A systematic procedure for finding Perfect Bayesian Equilibria in Incomplete Information Games

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Motivation

- Rapidly expanding literature on game theory and industrial organization analyzing incomplete information settings.
    - Solution concept commonly used : *Perfect Bayesian Equilibrium* (PBE).

- *Examples*:
    - Labor market [Spence, 1974]
    - Limit pricing [Milgrom and Roberts, 1982 and 1986]
    - Signaling with several incumbents [Harrington, 1986, and Bagwell and Ramey, 1991]
    - Warranties [Gal-Or, 1989]
    - Social preferences [Fong, 2008]

Motivation

- We know that, for a strategy profile to be part of a PBE, it must satisfy:
  - Sequential rationality, in an incomplete information context; and
  - Consistency of beliefs.
- How to apply these two conditions, and find all pure-strategy PBEs in incomplete information games?
  - We will describe a 5-step procedure...
  - that checks if a given strategy profile can be sustained as PBE.

Outline of the presentation

- Non-technical introduction to PBE.
  - Updating beliefs with Bayes' rule...
  - both in- and off-the-equilibrium path.
- General presentation of the 5-step procedure.
- Worked-out example, where we apply the procedure to a signaling game.

Definition of PBE

- A strategy profile for N players $(s_1, s_2, \ldots, s_N)$, and a system of beliefs over the nodes at all information sets, are a PBE if:
  1. Each player's strategies specify optimal actions, given the strategies of the other players, and given his beliefs.
  2. The beliefs are consistent with Bayes' rule, whenever possible.
- These two properties can be summarized into two: **sequential rationality**, and **consistency of beliefs**.
- Let us analyze each property separately.

Sequential rationality

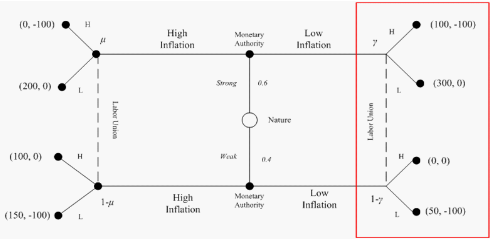
- We just need to extend the definition of sequential rationality in games of complete information to incomplete information settings, as follows:
  - At every node a player is called on to move, he must maximize his *expected* utility level,
  - given his own *beliefs* about the other players' types

Sequential rationality

- **Example:** Let us consider the following sequential game with incomplete information:
  - A monetary authority (such as the Federal Reserve Bank) privately observes its real degree of commitment with maintaining low inflation levels.
  - After knowing its type (either Strong or Weak), the monetary authority decides whether to announce that the expectation for inflation is either High or Low.
  - A labor union, observing the message sent by the monetary authority, decides whether to ask for high or low salary raises (denoted as H or L, respectively)

Sequential rationality

- Example:



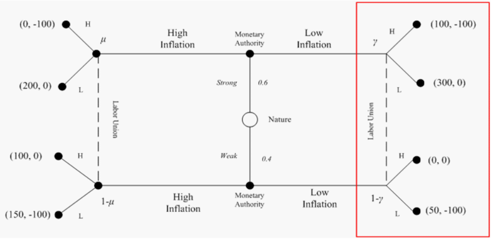
- After observing a low inflation announcement, the labor union responds with a high salary increase (H) if and only if

$$EU_{Labor}(H|LowInflation) > EU_{Labor}(L|LowInflation)$$

That is, if
$(-100)\gamma + 0(1-\gamma) > 0\gamma + (-100)(1-\gamma) \iff \gamma < \frac{1}{2}$

Sequential rationality

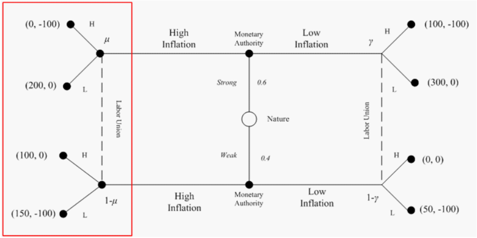
- Example:



- That is, the labor union responds with H only when it assigns a relatively low probability to the monetary authority being Strong.
- Alternatively, the lower right-hand corner is more likely.

Sequential rationality

- Example:



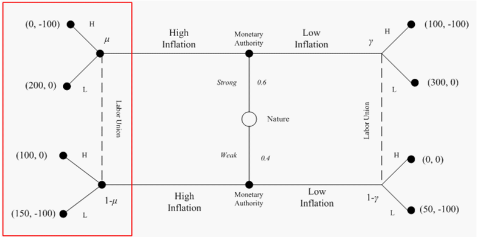
- Similarly, after observing high inflation, the labor union responds with H if and only if

$$EU_{Labor}(H|HighInflation) > EU_{Labor}(L|HighInflation)$$

That is, if
$(-100)\mu + 0(1-\mu) > 0\mu + (-100)(1-\mu) \iff \mu < \frac{1}{2}$.

Sequential rationality

- Example:



- Hence, upon observing high inflation...
  - the labor union responds with H if its beliefs assign a larger probability weight to the lower left-hand node.

Consistency of beliefs

- Players must update his beliefs using Bayes' rule.
- In our previous example, if the labor union observes a high inflation announcement, it updates beliefs μ as follows

$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}}$$

  - where α^{Strong} denotes the probability that a Strong monetary authority announces high inflation, and
  - α^{Weak} the probability that a Weak monetary authority announces high inflation.

Consistency of beliefs

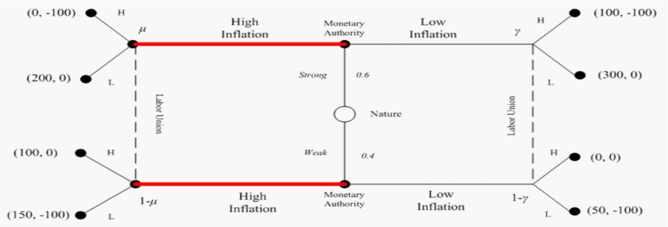
- For instance, if the Strong monetary authority announces high inflation with probability $\alpha^{Strong} = \frac{1}{8}$, and the Weak monetary authority with a lower probability $\alpha^{Weak} = \frac{1}{16.}$, then the labor union's updated beliefs become

$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6\frac{1}{8}}{0.6\frac{1}{8} + 0.4\frac{1}{16}} = 0.75$$

- Intuitively, since the Strong monetary authority is twice more likely to make such an announcement than the Weak type...
  - the updated (posterior) beliefs assign a **larger** probability to the high inflation message originating from a Strong monetary authority (0.75) than the prior probability did (0.6).

Consistency of beliefs

- If, instead, both types of monetary authority make such an announcement, i.e., $\alpha^{Strong} = \alpha^{Weak} = 1$,...



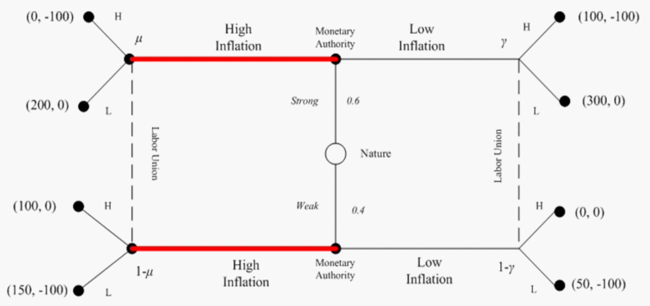
- Then, Bayes' rule provides us with beliefs that exactly **coincide** with the prior probability distribution:

$$\mu = \frac{p\alpha^{Strong}}{p\alpha^{Strong} + (1-p)\alpha^{Weak}} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 1} = \frac{0.6}{1} = 0.6$$

- Intuitively, the announcement becomes **uninformative**.

Consistency of beliefs

- **Off-the-equilibrium beliefs:** What about the beliefs in γ?



- In this case, the application of Bayes' rule yields...

$$\gamma = \frac{0.6\left(1 - \alpha^{Strong}\right)}{0.6\left(1 - \alpha^{Strong}\right) + 0.4\left(1 - \alpha^{Weak}\right)} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 0} = \frac{0}{0}$$

Consistency of beliefs

- **Off-the-equilibrium beliefs:**
- Hence, γ are indeterminate, and they can be arbitrarily specified, i.e., any value $\gamma \in [0, 1]$.
- For this reason, the definition of the PBE solution concept requires that "...beliefs must satisfy Bayes' rule, whenever possible."
  - Of course, it is only possible along the equilibrium path,
  - not off-the-equilibrium path, where beliefs are indeterminate.

Procedure to find PBEs

- The definition of PBE is hence relatively clear, but...
  - How can we find the set of PBEs in an incomplete information game?
- We will next describe a systematic 5-step procedure that helps us find all pure-strategy PBEs.

Procedure to find PBEs

1. Specify a strategy profile for the privately informed player, either separating or pooling.
    - In our above example, there are only four possible strategy profiles for the privately informed monetary authority: two separating strategy profiles, $High^S Low^W$ and $Low^S High^W$, and two pooling strategy profiles, $High^S High^W$ and $Low^S Low^W$.
        - (For future reference, it might be helpful to shade the branches corresponding to the strategy profile we test.)
2. Update the uninformed player's beliefs using Bayes' rule, whenever possible.
    - In our above example, we need to specify beliefs μ and γ, which arise after the labor union observes a high or a low inflation announcement, respectively.

Procedure to find PBEs - Cont'd

3. Given the uninformed player's updated beliefs, find his optimal response.
    - In our above example, we first determine the optimal response of the labor union (H or L) upon observing a high-inflation announcement (given its updated belief μ),
    - we then determine its optimal response (H or L) after observing a low-inflation announcement (given its updated belief γ).
        - (Also for future reference, it might be helpful to shade the branches corresponding to the optimal responses we just found.)

Procedure to find PBEs - Cont'd

4. Given the optimal response of the uninformed player, find the optimal action (message) for the informed player.
   - In our previous example, we first check if the Strong monetary authority prefers to make a high or low inflation announcement (given the labor union's responses determined in step 3).
   - We then operate similarly for the Weak type of monetary authority.

Procedure to find PBEs - Cont'd

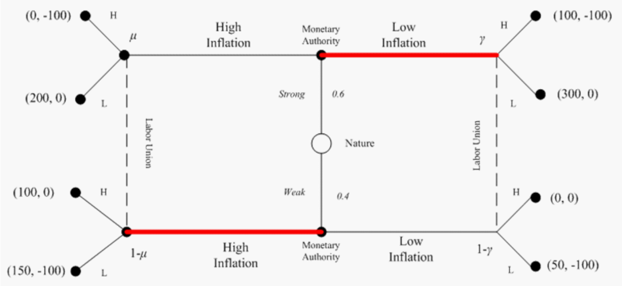
5. Then check if this strategy profile for the informed player coincides with the profile suggested in step 1.
   - If it coincides, then this strategy profile, updated beliefs and optimal responses **can** be supported as a PBE of the incomplete information game.
   - Otherwise, we say that this strategy profile **cannot** be sustained as a PBE of the game.

Procedure to find PBEs - Cont'd

- Let us next separately apply this procedure to test each of the four candidate strategy profiles:
  - two separating strategy profiles:
    - $High^S Low^W$, and
    - $Low^S High^W$.
  - And two pooling strategy profiles:
    - $High^S High^W$, and
    - $Low^S Low^W$.

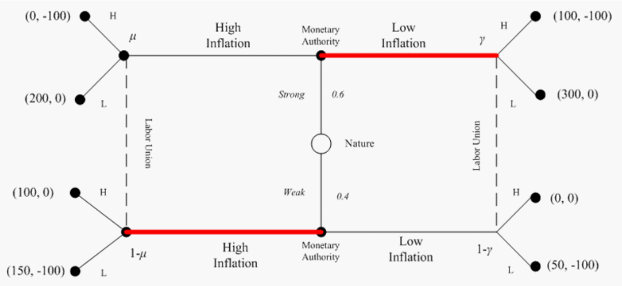
Separating equilibrium with (Low,High)

- Let us first check separating strategy profile: $Low^S High^W$.



- **Step #1:** Specifying strategy profile $Low^S High^W$ that we will test.
    - (See shaded branches in the figure.)

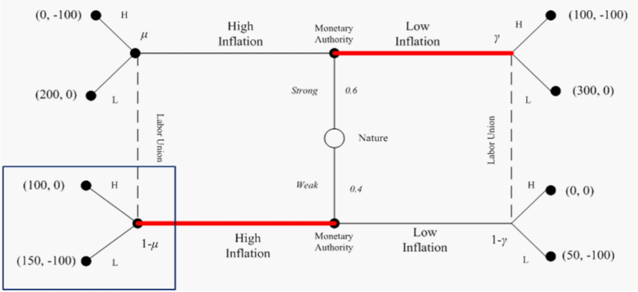
Separating equilibrium with (Low,High)



- **Step #2:** Updating beliefs
  - (a) After high inflation announcement (left-hand side)

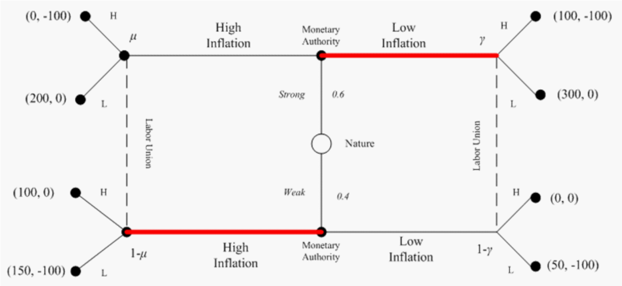
$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 1} = 0$$

Separating equilibrium with (Low,High)



- **Step #2:** Updating beliefs
  - This implies that after high inflation...
  - the labor union restricts its belief to the lower left-hand corner (see box), since $\mu = 0$ and $1 - \mu = 1$

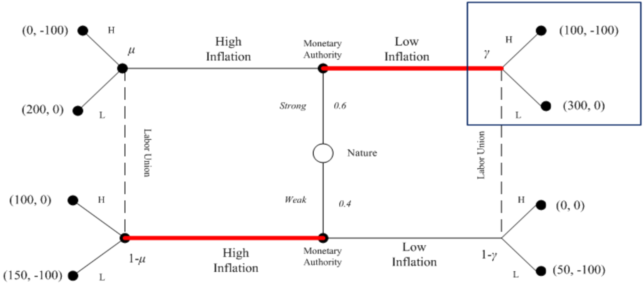
Separating equilibrium with (Low,High)



- **Step #2:** Updating beliefs
  - (b) After low inflation announcement (right-hand side)

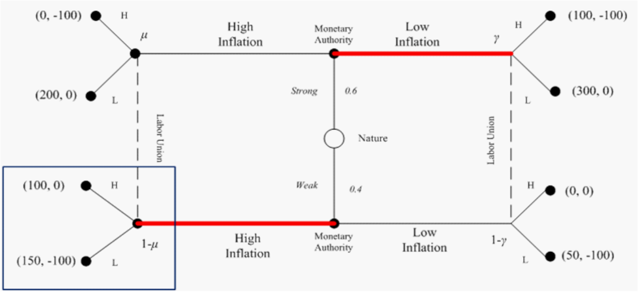
$$\gamma = \frac{0.6\left(1-\alpha^{Strong}\right)}{0.6\left(1-\alpha^{Strong}\right)+0.4\left(1-\alpha^{Weak}\right)} = \frac{0.6\times 1}{0.6\times 1+0.4\times 0} = 1$$

Separating equilibrium with (Low,High)



- **Step #2:** Updating beliefs
  - This implies that, after low inflation...
  - the labor union restricts its belief to the upper right-hand corner (see box), since $\gamma = 1$ and $1 - \gamma = 0$.

Separating equilibrium with (Low,High)

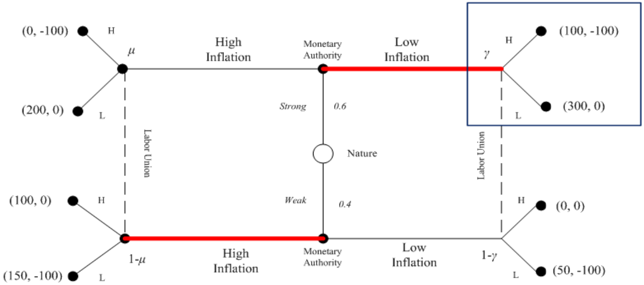


- **Step #3:** Optimal response
  - (a) After high inflation announcement, respond with H since

$$0 > -100$$

in the lower left-hand corner of the figure (see blue box).

Separating equilibrium with (Low,High)



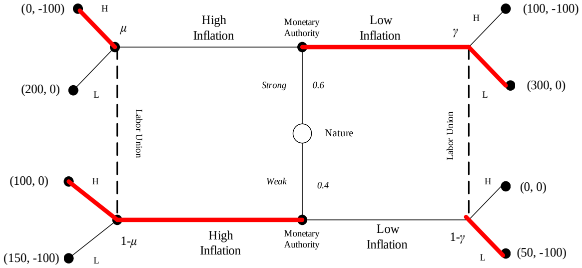
- **Step #3:** Optimal response
  - (b) After low inflation announcement, respond with L since

$$0 > -100$$

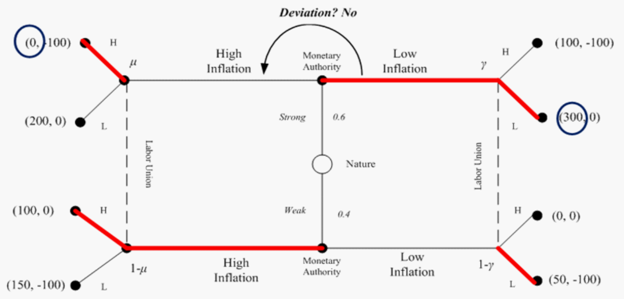
in the upper right-hand corner of the figure (see box).

Separating equilibrium with (Low,High)

- We can hence summarize the optimal responses we just found, by shading them in the figure:
  - H after high inflation, but L after low inflation.

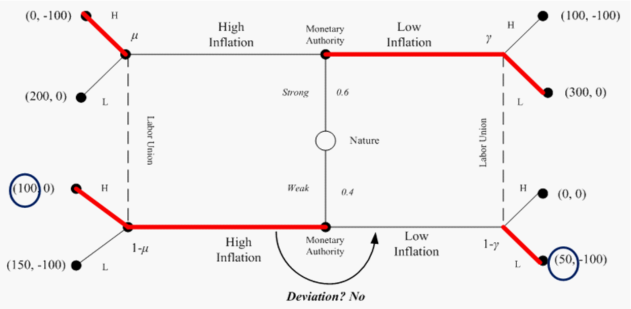


Separating equilibrium with (Low,High)



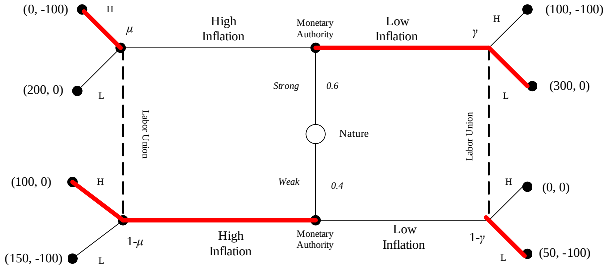
- **Step #4:** Optimal messages by the informed player
  - (a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is $300,
  - while if it deviates, its payoff decreases to $0.
  - (No incentives to deviate).

Separating equilibrium with (Low,High)



- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is \$100,
  - while if it deviates, its payoff decreases to \$50.
  - (No incentives to deviate either).

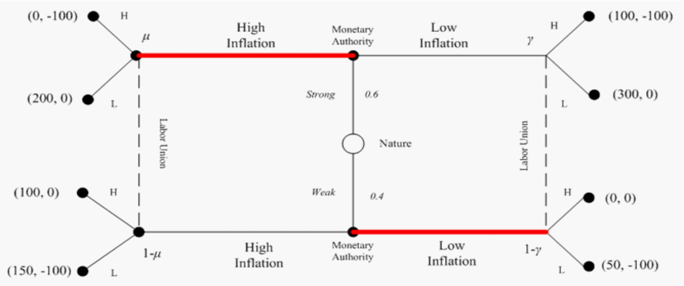
Separating equilibrium with (Low,High)



- Since no type of privately informed player (monetary authority) has incentives to deviate,
  - The separating strategy profile $Low^S High^W$ **can** be sustained as a PBE.

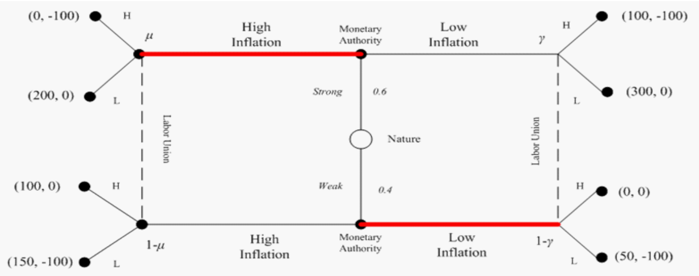
Separating equilibrium with (High,Low)

- Let us now check the opposite separating strategy profile: $High^S Low^W$.



- **Step #1:** Specifying strategy profile $High^S Low^W$ that we will test.
  - (See shaded branches in the figure.)

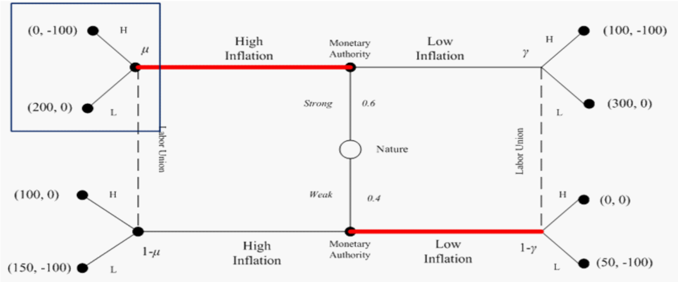
Separating equilibrium with (High,Low)



- **Step #2:** Updating beliefs
  - (a) After high inflation announcement

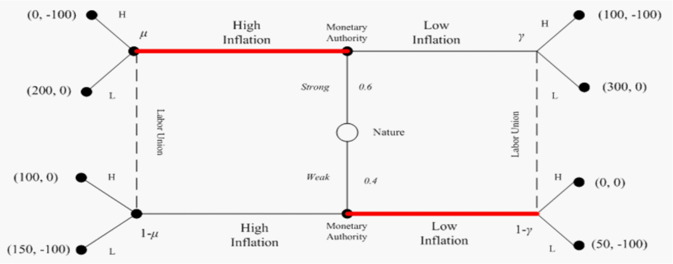
$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 0} = 1$$

Separating equilibrium with (High,Low)



- **Step #2:** Updating beliefs
  - Hence, after high inflation...
  - the labor union restricts its beliefs to $\mu = 1$ in the upper left-hand corner (see box).

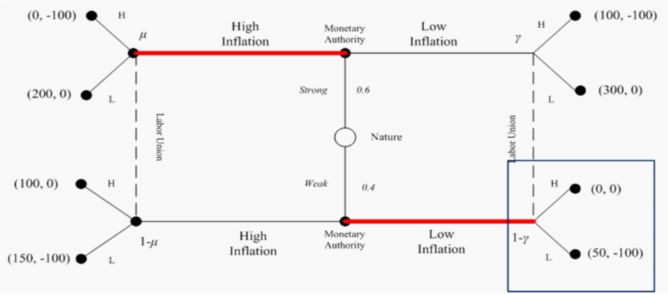
Separating equilibrium with (High,Low)



- **Step #2:** Updating beliefs
  - (b) After low inflation announcement

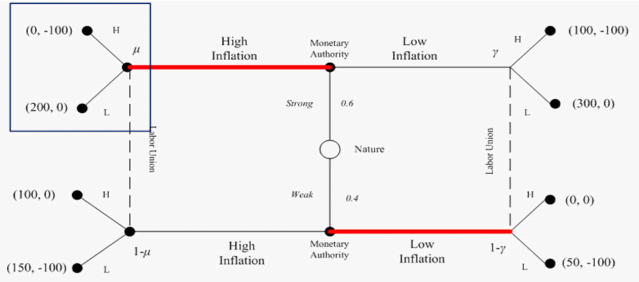
$$\gamma = \frac{0.6\left(1-\alpha^{Strong}\right)}{0.6\left(1-\alpha^{Strong}\right)+0.4\left(1-\alpha^{Weak}\right)} = \frac{0.6\times 0}{0.6\times 0+0.4\times 1} = 0$$

Separating equilibrium with (High,Low)



- **Step #2:** Updating beliefs
  - Hence, after low inflation...
  - the labor union restricts its beliefs to $\gamma = 0$ (i.e., $1 - \gamma = 1$) in the lower right-hand corner (see box).

Separating equilibrium with (High,Low)

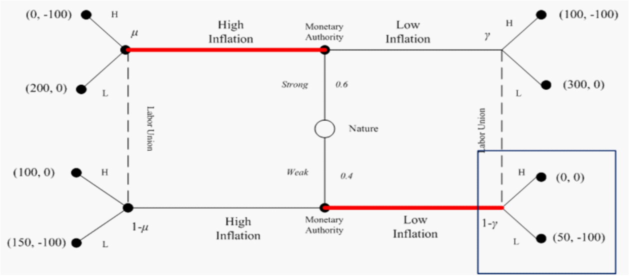


- **Step #3:** Optimal response
  - (a) After high inflation announcement, respond with L since

$$0 > -100$$

in the upper left-hand corner of the figure (see box).

Separating equilibrium with (High,Low)



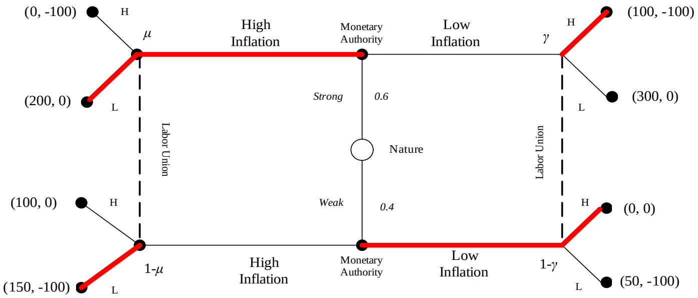
- **Step #3:** Optimal response
  - (a) After low inflation announcement, respond with H since

$$0 > -100$$

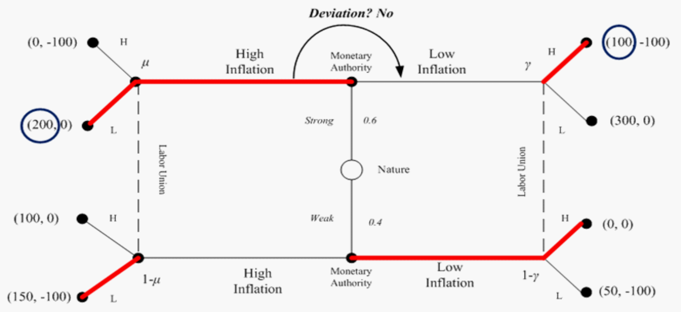
in the lower right-hand corner of the figure (see box).

Separating equilibrium with (High,Low)

- Summarizing the optimal responses we just found:
    - L after high inflation, but H after high inflation.

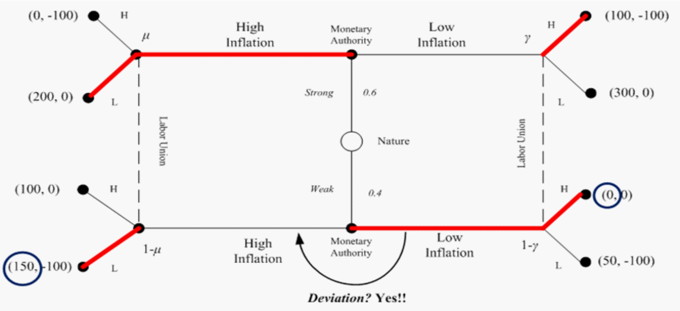


Separating equilibrium with (High,Low)



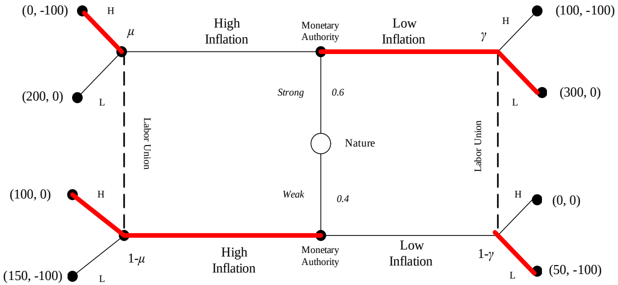
- **Step #4:** Optimal messages of the informed player
  - (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is \$200,
  - while if it deviates, its payoff decreases to \$100.
  - (No incentives to deviate).

Separating equilibrium with (High,Low)



- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses Low (as prescribed), its payoff is \$0,
  - while if it deviates, its payoff **increases** to \$150.
  - (Incentives to deviate!!).

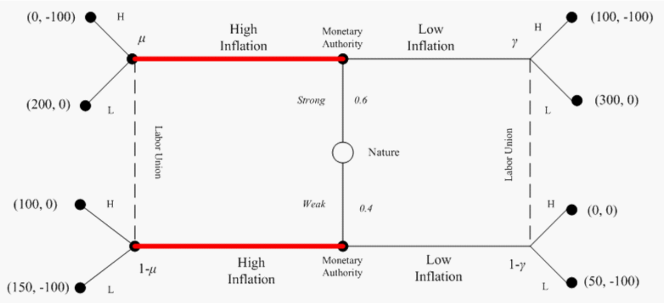
Separating equilibrium with (High,Low)



- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
  - The separating strategy profile $High^S Low^W$ **cannot** be sustained as a PBE.

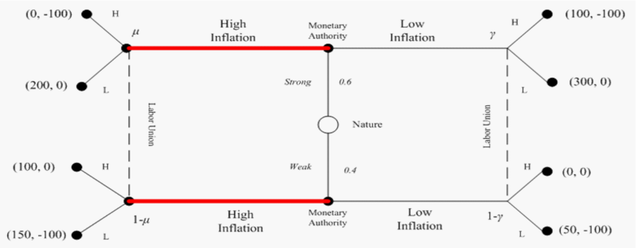
Pooling equilibrium with (High,High)

- Let us now test the pooling strategy profile $High^S High^W$.



- **Step #1:** Specifying strategy profile $High^S High^W$ that we will test.
  - (See shaded branches in the figure.)

Pooling equilibrium with (High,High)

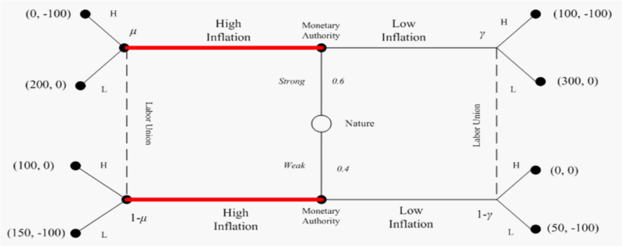


- **Step #2:** Updating beliefs
  - (a) After high inflation announcement

$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 1} = 0.6$$

so the high inflation announcement is uninformative.

Pooling equilibrium with (High,High)

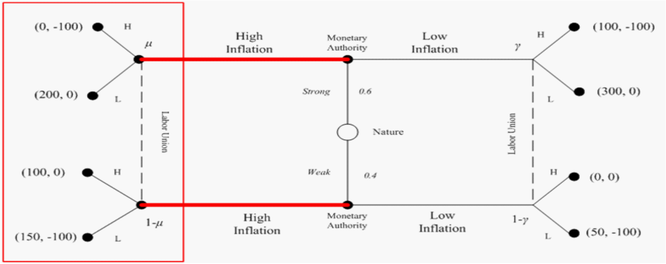


- **Step #2:** Updating beliefs
  - (b) After low inflation announcement (off-the-equilibrium path)

$$\gamma = \frac{0.6\left(1-\alpha^{Strong}\right)}{0.6\left(1-\alpha^{Strong}\right)+0.4\left(1-\alpha^{Weak}\right)} = \frac{0.6\times 0}{0.6\times 0+0.4\times 0} = \frac{0}{0}$$

hence, $\gamma \in [0,1]$.

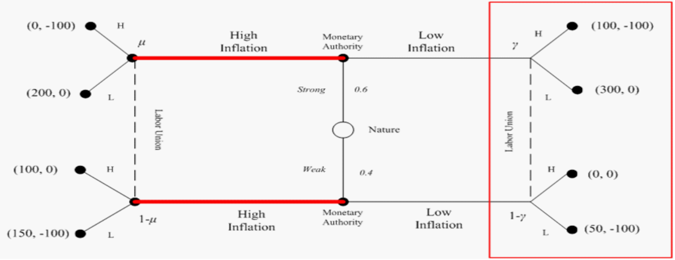
Pooling equilibrium with (High,High)



- **Step #3:** Optimal response
  - (a) After high inflation announcement (along the equil. path), respond with L since

$$EU_{Labor}(H|High) = 0.6 \times (-100) + 0.4 \times 0 = -60$$
$$EU_{Labor}(L|High) = 0.6 \times 0 + 0.4 \times (-100) = -40$$

Pooling equilibrium with (High,High)



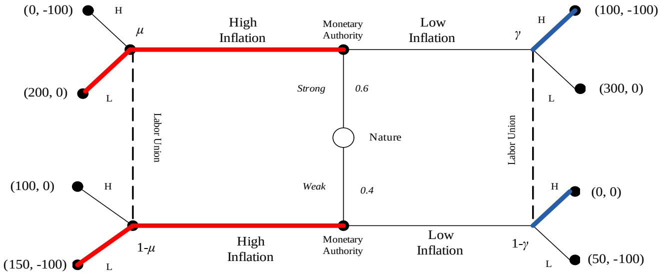
- **Step #3:** Optimal response
  - (a) After low inflation announcement (off-the-equil.),

$$EU_{Labor}(H|Low) = \gamma \times (-100) + (1-\gamma) \times 0 = -100\gamma$$
$$EU_{Labor}(L|Low) = \gamma \times 0 + (1-\gamma) \times (-100) = -100 + 100\gamma$$

i.e., respond with H if $\gamma < \frac{1}{2}$.

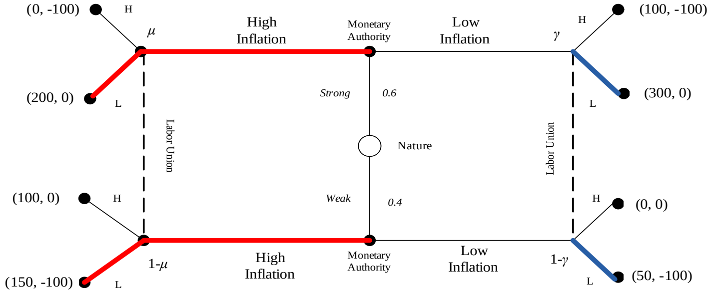
Pooling equilibrium with (High,High)

- Summarizing the optimal responses we found...
  - Note that we need to divide our analysis into two cases:
  - **Case 1**, where $\gamma < \frac{1}{2}$, implying that the labor union responds with H after observing low inflation (right-hand side).



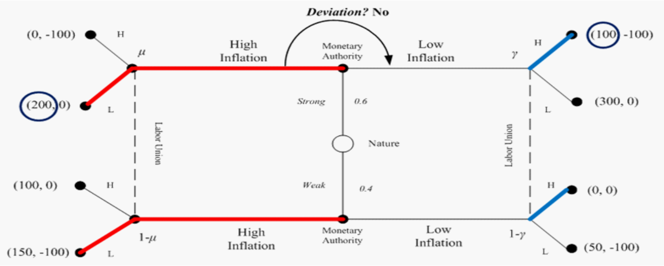
Pooling equilibrium with (High,High)

- and...
  - **Case 2**, where $\gamma \geq \frac{1}{2}$, implying that the labor union responds with L after observing low inflation (right-hand side).



Pooling equilibrium with (High,High)

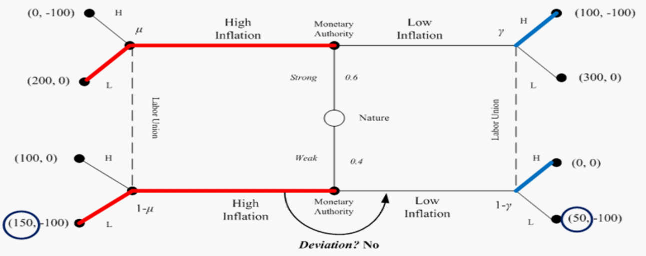
Case 1, where $\gamma < \frac{1}{2}$



- **Step #4:** Optimal messages
  - (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is \$200,
  - while if it deviates, its payoff decreases to \$100.
  - (No incentives to deviate).

Pooling equilibrium with (High,High)

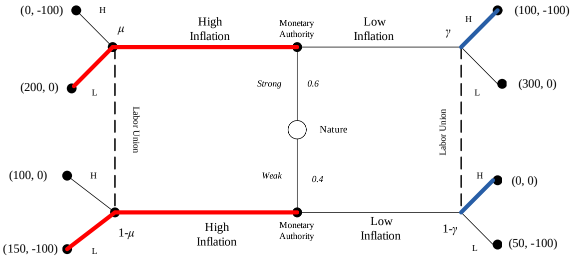
Case 1, where $\gamma < \frac{1}{2}$



- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is \$150,
  - while if it deviates, its payoff drops to \$0.
  - (No incentives to deviate either).

Pooling equilibrium with (High,High)

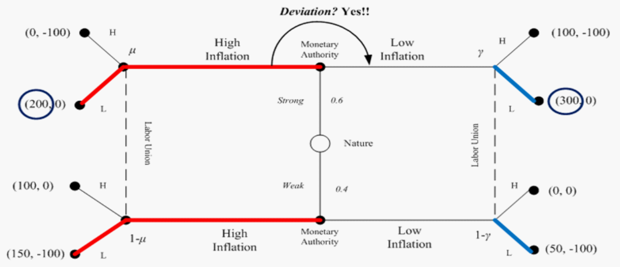
Case 1, where $\gamma < \frac{1}{2}$



- No type of monetary authority has incentives to deviate.
- Hence, the pooling strategy profile $High^S High^W$ **can** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma < \frac{1}{2}$.

Pooling equilibrium with (High,High)

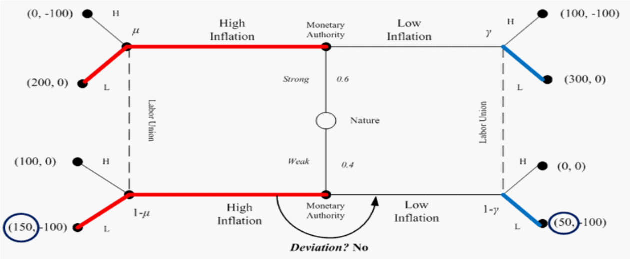
Case 2, where $\gamma \geq \frac{1}{2}$



- **Step #4:** Optimal messages
  - (a) When the monetary authority is Strong, if it chooses High (as prescribed), its payoff is \$200,
  - while if it deviates, its payoff **increases** to \$300.
  - (Incentives to deviate!!).

Pooling equilibrium with (High,High)

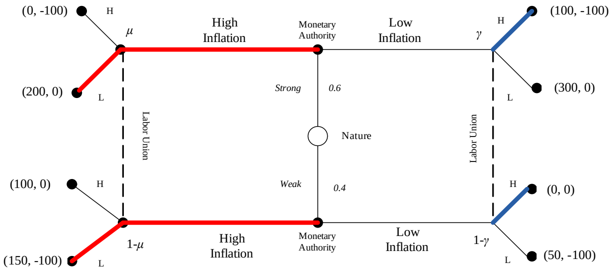
Case 2, where $\gamma \geq \frac{1}{2}$



- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is \$150,
  - while if it deviates, its payoff drops to \$50.
  - (No incentives to deviate).

Pooling equilibrium with (High,High)

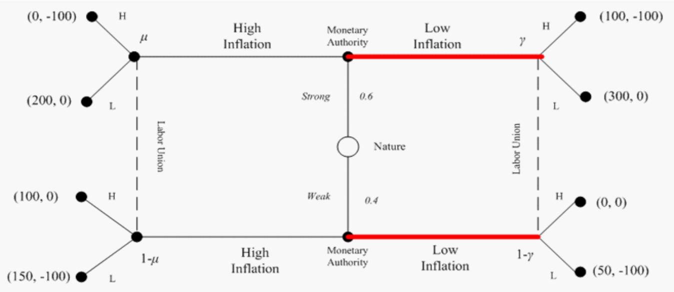
Case 2, where $\gamma \geq \frac{1}{2}$



- Since we found one type of privately informed player (the Strong monetary authority) who has incentives to deviate...
  - The pooling strategy profile $High^S High^W$ **cannot** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma \geq \frac{1}{2}$.

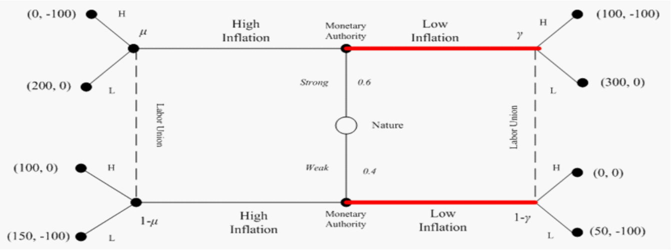
Pooling equilibrium with (Low,Low)

- Let us now examine the opposite pooling strategy profile.



- **Step #1:** Specifying strategy profile $Low^S Low^W$ that we will test.
  - (See shaded branches in the figure.)

Pooling equilibrium with (Low,Low)

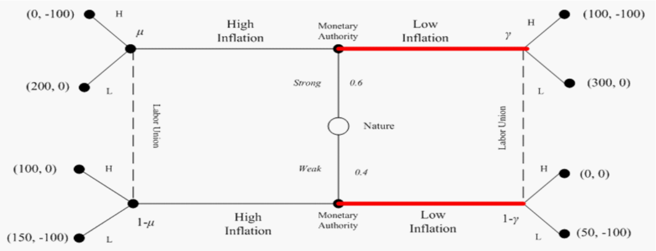


- **Step #2:** Updating beliefs
  - (a) After a low inflation announcement

$$\gamma = \frac{0.6\left(1-\alpha^{Strong}\right)}{0.6\left(1-\alpha^{Strong}\right)+0.4\left(1-\alpha^{Weak}\right)} = \frac{0.6\times 1}{0.6\times 1+0.4\times 1} = 0.6$$

so posterior and prior beliefs coincide.

Pooling equilibrium with (Low,Low)

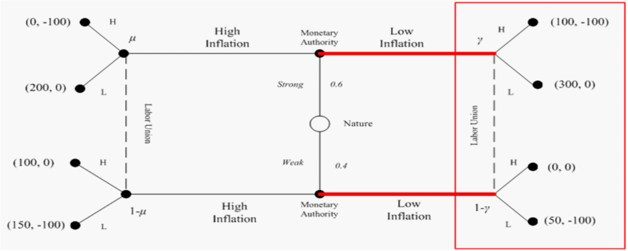


- **Step #2:** Updating beliefs
  - (b) After a high inflation announcement (off-the-equil. path)

$$\mu = \frac{0.6\alpha^{Strong}}{0.6\alpha^{Strong} + 0.4\alpha^{Weak}} = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 0} = \frac{0}{0}$$

hence, $\mu \in [0, 1]$.

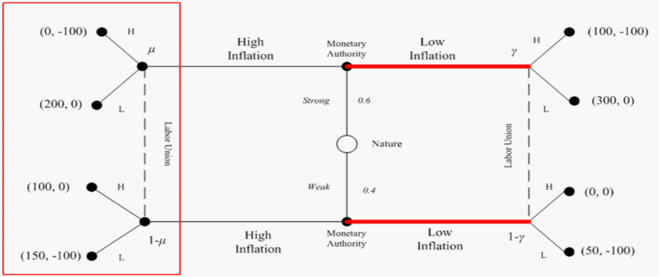
Pooling equilibrium with (Low,Low)



- **Step #3:** Optimal response
  - (a) After a low inflation announcement (along the equilibrium path), respond with L since

$$
\begin{aligned}
EU_{Labor}(H|Low) &= 0.6 \times (-100) + 0.4 \times 0 = -60 \\
EU_{Labor}(L|Low) &= 0.6 \times 0 + 0.4 \times (-100) = -40
\end{aligned}
$$

Pooling equilibrium with (Low,Low)



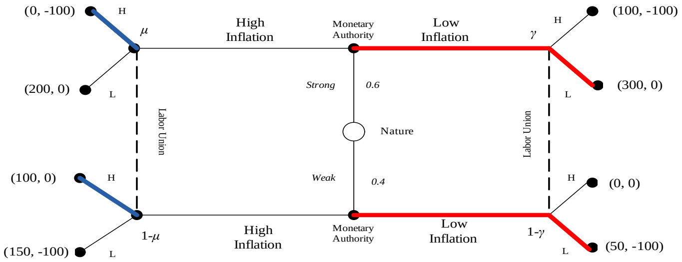
- **Step #3:** Optimal response
  - (a) After a high inflation announcement (off-the-equil.),

$$
\begin{aligned}
EU_{Labor}(H|Low) &= \mu \times (-100) + (1-\mu) \times 0 = -100\mu \\
EU_{Labor}(L|Low) &= \mu \times 0 + (1-\mu) \times (-100) = -100 + 100\mu
\end{aligned}
$$

i.e., respond with H if $\mu < \frac{1}{2}$.

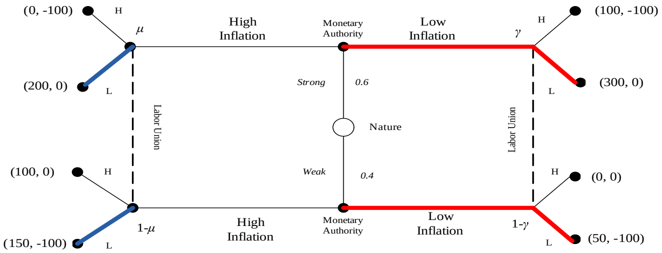
Pooling equilibrium with (Low,Low)

- Summarizing the optimal responses we found...
  - Note that we need to divide our analysis into two cases:
  - **Case 1**, where $\mu < \frac{1}{2}$, implying that the labor union responds with H after observing high inflation (left-hand side).



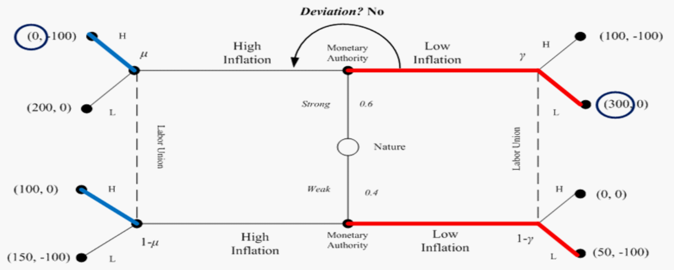
Pooling equilibrium with (Low,Low)

- and...
  - **Case 2**, where $\mu \geq \frac{1}{2}$, implying that the labor union responds with L after observing high inflation (left-hand side).



Pooling equilibrium with (Low,Low)

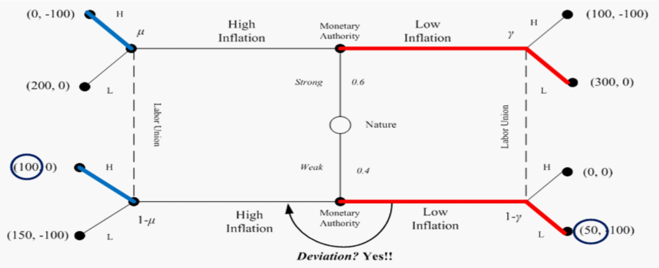
Case 1, where $\mu < \frac{1}{2}$



- **Step #4:** Optimal messages
  - (a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is $300,
  - while if it deviates, its payoff decreases to $200.
  - (No incentives to deviate).

Pooling equilibrium with (Low,Low)

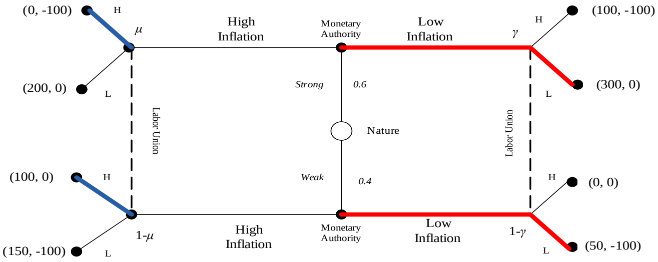
Case 1, where $\mu < \frac{1}{2}$



- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses High (as prescribed), its payoff is $50,
  - while if it deviates, its payoff **increases** to $100.
  - (Incentives to deviate!!).

Pooling equilibrium with (Low,Low)

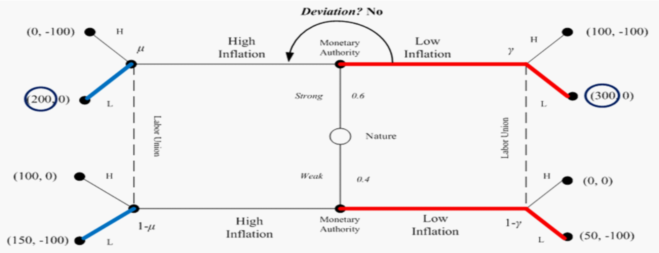
Case 1, where $\mu < \frac{1}{2}$



- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
  - The pooling strategy profile $Low^S Low^W$ **cannot** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu < \frac{1}{2}$.

Pooling equilibrium with (Low,Low)

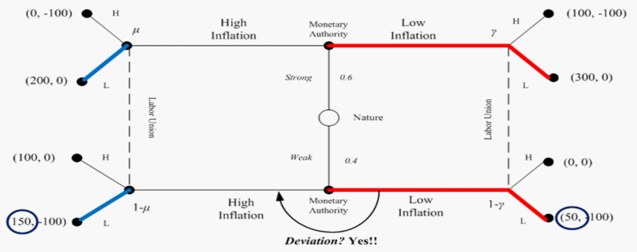
Case 2, where $\mu \geq \frac{1}{2}$



- **Step #4:** Optimal messages
  - (a) When the monetary authority is Strong, if it chooses Low (as prescribed), its payoff is \$300,
  - while if it deviates, its payoff decreases to \$200.
  - (No incentives to deviate).

Pooling equilibrium with (Low,Low)

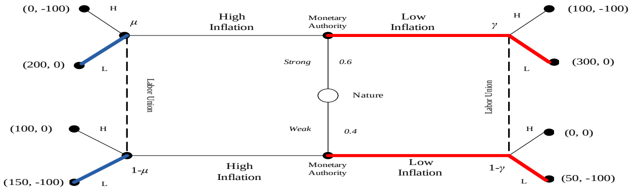
Case 2, where $\mu \geq \frac{1}{2}$



- **Step #4:** Optimal messages
  - (b) When the monetary authority is Weak, if it chooses Low (as prescribed), its payoff is \$50,
  - while if it deviates, its payoff **increases** to \$150.
  - (Incentives to deviate!!).

Pooling equilibrium with (Low,Low)

Case 2, where $\mu \geq \frac{1}{2}$



- Since we found one type of privately informed player (the Weak monetary authority) who has incentives to deviate...
  - The pooling strategy profile $Low^S Low^W$ **cannot** be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu \geq \frac{1}{2}$.
- Hence, the pooling strategy profile $Low^S Low^W$ **cannot** be sustained as a PBE for **any** off-the-equilibrium beliefs μ.