EconS 301

Math Review

Math Concepts

Functions: Functions describe the relationship between input variables and outputs

$$y = f(x)$$

where x is some input and y is some output.

Example: *x* could number of Bananas consumed and *y* could be utility derived from consuming the bananas. The function could tell us that when 4 bananas are consumed, the consumer's utility is 10.

OR

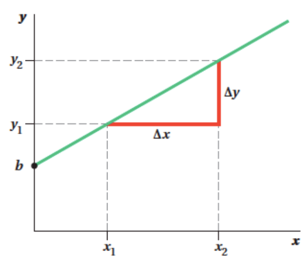
x could be number of workers hired to make shoes and *y* could be number of shoes produced. If the functional relationship were y = 3x then when 5 workers are hired the output is 15 shoes.

Lines and Curves: One important function is the one which describes a line graphed on an x-y plane, written in "slope-intercept form"

$$y = mx + b$$

where m is the slope of the curve and b is the intercept. The slope of a line describes the rate at which y changes as x varies. For a line, this value is constant and equal to

Figure A.1 Slope and Intercept of a Line



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Lines and Curves

- The slope of a line helps us answer two questions
- How steep or flat is the line?
- Is the relationship between x and y positive or negative?

While lines are straight by definition, curves can represent more complex relationships between y and x

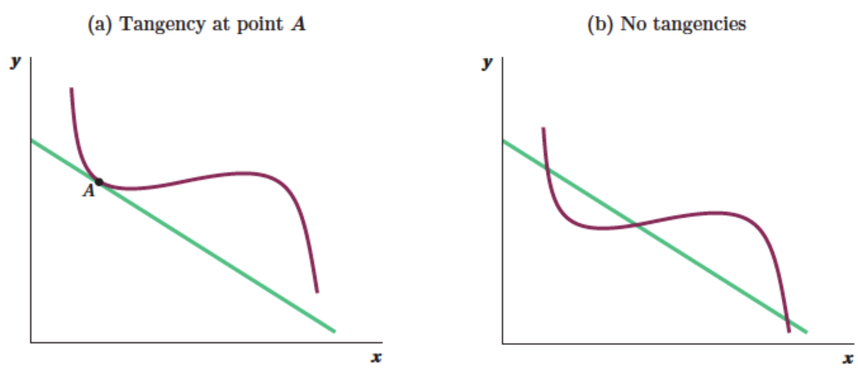
Tangency

The point at which a given line and curve touch without crossing is the tangent point, and is characterized by both the line and curve having the exact same slope

- Understanding tangency is important in microeconomics
- This concept is used to solve optimization problems such as profit or utility maximization, or cost minimization

Line and curve graph

Figure A.2 Tangency



Question:

A line has a slope of – 2 and an intercept of 10.

Answer the following:

a. Write the line in slope-intercept form.

b. Draw the line on a Cartesian plane (x-y axis).

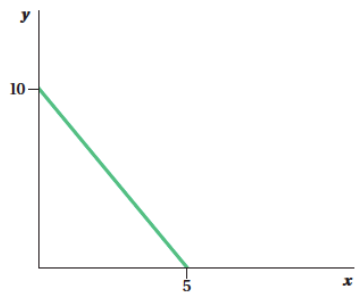
c. Draw a curve tangent to the line.

Answer:

a. The slope-intercept form of a line can be expressed as $y = mx + b$, where m is the slope and b is the intercept

$$y = -2x + 10$$

a. Draw the line on a Cartesian plane (x-y axis).



Calculus

First Derivatives

As mentioned, slopes are very important for economic analysis

- For a line, we can use the approach in the previous slides to determine the slope; along a curve, however, the slope is not constant
- To determine the slope of a curve, we need help from calculus in the form of a *first derivative*

Our function relating x to y is given as

$$y = f(x)$$

The first derivative describes the slope of this curve, and is written as

$$f'(x) = \frac{df(x)}{dx} = \frac{dy}{dx}$$

First Derivatives

There are a few rules for derivatives that work for specific functional forms

Derivative of a constant

A curve that is a constant is represented by: $y = f(x) = c$

For this curve, its slope is equal to zero: $\frac{df(x)}{dx} = 0$

This is because for this line, y does not change with x; thus, the slope must be zero and the derivative is zero.

The Power Rule for Derivatives

This is one of the most important rules, as it can be used for any curve that is described by x raised to an exponent

$$f(x) = cx^{\alpha}$$

For these curves, the derivative takes the form

$$\frac{df(x)}{dx} = c\alpha x^{\alpha-1}$$

In other words, you multiply x by its exponent and subtract 1 from the exponent you began with. For instance, if

$$f(x) = 3x^4 \rightarrow \frac{df(x)}{dx} = 3 \times 4 \times x^{4-1} = 12x^3$$

Addition and Subtraction Rules for Derivatives

Taking the derivative of a function of the form

$$f(x) = g(x) + h(x)$$

is as simple as taking the derivative of each sub-function and then adding them together

$$\frac{df(x)}{dx} = \frac{dg(x)}{dx} + \frac{dh(x)}{dx}$$

The same rule applies for the subtraction of derivatives

Questions:

Take the derivative of the following:

$a)$ $5x^2 + 4$

$b)$ $6x^3 + 2x^2$

$c)$ $3x^2 - x$

Answers:

a) $10x$
b) $18x^2 + 4x$
c) $6x - 1$

Partial Derivatives

So far we have only considered single-variable equations. Often, economic equations are multi-variable (for instance, demand for a product may be a function of both price and income)

$$z = f(x, y)$$

Given this equation, we can find two *partial derivatives*, one for x and one for y

$$f_x(x, y) = \frac{\partial f(x, y)}{\partial x} = \frac{\partial z}{\partial x}, \quad f_y(x, y) = \frac{\partial f(x, y)}{\partial y} = \frac{\partial z}{\partial y}$$

Partial Derivatives

Calculating partial derivatives is the same as calculating full derivatives of single-variable equations. Consider a Cobb-Douglas production function

$$f(x,y) = x^{\alpha} y^{1-\alpha}$$

We can use the power rule to find the two partial derivatives

$$f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = \alpha x^{\alpha-1} y^{1-\alpha}$$
$$f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = (1-\alpha) x^{\alpha} y^{-\alpha}$$

Basic algebra with exponents

Suppose that we want to find the ratio $\frac{f_x}{f_y}$ using the partial derivatives from the previous example. Therefore

$$\frac{f_x}{f_y} = \frac{\alpha x^{\alpha-1} y^{1-\alpha}}{(1-\alpha) x^{\alpha} y^{-\alpha}} \rightarrow \frac{\alpha x^{\alpha} x^{-1} y^{1} y^{-\alpha}}{(1-\alpha) x^{\alpha} y^{-\alpha}}$$

The exponents can be broken apart showing that x^{α} and $y^{-\alpha}$ appear in both the numerator and denominator so they cancel out and we get

$$\frac{\alpha x^{-1} y^{1}}{(1-\alpha)} \rightarrow \frac{\alpha y}{(1-\alpha) x}$$

Since $x^{-1} = \frac{1}{x}$

Unconstrained Optimization Problem

One of the most important uses of the calculus introduced here is unconstrained optimization. We begin with a function of the form

$$y = f(x)$$

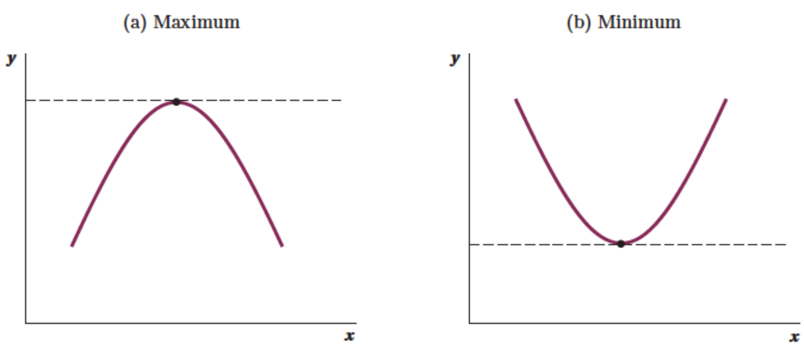
We then solve for the *first-order condition*, which is equivalent to taking the derivative of a function and setting it equal to zero

$$\frac{df(x)}{dx} = 0$$

Question: What does this tell us?

- Remember that the derivative of a function is equal to its slope
- When the slope is equal to zero, the line tangent to the curve is horizontal
- This means that the function must be at either a maximum or a minimum

Figure A.4 Optima



Question:

Find the maximum of the function of one variable, $\mathrm{f(x)} = 12x - 3x^2$.

Solution:

First, solve the first-order condition by setting the first derivative equal to zero

$$\frac{df(x)}{dx} = 12 - 6x = 0 \xrightarrow{\text{yields}} x = 2$$

The first-order condition indicates that $x = 2$ is a maximum.

Second order conditions

In the previous example you were told the equation had a maximum. But, suppose you were uncertain about whether the function had a maximum or a minimum.

For example optimize the following function $\mathrm{f(x)} = 3x^2 - 12x$

First, solve the first-order condition by setting the first derivative equal to zero

$$\frac{df(x)}{dx} = -12 + 6x = 0 \xrightarrow{\text{yields}} x = 2$$

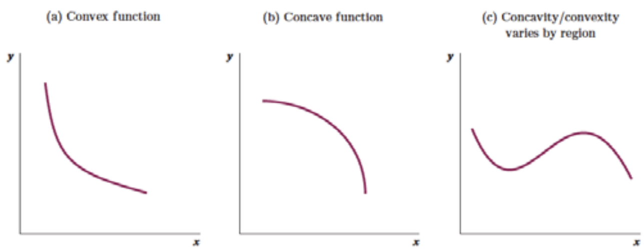
To determine whether this is a maximum or a minimum we need to find the derivative of a derivative. This is referred to as the *second derivative*.

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d^2 y}{dx^2}$$

What information does the second derivative give us?

- The first derivative tells us the slope of the function, while the second derivative tells us the slope of the slope, or the rate at which the slope changes at a given point on the curve
- In other words, the second derivative describes the *curvature* of the curve
- This can tell us whether a given curve is *concave* or *convex* to the origin

Figure A.3 Convexity and Concavity



Over some range of *x*, a curve is said to be *convex* if its second derivative is greater than zero

$$\frac{d^2 f(x)}{dx^2} > 0$$

Similarly, over some range of *x*, a curve is said to be *concave* if its second derivative is less than zero

$$\frac{d^2 f(x)}{dx^2} < 0$$

Back to the example

$$\frac{df(x)}{dx} = -12 + 6x \rightarrow \frac{d^2 f(x)}{dx^2} = 6$$

Which means this is a minimum.

Lets also verify the answer in the maximization problem. Remember,

$\frac{df(x)}{dx} = 12 - 6x \rightarrow \frac{d^2 f(x)}{dx^2} = -6$ Therefore this is a maximum.