3}}K^{\frac{1}{3}} \\
&= \lambda\left[5L^{\frac{2}{3}}K^{\frac{1}{3}}\right] \\
&= \lambda Q
\end{aligned}
$$

Thus, we have constant returns to scale.

b) What is the marginal rate of technical substitution of L for K for this production function?

Answer

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{\left(\frac{10}{3}\right)L^{-\frac{1}{3}}K^{\frac{1}{3}}}{\left(\frac{5}{3}\right)L^{\frac{2}{3}}K^{-\frac{2}{3}}} = \frac{2K}{L}$$

c) What is the elasticity of substitution for this production function?

Answer

The formula for the elasticity of substitution is given by,

$$\sigma = \frac{\textit{percentage change in capital to labor ratio}}{\textit{percentage change in } MRTS_{L,K}} = \left(\frac{\Delta\left(\frac{K}{L}\right)}{\Delta MRTS_{L,K}}\right)\left(\frac{MRTS_{L,K}}{\frac{K}{L}}\right) = \frac{d\left(\frac{K}{L}\right)}{dMRTS_{L,K}}\left(\frac{MRTS_{L,K}}{\frac{K}{L}}\right)$$

So from part (b) we know,

$$MRTS_{L,K} = \frac{2K}{L} \rightarrow \frac{MRTS_{L,K}}{\left(\frac{K}{L}\right)} = 2 \text{ and } \frac{K}{L} = \frac{MRTS_{L,K}}{2}. \text{ So, } \frac{d\left(\frac{K}{L}\right)}{dMRTS_{L,K}} = \frac{1}{2}.$$

$$\text{Thus, } \sigma = \frac{d\left(\frac{K}{L}\right)}{dMRTS_{L,K}}\left(\frac{MRTS_{L,K}}{\frac{K}{L}}\right) = \left(\frac{1}{2}\right)2 = 1.$$

Exercise 4

Suppose a firm's production function initially took the form $Q = 500(L + 3K)$. However, as a result of a manufacturing innovation, its production function is now $Q = 1000(0.5L + 10K)$.

a) Show that the innovation has resulted in technological progress in the sense defined in the text.

Answer

We simply need to show that given a fixed combination of inputs, the quantity produced will increase as a result of the innovation. Assume a fixed level of labor and capital at 2 and 2 units respectively. Now we just calculate the quantities produced before the innovation and after the innovation using these input levels. Before the innovation, $Q = 500(L + 3K) = 500(2 + 3(6)) = 4000$. And after the innovation, $Q = 1000(0.5L + 10K) = 1000(1 + 20) = 21000$. So we obviously have more output after the innovation, thus we have technological progress.

b) Is the technological progress neutral, labor saving, of capital saving?

Answer

We simply need to look at the marginal products of labor and capital before and after the innovation. Before the innovation we have $MP_L = 500$ and $MP_K = 1500$. After the innovation we have $MP_L = 500$ and $MP_K = 10000$. So, obviously the marginal product of capital increased relative to the marginal product of labor, which remained constant. Thus, we have labor saving technology (ie they will use less labor and more capital).

MULTIPLE CHOICE EXERCISES

1. Identify the truthfulness of the following statements.

   I. When the marginal product of labor is falling, the average product of labor is falling.

   II. When the marginal product curve lies above the average product curve, then average product is rising.

   a. Both I and II are true.
   b. Both I and II are false.
   c. I is true; II is false.
   d. I is false; II is true.

Answer

Although both statements sound similar, they are actually different. The first statement is false, since the marginal product curve can lie above the average product curve and still be decreasing. And since the marginal product curve is above the average product curve, the average product curve will be increasing. Thus the first statement is false. The second statement is true, since when the marginal product curve crosses the average product curve at its highest point (maximum), thus since the marginal product curve lies below it, then it must be decreasing from its maximum. Thus the correct choice is D. Refer to page 191 in Besanko for a graph.

2. Which one of these is false when compared to the relationship between marginal and average product

   a. When average product is increasing in labor, marginal product is greater than average product. That is, if AP_L increases in L, then $MP_L > AP_L$.
   b. When average product is decreasing in labor, marginal product is less than average product. That is, if AP_L decreases in L, then $MP_L < AP_L$.
   c. The relationship between MP_L and AP_L is not the same as the relationship between the marginal of anything and the average of anything.
   d. When average product neither increases nor decreases in labor because we are at a point at which AP_L is at a maximum, then marginal product is equal to average product.

Answer

The only statement that is false is C, since the concept of average and marginal is mathematical and doesn't depend on what you are comparing.

3. The $MRTS_{L,K} =$

a. MP_K / MP_L

b. $-\Delta L / \Delta K$

c. MP_L / MP_K

d. $-MP_K / MP_L$

Answer

The correct answer is C, since from our previous problems we know $MRTS_{L,K} = \dfrac{MP_L}{MP_K}$

4. The production function $Q(L,K,M) = 25K^{0.5}L^{0.5}M^{0.5}$ exhibits

a. decreasing returns to scale.
b. constant returns to scale.
c. increasing returns to scale.
d. either decreasing or constant returns to scale, but more information is needed to determine which one.

Answer

$$\begin{aligned}
Q_\lambda &= 25(\lambda K)^{\frac{1}{2}}(\lambda L)^{\frac{1}{2}}(\lambda M)^{\frac{1}{2}} \\
&= 25\lambda^{\frac{1}{2}}\lambda^{\frac{1}{2}}\lambda^{\frac{1}{2}}K^{\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}} \\
&= 25\lambda^{\frac{3}{2}}K^{\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}} \\
&= \lambda^{\frac{3}{2}}\left[25K^{\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}}\right] \\
&= \lambda^{\frac{3}{2}}Q
\end{aligned}$$

Thus we have increasing returns to scale, so the answer is C.