Exercises – Recitation #3

Exercise 1. Find the demanded bundle for a consumer whose utility function is $u(x_1,x_2)= x_1^{3/2}x_2$ and her budget constraint is $3x_1+4x_2=100$.

Solution. Making a log transformation of the utility function, $\ln u(x_1, x_2) = \frac{3}{2}\ln x_1 + \ln x_2$

Write the Lagrangian

$$L(x,\lambda) = \frac{3}{2}\ln x_1 + \ln x_2 - \lambda(3x_1 + 4x_2 - 100)$$

(We can transform *u* this way, because the Ln function is strictly increasing.) Now, equating the derivatives with respect to x_1, x_2, and λ to zero, we get three equations in three unknowns

$$\frac{3}{2x_1} = 3\lambda,$$

$$\frac{1}{x_2} = 4\lambda,$$

$$3x_1 + 4x_2 = 100.$$

Solving, we get that the Walrasian demands at price $p_1 = 3, p_2 = 4$ and income $m = 100$ are

$$x_1(3,4,100) = 20 \text{, and } x_2(3,4,100) = 10 .$$

Note that if you are going to interpret the Lagrange multiplier as the marginal utility of income, you must be explicit as to which utility function you are referring to. Thus, the marginal utility of income can be measured in original '*utils*' or in 'ln *utils*'. Let *u**=ln*u* and, correspondingly, v*=lnv; then

$$\lambda = \frac{\partial v^*(p,m)}{\partial m} \; and \;\; \mu = \frac{\partial u^*(p,m)}{\partial m}$$

Where μ denotes the Lagrange multiplier in the Lagrangian

$$\ell(x,\mu) = x_1^{3/2}x_2 - \mu(3x_1 + 4x_2 - 100).$$

Check that in this problem we'd get $\mu = \frac{20^{3/2}}{4}$, $\lambda = \frac{1}{40}$, and $v(3,4,100) = 20^{3/2}10$.

Exercise 2. Use the utility function $u(x_1,x_2)= x_1^{1/2}x_2^{1/3}$ and the budget constraint $m=p_1x_1+p_2x_2$ to calculate the Walrasian demand, the indirect utility function, the Hicksian demand, and the expenditure function.

Solution. The Lagrangian for the utility maximization problem is

$$\ell(x,\lambda) = x_1^{1/2}x_2^{1/3} - \lambda(p_1x_1 + p_2x_2 - m),$$

Taking derivatives,

$$\frac{1}{2}x_1^{-1/2}x_2^{1/3} = \lambda p_1,$$
$$\frac{1}{3}x_1^{1/2}x_2^{-2/3} = \lambda p_2,$$
$$p_1x_1 + p_2x_2 = m.$$

Solving, we get

$$x_1(p,m) = \frac{3}{5}\frac{m}{p_1}, x_2(p,m) = \frac{2}{5}\frac{m}{p_2}.$$

Plugging these demands into the utility function, we get the indirect utility function

$$v(p,m) = U\left(x(p,m)\right) = \left(\frac{3}{5}\frac{m}{p_1}\right)^{1/2}\left(\frac{2}{5}\frac{m}{p_2}\right)^{1/3} = \left(\frac{m}{5}\right)^{5/6}\left(\frac{3}{p_1}\right)^{1/2}\left(\frac{2}{p_2}\right)^{1/3}.$$

Rewrite the above expression replacing *v(p, m)* by *u* and *m* by *e(p, u)*. Then solve it for *e(.)* to get

$$e(p,u) = 5\left(\frac{p_1}{3}\right)^{3/5}\left(\frac{p_2}{2}\right)^{2/5}u^{6/5}$$

Finally, since $h_i = \partial e / \partial p_i$, the Hicksian demands are

$$h_1(p,u) = \left(\frac{p_1}{3}\right)^{-2/5}\left(\frac{p_2}{2}\right)^{2/5}u^{6/5},$$
$$h_2(p,u) = \left(\frac{p_1}{3}\right)^{3/5}\left(\frac{p_2}{2}\right)^{-3/5}u^{6/5}.$$

Exercise 3. Consider a two-period model with Dave's utility given by $u\left(x_1, x_2\right)$ where x_1 represents his consumption during the first period and x_2 is his second period's consumption. Dave is endowed with $\left(\bar{x}_1, \bar{x}_2\right)$ which he could consume in each period, but he could also trade present consumption for future consumption and vice versa. Thus, his budget constraint is

$$p_1x_1 + p_2x_2 = p_1\bar{x}_1 + p_2\bar{x}_2,$$

where p_1 and p_2 are the first and second period prices respectively.

a) Derive the Slutsky equation in this model. (Note that now Dave's income depends on the value of his endowment which, in turn, depends on prices: $m = p_1\bar{x}_1 + p_2\bar{x}_2$.)

Solution. Differentiate the identity $h_j(p,u) \equiv x_j\left(p, e(p,u)\right)$ with respect to p_i to get

$$\frac{\partial h_j(p,u)}{\partial p_i} = \frac{\partial x_j(p,m)}{\partial p_i} + \frac{\partial x_j\left(p, e(p,u)\right)}{\partial m}\frac{\partial e(p,u)}{\partial p_i}$$

We must be careful with this last term. Look at the expenditure minimization problem

$$e(p,u) = \min\left\{p(x - \bar{x}) : u(x) = u\right\} = ph(p,u) - p\bar{x}$$

By the envelope theorem, we have

$$\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u) - \overline{x}_i = x_i\left(p, e(p,u)\right) - \overline{x}_i$$

Therefore, we have

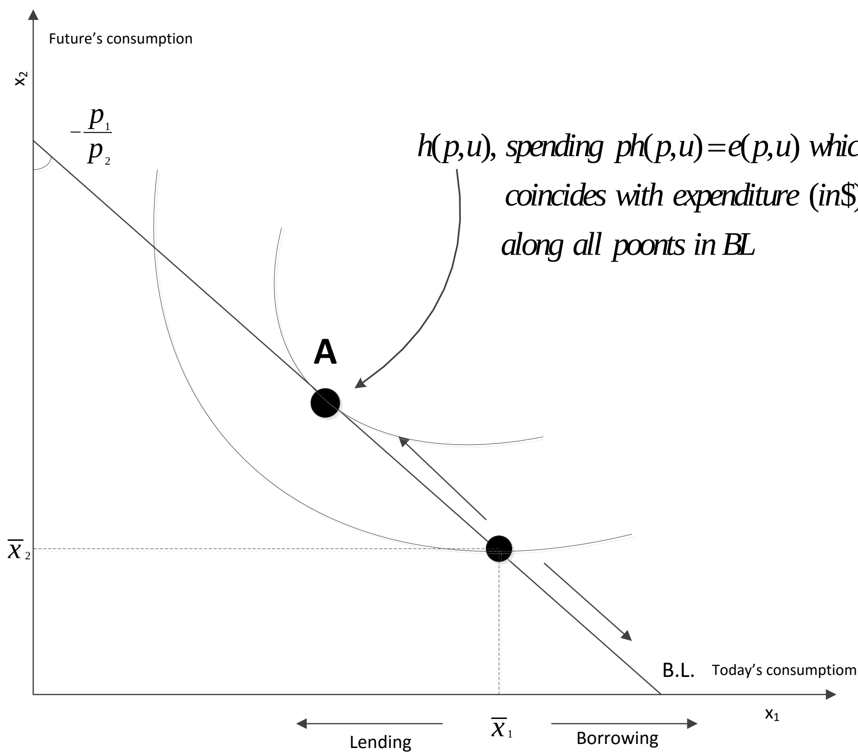
$$\frac{\partial h_j(p,u)}{\partial p_i} = \frac{\partial x_j(p,m)}{\partial p_i} + \frac{\partial x_j\left(p,e(p,u)\right)}{\partial m}\left(x_i(p,m) - \overline{x}_i\right)$$

And reorganizing we get the Slutsky equation

$$\frac{\partial x_j(p,m)}{\partial p_i} = \frac{\partial h_j(p,u)}{\partial p_i} + \frac{\partial x_j\left(p,e(p,u)\right)}{\partial m}\left(\overline{x}_i - x_i(p,m)\right)$$

b) Assume that Dave's optimal choice is such that $x_1 < \overline{x}$. If p_1 goes down, will Dave be better off or worse off? What if p_2 goes down?

Solution. The following picture depicts Dave's optimal allocation $h(p,u)$ for a given price vector p_1 / p_2.

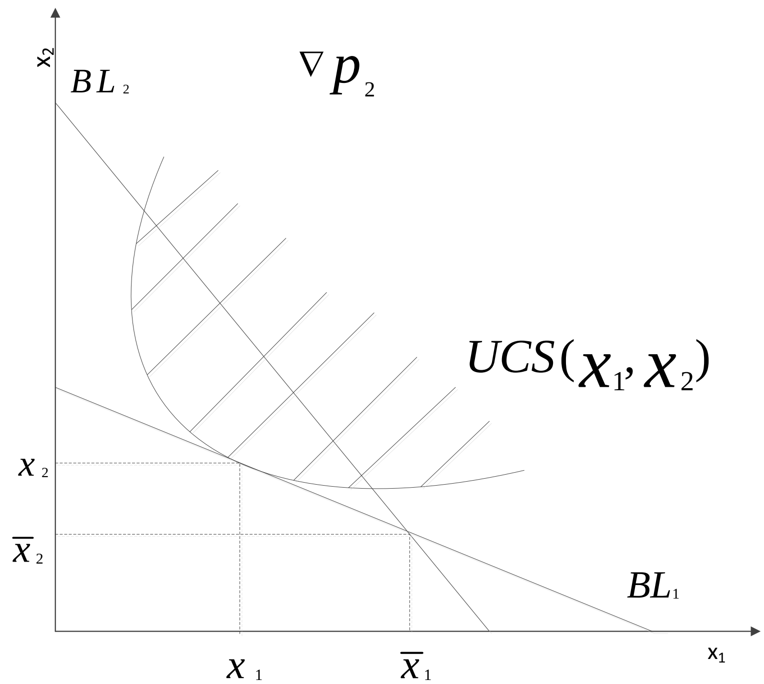


Intuitively, $p \cdot x - p \cdot \overline{x}$ measures the extra amount of money that Dave needs to spend after selling his endowment $\overline{x}$, in order to acquire his optimal consumption bundle $h(p,u)$. Hence, Dave minimizes the expenditure $p \cdot x - p \cdot \overline{x}$ at the optimal bundle $h(p,u)$, i.e., at point A of the figure.

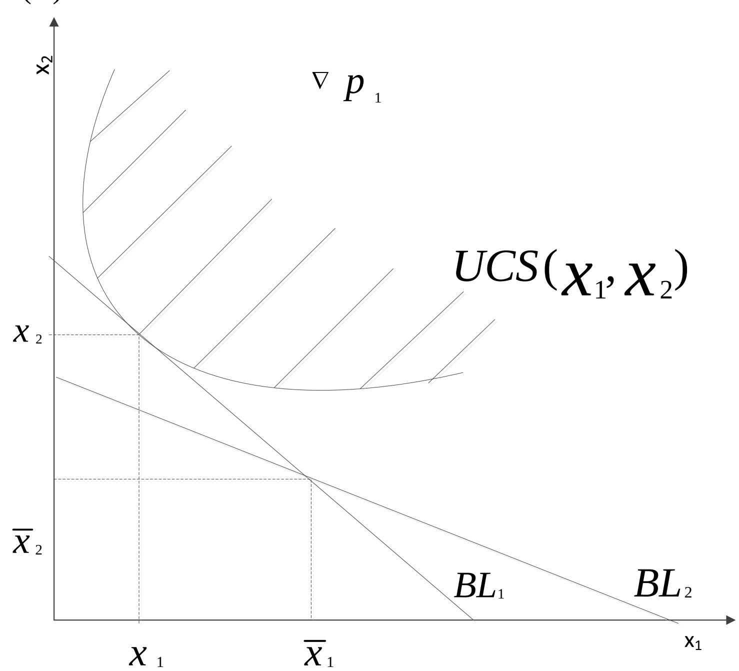
Therefore, the expenditure function of his EMP is $e(p,u) = p \cdot h(p,u) - p \cdot \overline{x}$.

Differentiating with respect to p_i, we obtain $h_i(p,u) - \overline{x}_i = x_i(p,e(p,u)) - \overline{x}_i$

When p_2 goes down, Dave is better off; since there is a region of the new budget line that lies of the $UCS(x)$, i.e., the set of bundle for which Dave is better off than at his original bundle X.



When p_1 goes down, Dave is worse off; since the new budget line, BL_2, unambiguously lies below the UCS(X)

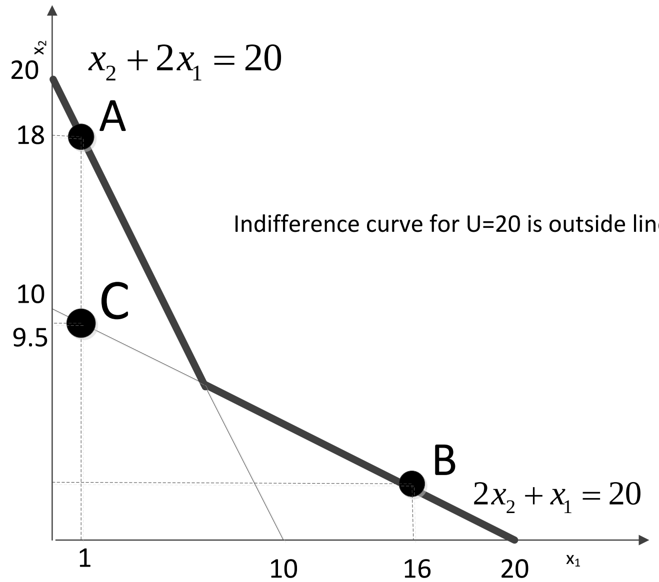


Exercise 4. The utility function is $u(x_1, x_2) = \min\{x_2 + 2x_1, x_1 + 2x_2\}$.

a) Draw the indifference curve for $u(x_1, x_2) = 20$. Shade the area where $u(x_1, x_2) \geq 20$.

Solution.

Depicting an indifferent curve. For a given utility level $u = 20$, consider $x_2 + 2x_1 = 20$ and $x_1 + 2x_2 = 20$. Solving for x_2, we obtain $x_2 = 20 - 2x_1$ and $x_2 = 10 - \frac{x_1}{2}$, respectively. We plot these two lines in the picture below. Line $x_2 = 20 - 2x_1$ originates at 20 and has a slope of -2, whereas $x_2 = 10 - \frac{x_1}{2}$ originates at 10 and has a slope of -1/2.



The indifference curve is the northeast boundary of these two lines (i.e., the upper envelope). In particularly, for a bundle $(x_1, x_2) = (1,18)$, located at point *A* in the figure, the consumer's utility is

$$\min\{18 + 2\times 1, 1 + 2\times 18\} = \min\{20, 37\} = 20.$$

Similarly, bundle *B* in the other extreme of the figure, i.e., $(x_1, x_2) = (16, 2)$, yields a utility level of

$$\min\{2 + 2\times 16, 16 + 2\times 2\} = \min\{34, 20\} = 20.$$

Note that bundles in the southeast boundary, such as $C = (1, 9.5)$, only provide a utility of

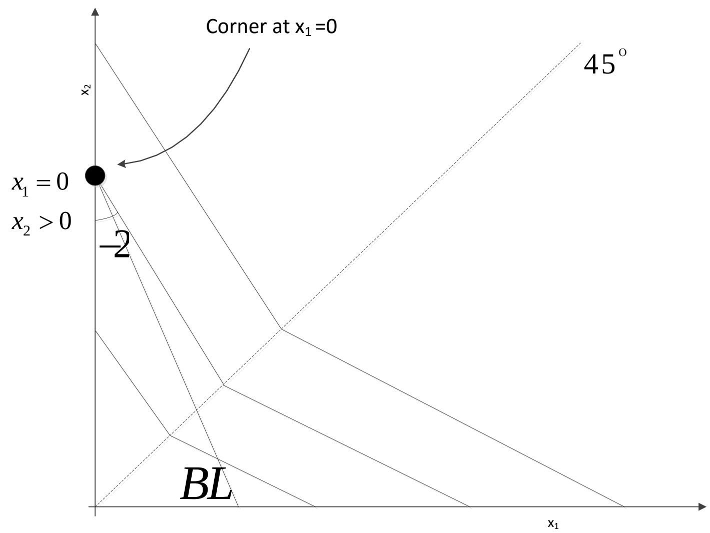
$$\min\{9.5 + 2\times 1, 1 + 2\times 9.5\} = \min\{11.5, 20\} = 11.5 < 20$$

So the southwest boundary of the two lines cannot be the indifference curve of $u = 20$. If we wanted to depict the indifference curve associated to a utility of $u = 11.5$, we would need two lines parallel to the thick lines in the figure but shifted inwards towards to origin so they cross point C.

Upper contour set. Finally, the upper contour set contains all those bundles to the northeast of the indifference curve we just depicted

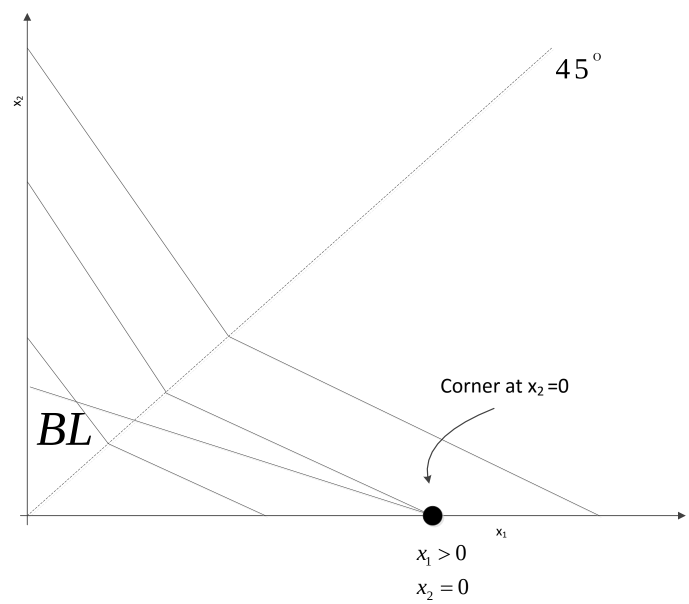
b) For what values of p_1 / p_2 will the unique optimum be $x_1 = 0$?

Solution. The slope of a budget line is $-p_1 / p_2$. If the budget line is steeper than 2, in absolute value, e.g., -5, then we have $x_1 = 0$ as illustrated in the next figure. Hence, the condition is $p_1 / p_2 > 2$.



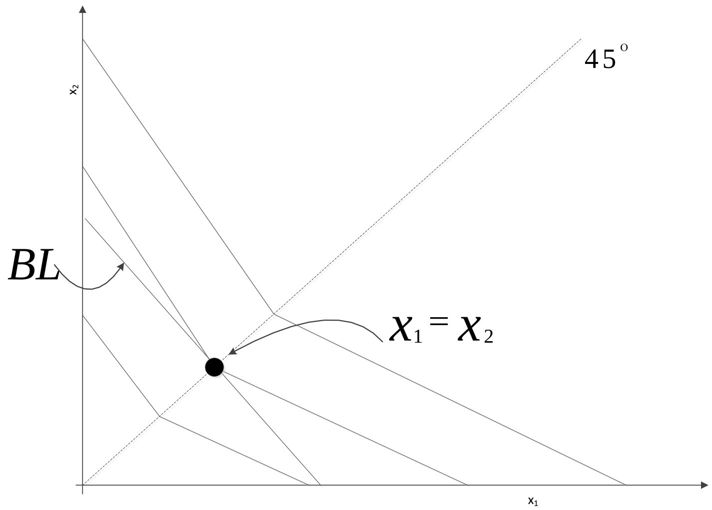
c) For what values of p_1 / p_2 will the unique optimum $x_2 = 0$?

Solution. Similarly, if the budget line is flatter than $\frac{1}{2}$ in absolute value, e.g., $-\frac{1}{4}$, x_2 will equal 0, as illustrated in the next figure. Therefore, the condition is $p_1 / p_2 < 1/2$.



d) If neither x_1 nor x_2 is equal to zero, and the optimum is unique, what must be the value of x_1 / x_2?

Solution. If the optimum is unique, it must occur where at the kink $x_2 - 2x_1 = x_1 - 2x_2$. Since line $x_2 - 2x_1$ crosses $x_1 - 2x_2$ at the 45°-line, the interior optimum occurs at $x_1 = x_2$, so that $x_1 / x_2 = 1$.



Plugging this result, $x_1 = x_2$, into the budget line, we obtain $p_1 x_2 + p_2 x_2 = w$. Solving for x_2, yields a Walrasian demand of

$$x_2 = \frac{w}{p_1 + p_2}.$$

which coincides with the Walrasian demand of good 1 since $x_1 = x_2$ at the kink.

Exercise 5. Under current tax law some individuals can save up to $2,000 a year in an Individual Retirement Account (I.R.A.), a savings vehicle that has an especially favorable tax treatment. Consider an individual at a specific point in time who has income *Y*, which he or she wants to spend on consumption, *C*, I.R.S. savings, S_1, or ordinary savings S_2. Suppose that the "reduced form" utility function is taken to be:

$$U(C, S_1, S_2) = S_1^{\alpha} S_2^{\beta} C^{\gamma}.$$

(This is a reduced form since the parameters are not truly exogenous taste parameters, but also include the tax treatment of the assets, etc.) The budget constraint of the consumer is given by:

$$C + S_1 + S_2 = Y,$$

and the limit that he or she can contribute to the I.R.A. is denoted by *L*.

a) Derive the demand functions for S_1 and S_2 for a consumer for whom the limit *L* is *not* binding.

Solution. Building the Lagrangian, we obtain:

$$L = \alpha \ln s_1 + \beta \ln s_2 + \gamma \ln C + \lambda(Y - C - s_1 - s_2) .$$

Take all derivatives with respect to λ, C, s_1, s_2 to find s_1, s_2. This is an ordinary Cobb-Douglas demand:

$$S_1 = \frac{\alpha}{\alpha+\beta+\gamma} Y \text{ and } S_2 = \frac{\beta}{\alpha+\beta+\gamma} Y .$$

b) Derive the demand function S_1 and S_2 for a consumer for whom the limit *L is* binding.

Solution. Since s_2 has reached the maximum allowed, L, we plug $S_2=L$ in the utility function $U(C, S_1, L) = S_1^{\alpha} L^{\beta} C^{\gamma}$. Note that the L term is just a constant, so applying the standard Cobb-Douglas formula $S_1 = \frac{\alpha}{\alpha+\gamma} Y$.

Exercise 3.E.7. Show that if a preference relation is quasilinear with respect to good 1, the Hicksian demand functions for the remaining goods 2, 3, …, *L* do not depend on *u*. What is the form of the expenditure function in this case?

Solution. Exercise 3.C.5(b) in MWG shows that every quasilinear preference with respect to good 1 can be represented by a utility function of the form $u(x) = x_1 + \tilde{u}(x_2, \ldots, x_L)$. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^L$. We shall prove that for every $p >> 0$ with $p_1 = 1,\ u \in \mathbb{R},\ \alpha \in \mathbb{R},$ and $x \in (-\infty, \infty) \times \mathbb{R}_+^{L-1},$ if $x = h(p,u)$, then $x + \alpha e_1 = h(p, u+\alpha).$ Note first that $u(x+\alpha e_1) \geq u + \alpha,$ that is, $x + \alpha e_1$ satisfies the constraint of the EMP for $(p, u+\alpha).$ Let $y \in \mathbb{R}_+^L$ and $u(y) \geq u + \alpha$. Then $u(y - \alpha e_1) \geq u.$ Hence, $p \cdot (y - \alpha e_1) \geq p \cdot x.$ Thus $p \cdot y \geq p \cdot (x + \alpha e_1).$ Hence $x + \alpha e_1 = h(p, u+\alpha).$

Therefore, for every good $\ell \in \{2 \ldots, L\},\ u \in \mathbb{R},$ and $u' \in \mathbb{R},\ h_\ell(p,u) = h_\ell(p,u').$ That is, the hicksian demand functions for goods $2, \ldots L$ are independent of the utility level that the individual must reach in his EMP. Thus, if we define the hicksian demand of reaching a zero utility level as $\tilde{h}(p) = h(p,0),$ then the hicksian demand of reaching a positive utility level $u > 0$, $h(p,u)$, is $h(p,u) = \tilde{h}(p) + ue_1,$ where the positive utility originates from units of good 1.
We can extend the above argument by saying that the hicksian of reaching an even farther utility level $u + \alpha, h(p, u+\alpha)$, is $h(p, u+\alpha) = h(p,u) + \alpha e_1,$ that is, the hicksian from reaching utility level u plus additional units of good 1. Thus, we have that the expenditure function of such hicksian demand, $h(p, u+\alpha)$ is $e(p, u+\alpha) = e(p,u) + \alpha$, which indicates that, in order to increase the utility level from u to $u+\alpha$, the consumer must increase his minimal expenditure from $e(p,u)$ to $e(p,u)+\alpha$. Thus, if we define the expenditure of reaching a zero utility level as $\tilde{e}(p) = e(p,0),$ then the minimal expenditure of reaching a positive utility level $u > 0$ is $e(p,u) = \tilde{e}(p) + u$.

Exericse 3.E.8. For the Cobb-Douglas utility function, verify that the following relationships in (3.E.1) and (3.E.3) respectively hold.

e(p,v(p,w))=w and *v(p,e(p,u))=u*, and
h(p,u)=x(p,e(p,u)) and *x(p,w)=h(p,v(p,w))*

Note that the expenditure function can be derived by simply inverting the indirect utility function, and vice versa.

Solution. We use the utility function $u(x)=x_1^{\alpha}x_2^{(1-\alpha)}$. To prove (3.E.1),

$$e\left(p,v\left(p,w\right)\right)=\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}\left(\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}w\right)=w,$$

$$v\left(p,e\left(p,u\right)\right)=\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}\left(\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u\right)=u.$$

To prove (3.E.3),

$$\begin{aligned}x\left(p,e\left(p,u\right)\right)&=\left(\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u\right)\left(\alpha/p_1,(1-\alpha)/p_2\right)\\&=\left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}u,\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}u\right)=h\left(p,u\right),\end{aligned}$$

$$\begin{aligned}h\left(p,v\left(p,w\right)\right)&=\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}w\left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha},\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}\right)\\&=w\left(\alpha/p_1,\ (1-\alpha)/p_2\right)=x\left(p,w\right).\end{aligned}$$

Exercise 3.E.9. Use the relations in 3.E.1:

e(p,v(p,w))=w and *v(p,e(p,u))=u*

to show that the properties of the indirect utility function *e(p,u)* identified in Proposition 3.E.2:

1. Homogeneous of degree one in prices.
2. Strictly increasing in *u* and nondecreasing in p_k for any good *k*.
3. Concave in prices.
4. Continuous in *p* and *w*.

imply the properties of the expenditure function *v*(*p*,*w*) identified in Proposition 3.D.3:

1. Homogeneity of degree zero.

2. Strictly increasing in w and nonincreasing in p_k for any good k.

3. Quasiconvex; that is, the set $\{(p,w): v(p,w)\le v\}$ is convex for any v.

4. Continuous in p and w.

Likewise, use the relations

$$e(p,v(p,w))=w \text{ and } v(p,e(p,u))=u$$

to prove that the properties of $v(p,w)$ identified in Proposition 3.D.3 imply the properties of $e(p,u)$ identified in Proposition 3.E.2.

Solution. First, we shall prove that Proposition 3.D.3 implies Proposition 3.E.2 via (3.E.1). Let $p>>0,\ p'>>0,\ u\in\mathbb{R},\ u'\in\mathbb{R},$ and $\alpha\ge 0$.

(i) Homogeneity of degree one in p: Let $\alpha>0$. Define $w=e(p,u)$, then $u=v(p,w)$ by the second relation of (3.E.1). Hence

$$e(\alpha p,u)=e\big(\alpha p,v(p,w)\big)=e\big(\alpha p,v(\alpha p,\alpha w)\big)=\alpha w=\alpha e(p,u),$$

where the second equality follows from the homogeneity of $v(\cdot,\cdot)$ and the third from the first relation of (3.E.1).

(ii) Monotonicity: Let $u'>u$. Define $w=e(p,u)$ and $w'=e(p,u')$, then $u=v(p,w)$ and $u'=v(p,w')$. By the monotonicity of $v(\cdot,\cdot)$ in w, we must have $w'>w$, that is, $e(p',u)>e(p,u)$.

Next let $p'\ge p$. Define $w=e(p,u)$ and $w'=e(p',u)$, then, by the second relation of (3.E.1), $u=v(p,w)=v(p',w')$. By the monotonicity of $v(\cdot,\cdot)$, we must have $w'\ge w$, that is, $e(p',u)\ge e(p,u)$.

(iii) Concavity: Let $\alpha\in[0,1]$. Define $w=e(p,u)$ and $w'=e(p',u)$, then $u=v(p,w)=v(p',w)$. Define $p''=\alpha p+(1-\alpha)p''$ and $w''=\alpha w+(1-\alpha)w'$. Then, by the quasiconvexity of $v(\cdot,\cdot)$, $v(p'',w'')\le u=v(p'',e(p'',u))$. Hence, by the monotonicity of $v(\cdot,\cdot)$ in w and the second relation of (3.E.1), $w''\le e(p'',u)$. That is,

$$e\big(\alpha p+(1-\alpha)p',u\big)\ge \alpha e(p,u)+(1-\alpha)e(p',u).$$

(iv) Continuity: It is sufficient to prove the following statement: For any sequence $\left\{\left(p^n,u^n\right)\right\}_{n=1}^{\infty}$ with $\left(p^n,u^n\right)\to(p,u)$ and any w, if $e\left(p^n,u^n\right)\le w$ for every n, then

$e(p,u) \leq w$; if $e\left(p^n,u^n\right) \geq w$ for every *n*, then $e(p,u) \geq w$. Suppose $e\left(p^n,u^n\right) \leq w$ for every *n*. Then, by the monotonicity of $v(\cdot,\cdot)$ in *w*, and the second relation of (3.E.1), we have $u^n \leq v\left(p^n,w\right)$ for every *n*. By the continuity of $v(\cdot,\cdot)$, $u \leq v(p,w)$. By the second relation of (3.E.1) and the monotonicity of $v(\cdot,\cdot)$ in *w*, we must have $e(p,u) \leq w$. The same argument can be applied for the case with $e\left(p^n,u^n\right) \geq w$ for every *n*.

Let's next prove that Proposition 3.E.2 implies Proposition 3.D.3 via (3.E.1). Let $p >> 0$, $p' >> 0$, $w \in \mathbb{R}$, $w' \in \mathbb{R}$, and $\alpha \geq 0$.

i. ***Homogeneity:*** Let $\alpha > 0$. Define $u = v(p,w)$. Then, by the first relation of (3.E.1), $e(p,u) = w$. Hence

$$v(\alpha p, \alpha w) = v\left(\alpha p, \alpha e(p,w)\right) = v\left(\alpha p, e(\alpha p,u)\right) = u = v(p,w),$$

where the second equality follows from the homogeneity of $e(\cdot,\cdot)$ and the third from the second relation of (3.E.1).

ii. ***Monotonicity:*** Let $w' > w$. Define $u = v(p,w)$ and $u' = v(p,w')$, then $e(p,u) = w$ and $e(p,u') = w'$. By the monotonicity of $e(\cdot,\cdot)$ and $w' > w$, we must have $u' > u$, that is, $v(p,w') > v(p,w)$.

Next, assume that $p' \geq p$. Define $u = v(p,w)$ and $u' = v(p',w)$, then $e(p,u) = e(p',u') = w$. By the monotonicity of $e(\cdot,\cdot)$ and $p' \geq p$, we must have $u' \leq u$, that is, $v(p,w) \geq v(p',w)$.

iii. ***Quasiconvexity:*** Quasiconvexity means that the lower contour set (LCS) is convex. Let $\alpha \in [0,1]$. Define $u = v(p,w)$ and $u' = v(p',w')$. Then $e(p,u) = w$ and $e(p',u') = w'$. Without loss of generality, assume that $u' \geq u$. Define $p'' = \alpha p + (1-\alpha) p'$ and. $w'' = \alpha w + (1-\alpha) w'$ Then

$$e(p'',u')$$

$$\geq \alpha e(p,u') + (1-\alpha)e(p',u')$$

$$\geq \alpha e(p,u) + (1-\alpha)e(p',u')$$

$$= \alpha w + (1-\alpha)w' = w'',$$

where the first inequality follows from the concavity of $e(\cdot,u)$ the second from the monotonicity of $e(\cdot,\cdot)$ in u and $u' \geq u$. We must thus have $v(p'',w'') \leq u' = v(p',w')$.

iv. ***Continuity:*** It is sufficient to prove the following statement. For any sequence $\{(p^n,w^n)\}_{n=1}^{\infty}$ with $(p^n,w^n) \to (p,w)$ and any u, if $v(p^n,w^n) \leq u$ for every n, then $v(p,w) \leq u$; if $v(p^n,w^n) \geq u$ for every n, then $v(p,w) \geq u$. Suppose $v(p^n,w^n) \leq u$ for every n. Then, by the monotonicity of $e(\cdot,\cdot)$ in u and the first relation of (3.E.1), we have $w^n \leq e(p^n,u)$ for every n. By the continuity of $e(\cdot,\cdot)$, $w \leq e(p,u)$. We must thus have $v(p,w) \leq u$. The same argument can be applied for the case with $v(p^n,w^n) \geq u$ for every n.

Alternative: An alternative, simpler way to show the equivalence on the concavity/quasiconvexity and the continuity uses what is sometimes called the epigraph.

For the concavity/quasiconvexity, the concavity of $e(\cdot,u)$ is equivalent to the convexity of the set $\{(p,w): e(p,u) \geq w\}$ and the quasi-convexity of $v(\cdot)$ is the equivalent to the convexity of the set $\{(p,w): v(p,w) \leq u\}$ for every u. But (3.E.1) and the monotonicity imply that $v(p,w) \leq u$ if and only if $e(p,u) \geq w$. Hence the two sets coincide and the quasiconvexity of $v(\cdot)$ is equivalent to the concavity of $e(\cdot,u)$.

As for the continuity, the function $e(\cdot)$ is continuous if and only if both $\{(p,w,u): e(p,u) \leq w\}$ and $\{(p,w,u): e(p,u) \geq w\}$ are closed sets. The function $v(\cdot)$ is continuous if and only if both $\{(p,w,u): v(p,w) \geq u\}$ and $\{(p,w,u): v(p,w) \leq u\}$ are closed sets. But, again by (3.E.1) and the monotonicity,

$$\{(p,w,u):e(p,u)\leq w\}=\{(p,w,u):v(p,w)\geq u\};$$

$$\{(p,w,u):e(p,u)\geq w\}=\{(p,w,u):v(p,w)\leq u\}$$

Hence the continuity of $e(\cdot)$ is equivalent to that of $v(\cdot)$.

Microeconomic Theory – Recitation #3 – Exercises.

1. Jan's utility function for goods X and Y is $U = 7200X^{.75}Y^{.25}$. She must pay $90 for a unit of good X and $30 for a unit of good Y. Jan's income is $1200.

   a. Determine the amounts of goods X and Y Jan purchases to maximize her utility given her budget constraint.

Maximize $U = 7200\, X^{.75}\, Y^{.25}$
subject to $\overline{P}_{X,1}X + \overline{P}_{Y,1}Y = \overline{I}_1$

$$\mathcal{L} = 7200X^{.75}\,Y^{.25} + \lambda\left(\overline{I}_1 - \overline{P}_{X,1}X - \overline{P}_{Y,1}Y\right)$$

$$= 7200X^{.75}\,Y^{.25} + \lambda\overline{I}_1 - \overline{P}_{X,1}\lambda X - \overline{P}_{Y,1}\lambda Y$$

$$\frac{\partial\mathcal{L}}{\partial X} = 5400X^{-.25}\,Y^{.25} - \overline{P}_{X,1}\lambda = 0$$

$$\frac{\partial\mathcal{L}}{\partial Y} = 1800X^{.75}\,Y^{-.75} - \overline{P}_{Y,1}\lambda = 0$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} = \overline{I}_1 - \overline{P}_{X,1}X - \overline{P}_{Y,1}Y = 0$$

$$5400X^{-.25}\,Y^{.25} = \overline{P}_{X,1}\lambda$$

$$1800X^{.75}\,Y^{-.75} = \overline{P}_{Y,1}\lambda$$

$$\frac{5400X^{-.25}Y^{.25}}{1800X^{.75}Y^{-.75}} = \frac{\overline{P}_{X,1}\lambda}{\overline{P}_{Y,1}\lambda}$$

$$\frac{3Y}{X} = \frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}$$

$$Y = \frac{\overline{P}_{X,1}X}{3\overline{P}_{Y,1}}$$

$$\overline{I}_1 - \overline{P}_{X,1}X - \overline{P}_{Y,1}\left(\frac{\overline{P}_{X,1}X}{3\overline{P}_{Y,1}}\right) = 0$$

$$\overline{I}_1 - \overline{P}_{X,1}X - \frac{1}{3}\overline{P}_{X,1}X = 0$$

$$\overline{I}_1 - \frac{4}{3}\overline{P}_{X,1}X = 0$$

$$X = \frac{3\overline{I}_1}{4\overline{P}_{X,1}}$$

To determine the optimal value of X, substitute $\overline{P}_{X,1} = \$90$ and $\overline{I}_1 = \$1200$ into the above equation as follows.

$$X^* = \frac{3(1200)}{4(90)} = 10 \text{ units}$$

To determine the optimal value of Y, substitute $\overline{P}_{X,1} = \$90$, $\overline{P}_{Y,1} = \$30$ and $X^* = 10$ into the equation

$$Y^* = \frac{\overline{P}_{X,1}X^*}{3\overline{P}_{Y,1}} = \frac{(90)(10)}{3(30)} = 10 \text{ units.}$$

b. Determine the maximum amount of utility Jan receives.

To determine the maximum amount of utility Jan can receive, substitute $X^* = 10$ and $Y^* = 10$ into the objective function as follows.

$$U^* = 7200\,(X^*)^{.75}\,(Y^*)^{.25}$$
$$= 7200\,(10)^{.75}\,(10)^{.25}$$
$$= 72{,}000$$

To determine the optimal value of λ, substitute $X^* = 10$, $Y^* = 10$, $\overline{P}_{X,1} = \$90$ and $\overline{P}_{Y,1} = \$30$ into the equation for $\frac{\partial \mathcal{L}}{\partial X}$ or $\frac{\partial \mathcal{L}}{\partial Y}$ as follows.

$$\frac{\partial \mathcal{L}}{\partial X} = 5400X^{-.25}\,Y^{.25} - \overline{P}_{X,1}\lambda = 0$$
$$= 5400X^{-.25}\,Y^{.25} = \overline{P}_{X,1}\lambda$$

c. Determine the value of $\lambda*$ associated with this problem.

$$\lambda^* = \frac{5400(X^*)^{-.25}(Y^*)^{.25}}{\overline{P}_{X,1}}$$
$$= \frac{5400(10)^{-.25}(10)^{.25}}{90} = 60$$

d. Interpret the value of $\lambda*$ you computed in part c. as it specifically applies to Jan.

The value of $\lambda^* = 60$ indicates that if Jan's income increases by one dollar, from \$1200 to \$1201, her utility will increase by the value of $\lambda^* = 60$, from 72,000 to 72,000 + 60 = 72,060. Conversely, if Jan's income decreases by one dollar from \$1200 to \$1199 her utility will decrease by the value of $\lambda^* = 60$, from 72,000 to 72,000 - 60 = 71,940.

2. a. Formulate the dual constrained expenditure minimization problem associated with 4.3 and determine the optimal amounts of goods X and Y Jan should purchase.

Minimize $E = \overline{P}_{X,1}X + \overline{P}_{Y,1}Y$

subject to $\overline{U}_1 = 7200X^{.75}Y^{.25}$

$$\mathcal{L}^D = \overline{P}_{X,1}X + \overline{P}_{Y,1}Y + \lambda^D\left(\overline{U}_1 - 7200X^{.75}Y^{.25}\right)$$

$$= \overline{P}_{X,1}X + \overline{P}_{Y,1}Y + \overline{U}_1\lambda^D - 7200\lambda^D X^{.75}Y^{.25}$$

$$\frac{\partial\mathcal{L}}{\partial X} = \overline{P}_{X,1} - 5400\lambda^D X^{-.25}Y^{.25} = 0$$

$$\frac{\partial\mathcal{L}}{\partial Y} = \overline{P}_{Y,1} - 1800\lambda^D X^{.75}Y^{-.75} = 0$$

$$\frac{\partial\mathcal{L}}{\partial \lambda^D} = \overline{U}_1 - 7200X^{.75}Y^{.25} = 0$$

$$\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}} = \frac{5400\lambda^D X^{-.25}Y^{.25}}{1800\lambda^D X^{.75}Y^{-.75}}$$

$$\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}} = \frac{3Y}{X}$$

$$Y = \frac{X\overline{P}_{X,1}}{3\overline{P}_{Y,1}}$$

$$\overline{U}_1 - 7200X^{.75}\left(\frac{X\overline{P}_{X,1}}{3\overline{P}_{Y,1}}\right)^{.25} = 0$$

$$\overline{U}_1 = 7200X\left(\frac{\overline{P}_{X,1}}{3\overline{P}_{Y,1}}\right)^{.25}$$

$$X = \frac{\overline{U}_1}{7200}\left(\frac{\overline{P}_{X,1}}{3\overline{P}_{Y,1}}\right)^{-.25}$$

From the corresponding primal problem in exercise 4.3, we determined that $\overline{U}_1 = 72{,}000$. Also $\overline{P}_{X,1} = \$90$ and $\overline{P}_{Y,1} = \$30$. Substitute these values into the above equation for X as follows.

$$X^* = \frac{72{,}000}{7200}\left(\frac{90}{3(30)}\right)^{-.25} = 10(1)^{-.25} = 10 \text{ units.}$$

To determine the optimal value of Y^*, substitute $X^* = 10$, $\overline{P}_{X,1} = \$90$, and $\overline{P}_{Y,1} = \$30$ into the equation for Y as follows.

$$Y^* = \frac{X^*\overline{P}_{X,1}}{3\overline{P}_{Y,1}} = \frac{10(90)}{3(30)} = \frac{900}{90} = 10 \text{ units.}$$

b. Determine the minimum amount of expenditure made by Jan.

Substitute $X^* = 10$ and $Y^* = 10$ into the objective function to determine the minimum expenditure necessary to purchase the optimal combination of goods as follows.

$$\begin{aligned} E^* &= \overline{P}_{X,1}X^* + \overline{P}_{Y,1}Y^* \\ &= (\$90)\,(10) + (\$30)\,(10) \\ &= \$900 + \$300 \\ &= \$1200 \end{aligned}$$

c. Determine the optimal value of λ^D and provide a written interpretation of this value as it specifically applies to Jan in this problem.

To determine the optimal value of λ^D, substitute $\overline{P}_{X,1} = \$90$, $\overline{P}_{Y,1} = \$30$, $X^* = 10$, and $Y^* = 10$ into the equations for either $\frac{\partial \mathcal{L}^D}{\partial X}$ or $\frac{\partial \mathcal{L}^D}{\partial Y}$ as follows.

$$\frac{\partial \mathcal{L}^D}{\partial X} = \overline{P}_{X,1} - 5400\lambda^D X^{-.25}Y^{.25} = 0$$

$$\lambda^D = \frac{\overline{P}_{X,1}X^{.25}Y^{-.25}}{5400}$$

$$\lambda^D = \frac{\overline{P}_{X,1}\left(X^*\right)^{.25}\left(Y^*\right)^{-.25}}{5400}$$

$$\lambda^D = \frac{90(10)^{.25}(10)^{-.25}}{5400} = \frac{1}{60} = 0.0167.$$

The value of $\lambda^D = \frac{1}{60} = 0.0167$ indicates that if the predetermined value of Jan's utility increases by one unit from 7200 to 7201 then the minimum expenditure necessary to achieve this higher level of utility will increase by \$0.0167 from \$1200 to \$1200 + \$0.0167 = \$1200.0167.

d. Compare the optimal values of X, Y and λ you computed in exercise 4.3 with those you computed in parts a. and c. of this exercise.

The optimal values of X and Y computed for the primal problem in exercise 4.3 are the same as those computed in the dual problem in exercise 4.4. The value of λ^* determined for the primal problem in exercise 4.3 is equal to the reciprocal of the value of λ^{D*} determined for the dual problem. Specifically, $\lambda^* = 60$ and $\lambda^{D*} = \frac{1}{60}$.

3. Raymond derives utility from consuming goods X and Y, where his utility function is $U = 80X^{.25}Y^{.25}$. He spends all of his income, I, on his purchases of goods X and Y, and he must pay prices of P_x and P_y for each unit of these goods, respectively. Assume that his income is \$3200, the unit price of good X is \$100, and the unit price of good Y is \$100.

a. Determine the amounts of goods X and Y that Raymond should purchase to maximize his utility given his budget constraint.

$$\text{Maximize } U = 80X^{.25}Y^{.25}$$
$$\text{subject to } \overline{P}_{X,1}X + \overline{P}_{Y,1}Y = \overline{I}_1$$
$$\mathcal{L} = 80X^{.25}Y^{.25} + \lambda\left(\overline{I}_1 - \overline{P}_{X,1}X - \overline{P}_{Y,1}Y\right)$$
$$= 80X^{.25}Y^{.25} + \lambda\overline{I}_1 - \lambda\overline{P}_{X,1}X - \lambda\overline{P}_{Y,1}Y$$
$$\frac{\partial\mathcal{L}}{\partial X} = 20X^{-.75}Y^{.25} - \lambda\overline{P}_{X,1} = 0$$
$$\frac{\partial\mathcal{L}}{\partial Y} = 20X^{.25}Y^{-.75} - \lambda\overline{P}_{Y,1} = 0$$
$$\frac{\partial\mathcal{L}}{\partial \lambda} = \overline{I}_1 - \overline{P}_{X,1}X - \overline{P}_{Y,1}Y = 0$$
$$\frac{20X^{-.75}Y^{.25}}{20X^{.25}Y^{-.75}} = \frac{\lambda\overline{P}_{X,1}}{\lambda\overline{P}_{Y,1}}$$
$$\frac{Y}{X} = \frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}$$
$$Y = \frac{\overline{P}_{X,1}X}{\overline{P}_{Y,1}}$$
$$\overline{I}_1 - \overline{P}_{X,1}X - \overline{P}_{Y,1}\left(\frac{\overline{P}_{X,1}X}{\overline{P}_{Y,1}}\right) = 0$$
$$\overline{I}_1 - \overline{P}_{X,1}X - \overline{P}_{X,1}X = 0$$
$$2\overline{P}_{X,1}X = \overline{I}_1$$
$$X = \frac{\overline{I}_1}{2\overline{P}_{X,1}}$$

Since $\overline{P}_{X,1} = \$100, \overline{P}_{Y,1} = \$100$, and $\overline{I}_1 = \$3200$ then

$$X^* = \frac{\overline{I}_1}{2\overline{P}_{X,1}} = \frac{\$3200}{2(\$100)} = 16 \text{ units}$$

and

$$Y^* = \frac{\overline{P}_{X,1}X^*}{\overline{P}_{Y,1}} = \frac{(\$100)(16)}{\$100} = 16 \text{ units.}$$

b. Determine the maximum amount of utility Raymond can receive.

To determine the maximum amount of utility Raymond receives, substitute $X^* = 16$ and $Y^* = 16$ into his utility function as follows.

$$U^* = 8\,(X^*)^{.25}\,(Y^*)^{.25}$$
$$= 80\,(16)^{.25}\,(16)^{.25}$$
$$= 80\,(16)^{.5}$$
$$= 320$$

4. Refer to your response to exercise 5.1.

a. Derive Raymond's own-price demand curve for good X.

Given the constrained utility maximization problem in part a. of exercise 5.1, recall it was determined that

$$X^* = \frac{\bar{I}_1}{2\bar{P}_{X,1}}.$$

Raymond's own-price demand curve for good X expresses his optimal level of consumption of good X as a function of the own price of good X, while holding the price of good Y and Raymond's income constant. Therefore, since $\bar{I}_1 = \$3200$, his own-price demand curve for good X is

$$X^* = X\,(P_X,\ \bar{P}_{Y,1}, \bar{I}_1) = \frac{\bar{I}_1}{2P_X} = \frac{3200}{2P_X} = \frac{1600}{P_X}.$$

b. Derive Raymond's own-price demand curve for good Y.

Given the constrained utility maximization problem in part a. of exercise 5.1, recall it was determined that

$$Y^* = \frac{\bar{P}_{X,1}X^*}{\bar{P}_{Y,1}},$$

where

$$X^* = \frac{\bar{I}_1}{2\bar{P}_{X,1}}.$$

Therefore, after substituting $X^* = \frac{\bar{I}_1}{2\bar{P}_{X,1}}$, into the equation for Y^*, we obtain

$$Y^* = \frac{\bar{P}_{X,1}\left(\frac{\bar{I}_1}{2\bar{P}_{X,1}}\right)}{\bar{P}_{Y,1}} = \frac{\bar{I}_1}{2\bar{P}_{Y,1}}.$$

Raymond's own-price demand curve for good Y expresses his optimal consumption level of good Y as a function of the own price of good Y, while holding the price of good X and Raymond's income constant. Therefore, since $\bar{I}_1 = \$3200$, his own-price demand curve for good Y is

$$Y^* = Y\,\left(\bar{P}_{X,1}, P_Y, \bar{I}_1\right) = \frac{\bar{I}_1}{2P_Y} = \frac{3200}{2P_Y} = \frac{1600}{P_Y}.$$

5. Refer to your responses to exercise 5.1.

a. Derive Raymond's Engel curve for good X.

Given the constrained utility maximization problem in part a. of exercise 5.1, recall it was determined that

$$X^* = \frac{\bar{I}_1}{2\bar{P}_{X,1}}.$$

Raymond's Engel curve for good X expresses his optimal consumption level of good X as a function of his income, while holding the prices of all goods constant. Therefore, since $\bar{P}_{X,1} = \$100$ then his Engel curve for good X is

$$X^* = X\left(\bar{P}_{X,1}\bar{P}_{Y,1}I\right) = \frac{I}{2\bar{P}_{X,1}} = \frac{I}{2(\$100)} = \frac{I}{200}.$$

b. Is good X a normal good or an inferior good? Justify your response mathematically.

To determine whether good X is normal or inferior, take the derivative of Raymond's Engel curve for good X, determined in part a., with respect to I.

$$X^* = X\left(\bar{P}_{X,1}, \bar{P}_{Y,1}, I\right) = \frac{I}{200}$$

$$\frac{\partial X}{\partial I} = \frac{1}{200} > 0$$

Since $\frac{\partial X}{\partial I} > 0$ good X is normal, indicating that as Raymond's income increases his optimal consumption level of good X rises.

c. Derive Raymond's Engel curve for good Y.

Given the constrained utility maximization problem in part a of exercise 5.1, recall it was determined that

$$Y^* = \frac{\overline{P}_{X,1}X^*}{\overline{P}_{Y,1}}$$

where

$$X^* = \frac{\overline{I}_1}{2\overline{P}_{X,1}}.$$

Therefore, after substituting $X^* = \frac{\overline{I}_1}{2\overline{P}_{X,1}}$ into the equation for Y^*, we obtain

$$Y^* = \frac{\overline{P}_{X,1}\left(\frac{\overline{I}_1}{2\overline{P}_{X,1}}\right)}{\overline{P}_{Y,1}} = \frac{\overline{I}_1}{2\overline{P}_{Y,1}}.$$

Raymond's Engel curve for good Y expresses his optimal consumption level of good Y as a function of his income, while holding the prices of all goods constant. Therefore, since $\overline{P}_{Y,1} = \$100$, his Engel curve for good Y is

$$Y^* = Y\left(\overline{P}_{X,1}, \overline{P}_{Y,1}, I\right) = \frac{I}{2\overline{P}_{Y,1}} = \frac{I}{2(100)} = \frac{I}{200}.$$

d. Is good Y a normal good or an inferior good? Justify your response mathematically.

To determine whether good Y is normal or inferior, take the derivative of Raymond's Engel curve for good Y, determined in part c., with respect to I.

$$Y^* = Y\left(\overline{P}_{X,1}, \overline{P}_{Y,1}, I\right) = \frac{I}{200}$$

$$\frac{\partial Y}{\partial I} = \frac{1}{200} > 0$$

Since $\frac{\partial Y}{\partial I} > 0$ good Y is normal, indicating that as Raymond's income increases his optimal consumption level of good Y rises.

6. Assume an individual's own-price demand function for good X is $X = X\left(P_x, P_y, I\right) = 200 - 4P_x - 1.5P_Y + 0.008I$ where of P_x and P_y denote the unit prices of goods X and Y, respectively, and I denotes the consumer's money income.

a. Compute the individual's cross-price demand curve for good X when the unit price of good X is $2 and the consumer's income is $40,000.

Substitute $\overline{P}_{X,1} = \$2$ and $\overline{I}_1 = \$40{,}000$ into the individual's own-price demand function to derive his cross-price demand curve.

$$X = X\left(\overline{P}_{X,1}, P_Y, \overline{I}_1\right) = 200 - 4\overline{P}_{X,1} - 1.5P_Y + 0.008\overline{I}_1$$
$$X = 200 - 4\,(2) - 1.5\,P_Y + 0.008\,(40{,}000)$$
$$= 200 - 8 - 1.5\,P_Y + 320$$
$$= 512 - 1.5\,P_Y$$

b. Are goods X and Y gross substitutes or gross complements? Justify your response mathematically.

To determine whether goods X and Y are gross substitutes of gross complements take the derivative of the cross-price demand curve with respect to P_Y as follows

$$X = 512 - 1.5\,P_Y$$
$$\frac{\partial X}{\partial P_Y} = -1.5$$

Since $\frac{\partial X}{\partial P_Y} < 0$ this indicates that goods X and Y are gross complements meaning that as the price of good Y rises, the quantity demanded of good X decreases.

7. Recall from exercise 5.1 Raymond's utility function, when he consumes goods X and Y, is $U = 80X^{.25}Y^{.25}$. Once again, assume the unit price of good X, P_x, is \$100, and the unit price of good Y, P_y, is \$100. Determine the quantities of goods X and Y Raymond should purchase that will minimize his expenditures on these goods and yield 320 units of utility to him.

$$\text{Minimize } E = \overline{P}_{X,1}X + \overline{P}_{Y,1}Y$$

$$\text{subject to } \overline{U}_1 = 80X^{.25}Y^{.25}$$

$$\mathcal{L} = \overline{P}_{X,1}X + \overline{P}_{Y,1}Y + \lambda\left(\overline{U}_1 - 80X^{.25}Y^{.25}\right)$$

$$= \overline{P}_{X,1}X + \overline{P}_{Y,1}Y + \lambda\overline{U}_1 - 80\lambda X^{.25}Y^{.25}$$

$$\frac{\partial \mathcal{L}}{\partial X} = \overline{P}_{X,1} - 20\lambda X^{-.75}Y^{.25} = 0$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \overline{P}_{Y,1} - 20\lambda X^{.25}Y^{-.75} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \overline{U}_1 - 80X^{.25}Y^{.25} = 0$$

$$\frac{20\lambda X^{-.75}Y^{.25}}{20\lambda X^{.25}Y^{-.75}} = \frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}$$

$$\frac{Y}{X} = \frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}$$

$$Y = \frac{\overline{P}_{X,1}X}{\overline{P}_{Y,1}}$$

$$\overline{U}_1 - 80X^{.25}\left(\frac{\overline{P}_{X,1}X}{\overline{P}_{Y,1}}\right)^{.25} = 0$$

$$\overline{U}_1 - 80X^{.50}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{.25} = 0$$

$$80X^{.50}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{.25} = \overline{U}_1$$

$$X^{.50} = \frac{\overline{U}_1}{80}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{-.25}$$

$$\left(X^{.50}\right)^2 = \left[\frac{\overline{U}_1}{80}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{-.25}\right]^2$$

$$X = \frac{\overline{U}_1^2}{(80)^2}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{-.50}$$

$$X^* = \frac{(320)^2}{(80)^2}\left(\frac{100}{100}\right)^{-.50} = \frac{102{,}400}{6400}(1)^{-.50} = 16 \text{ units}$$

$$Y^* = \frac{\overline{P}_{X,1}X^*}{\overline{P}_{Y,1}} = \frac{(100)(16)}{100} = 16 \text{ units}$$

8. Refer to your response to exercise 5.5.

a. Determine Raymond's compensated demand curve for good X.

Given the constrained expenditure minimization problem in exercise 5.5 it was determined that

$$X^* = \frac{\overline{U}_1^2}{(80)^2}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{-.50}$$

Raymond's compensated demand curve for good X expresses his optimal consumption level of good X as a function of the own price of good X while holding the price of good Y and Raymond's utility constant. Therefore, since $\overline{U}_1 = 320$ and $\overline{P}_{Y,1} = \$100$, his compensated demand curve for good X is

$$X' = X\left(P_X, \overline{P}_{Y,1}, \overline{U}_1\right) = \frac{\overline{U}_1^2}{(80)^2}\left(\frac{P_X}{\overline{P}_{Y,1}}\right)^{-.50}$$

$$= \frac{(320)^2}{(80)^2}\left(\frac{P_X}{100}\right)^{-.50}$$

$$= \left(\frac{102{,}400}{6400}\right)\left(\frac{10}{P_X^{.50}}\right)$$

$$= \frac{160}{P_X^{.50}}$$

$$= 160\, P_X^{-.50}.$$

b. Determine Raymond's compensated demand curve for good Y.

Given the constrained expenditure minimization problem in exercise 5.5 it was determined that

$$Y^* = \frac{\overline{P}_{X,1}X^*}{\overline{P}_{Y,1}}$$

where

$$X^* = \frac{\overline{U}_1^2}{(80)^2}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{-.50}.$$

After substituting $X^* = \frac{\overline{U}_1^2}{(80)^2}\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{-.50}$ into the equation for Y^* we obtain

$$Y^* = \left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)\left(\frac{\overline{U}_1^2}{80^2}\right)\left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{-.50}$$
$$= \left(\frac{\overline{P}_{X,1}}{\overline{P}_{Y,1}}\right)^{.50}\left(\frac{\overline{U}_1^2}{(80)^2}\right).$$

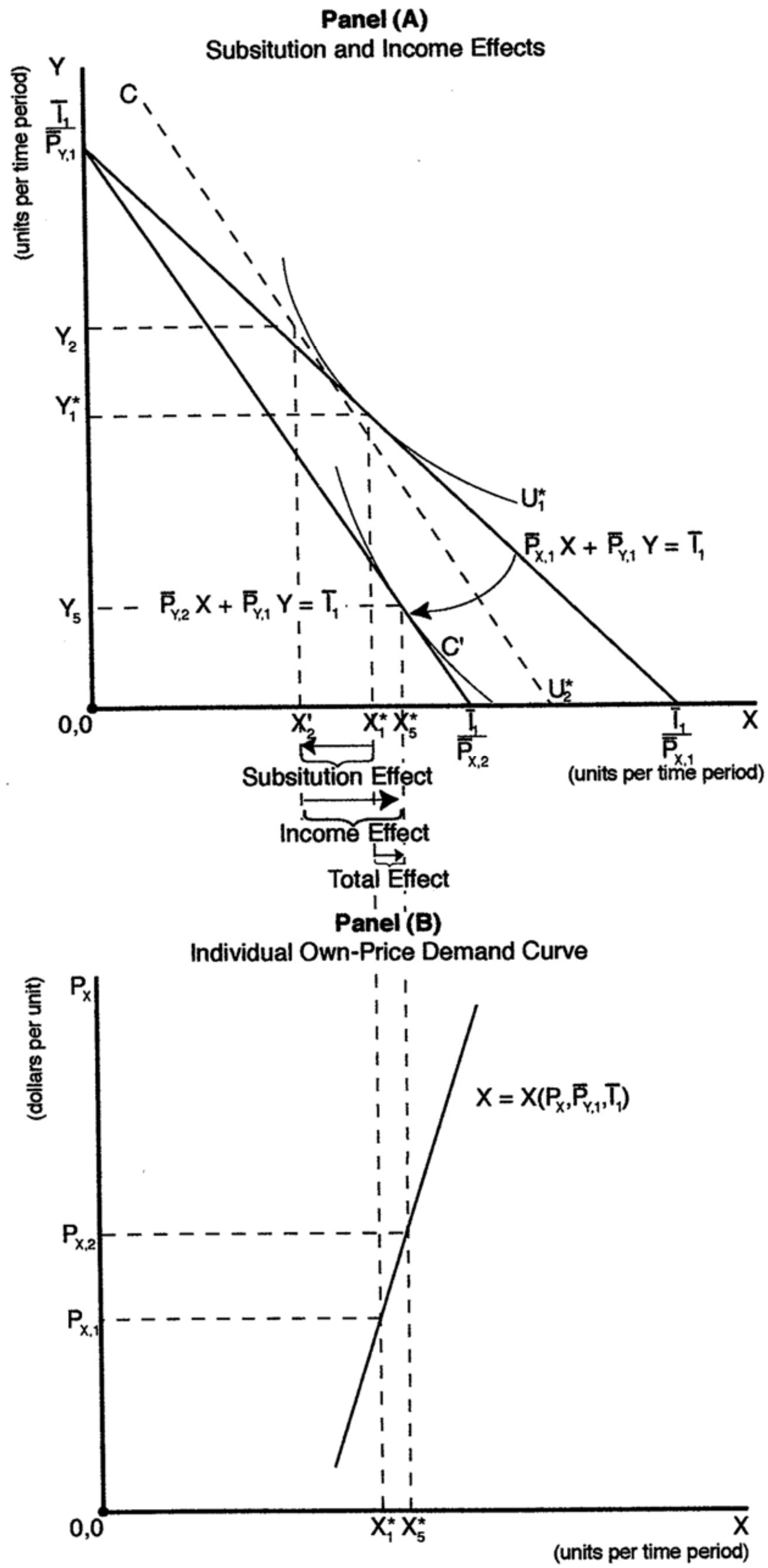
Raymond's compensated demand curve for good Y expresses his optimal consumption level of good Y as a function of the own price of good Y while holding the price of good X and Raymond's utility constant. Therefore, since $\overline{U}_1 = 320$ and $\overline{P}_{X,1} = \$100$, his compensated demand curve for good Y is

$$Y' = Y\left(\overline{P}_{X,1}, P_Y, \overline{U}_1\right) = \left(\frac{\overline{P}_{X,1}}{P_Y}\right)^{.50}\left(\frac{\overline{U}_1^2}{(80)^2}\right) = \left(\frac{100}{P_Y}\right)^{.50}\left(\frac{(320)^2}{(80)^2}\right) = \left(\frac{10}{P_Y^{.50}}\right)\left(\frac{102{,}400}{6400}\right)$$
$$= \frac{160}{P_Y^{.50}} = 160P_Y^{-.50}$$

9. Is it possible for an individual's demand curve for a good to be positively sloped? Support your response with an appropriate graphical analysis.

If an individual perceives a good as a Giffen good then the corresponding own-price demand curve is positively sloped, indicating that an increase in the price of this good results in an increase in the quantity demanded of the good. In the case of a Giffen good, the income effect dominates the substitution effect. As you can see in panel (A) in the figure below, an increase in the price of good X, from $\overline{P}_{X,1}$ to $\overline{P}_{X,2}$, results in a total effect of an increase in the quantity demanded of good X from X^*_1 to X^*_5. Specifically, the substitution effect, which reduces the consumer's purchase of good X from X^*_1 to X'_2, is overpowered by the income effect, which increases the quantity demanded of good X from X'_2 to X^*_5. The corresponding, positively sloped, individual own-price demand curve is shown in panel (B).

Figure 5.2


Panel (A)
Subsitution and Income Effects
Y
(units per time period)
C
Y2
Y1*
Y5
U1*
U2*
C'
0,0
X2'
X1*
X5*
X
(units per time period)
Subsitution Effect
Income Effect
Total Effect
Panel (B)
Individual Own-Price Demand Curve
PX
(dollars per unit)
PX,2
PX,1
X1*
X5*
X
(units per time period)