Auction Theory - An Introduction

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Introduction

- Auctions are a large part of the economic landscape:
  - Since Babylon in 500 BC, and Rome in 193 AC
  - Auction houses Shotheby's and Christie's founded in 1744 and 1766.

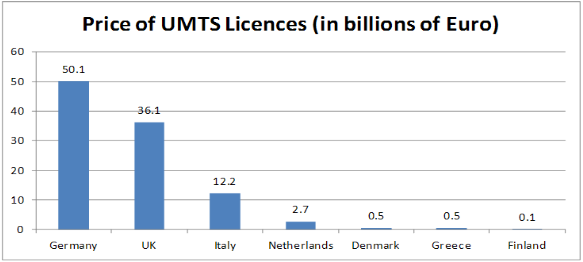
  - Munch's "The Scream," sold for US$119.9 million in 2012.

Introduction

- Auctions are a large part of the economic landscape:
  - More recently:
    - eBay: $11 billion in revenue, 27,000 employees.
    - Entry of more firms in this industry: QuiBids.com.

Introduction

- Also used by governments to sell:
  - Treasury bonds,
  - Air waves (3G technology):
    - British economists called the sale of the British 3G telecom licences "The Biggest Auction Ever" ($36 billion)
    - Several game theorists played an important role in designing the auction.



Overview

- Auctions as allocation mechanisms:
    - types of auctions, common ingredients, etc.
- First-price auction.
    - Optimal bidding function.
    - How is it affected by the introduction of more players?
    - How is it affected by risk aversion?
- Second-price auction.
- Efficiency.
- Common-value auctions.
    - The winner's curse.

Auctions

- N bidders, each bidder i with a valuation v_i for the object.
- One seller.
- We can design many different rules for the auction:
  1. **First price auction:** the winner is the bidder submitting the highest bid, and he/she must pay the *highest* bid (which is his/hers).
  2. **Second price auction:** the winner is the bidder submitting the highest bid, but he/she must pay the *second highest* bid.
  3. **Third price auction:** the winner is the bidder submitting the highest bid, but he/she must pay the *third highest* bid.
  4. **All-pay auction:** the winner is the bidder submitting the highest bid, but every single bidder must pay the price he/she submitted.

Auctions

- All auctions can be interpreted as allocation mechanisms with the following ingredients:
  1. **an allocation rule** (who gets the object):
     1. The allocation rule for most auctions determines the object is allocated to the individual submitting the highest bid.
     2. However, we could assign the object by a lottery, where $prob(win) = \frac{b_1}{b_1 + b_2 + \ldots + b_N}$ as in "Chinese auctions".
  2. **a payment rule** (how much every bidder must pay):
     1. The payment rule in the FPA determines that the individual submitting the highest bid pays his bid, while everybody else pays zero.
     2. The payment rule in the SPA determines that the individual submitting the highest bid pays the second highest bid, while everybody else pays zero.
     3. The payment rule in the APA determines that every individual must pay the bid he/she submitted.

Private valuations

- I know my own valuation for the object, v_i.
- I don't know your valuation for the object, v_j, but I know that it is drawn from a distribution function.

1. Easiest case:

$$v_j = \begin{cases} 10 \text{ with probability } 0.4, \text{ or} \\ 5 \text{ with probability } 0.6 \end{cases}$$

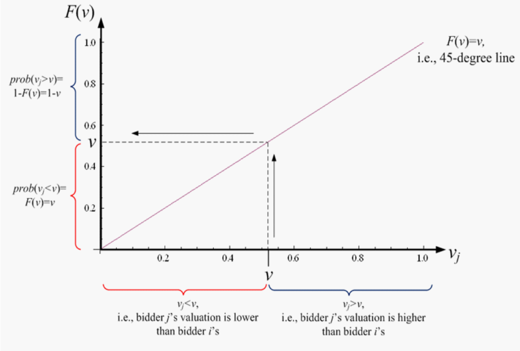
2. More generally,

$$F(v) = prob(v_j < v)$$

3. We will assume that every bidder's valuation for the object is drawn from a uniform distribution function between 0 and 1.

Private valuations

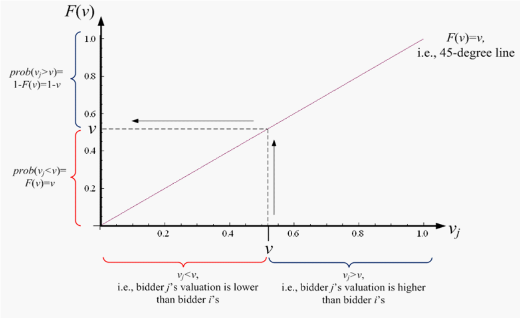
- Uniform distribution function $U[0, 1]$



- If bidder i's valuation is v, then all points in the horizontal axis where $v_j < v$, entail...
- Probability $prob(v_j < v) = F(v)$ in the vertical axis.

Private valuations

- Uniform distribution function $U[0, 1]$



- Similarly, valuations where $v_j > v$ (horizontal axis) entail:
- Probability $prob(v_j > v) = 1 - F(v)$ in the vertical axis.
  - Under a uniform distribution, implies $1 - F(v) = 1 - v$.

Private valuations

- Since all bidders are ex-ante symmetric...
- They will all be using the same bidding function:

$$b_i : [0, 1] \rightarrow \mathbb{R}_+ \quad \text{for every bidder } i$$

- They might, howver, submit different bids, depending on their privately observed valuation.
- **Example**:
  1. A valuation of $v_i = 0.4$ inserted into a bidding function $b_i(v_i) = \frac{v_i}{2}$, implies a bid of $b_i(0.4) = \$0.2$.
  2. A bidder with a higher valuation of $v_i = 0.9$ implies, in contrast, a bid of $b_i(0.9) = \frac{0.9}{2} = \$0.45$.
  3. Even if bidders are *symmetric* in the bidding function they use, they can be *asymmetric* in the actual bid they submit.

First-price auctions

- Let us start by ruling out bidding strategies that yield negative (or zero) payoffs, regardless of what your opponent does,
  - i.e., deleting dominated bidding strategies.
- Never bid **above your value**, $b_i > v_i$, since it yields a negative payoff if winning.

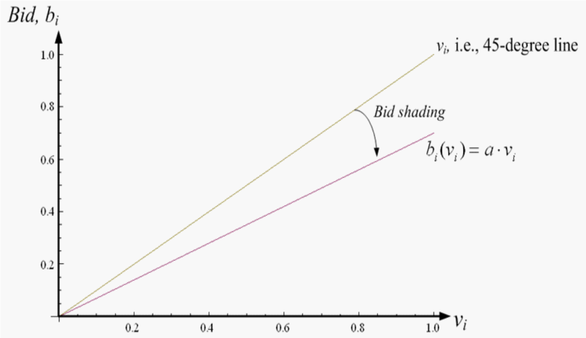
$$EU_i(b_i|v_i) = prob(win) \cdot \underbrace{(v_i - b_i)}_{-} + prob(lose) \cdot 0 < 0$$

- Never bid **your value**, $b_i = v_i$, since it yields a zero payoff if winning.

$$EU_i(b_i|v_i) = prob(win) \cdot \underbrace{(v_i - b_i)}_{0} + prob(lose) \cdot 0 = 0$$

First-price auctions

- Therefore, the only bidding strategies that can arise in equilibrium imply "bid shading,"
  - That is, $b_i < v_i$.
  - More specifically, $b_i(v_i) = a \cdot v_i$, where $a \in (0, 1)$.



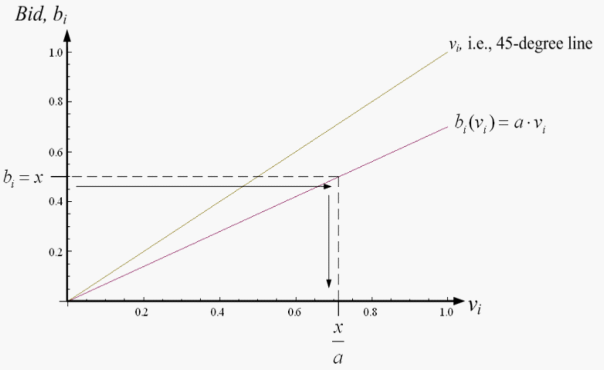
First-price auctions

- But, what is the precise value of parameter $a \in (0, 1)$.
  - That is, how much bid shadding?
- Before answering that question...
  - we must provide a more specific expression for the probability of winning in bidder i's expected utility of submitting a bid x,

$$EU_i(x|v_i) = prob(win) \cdot (v_i - x)$$

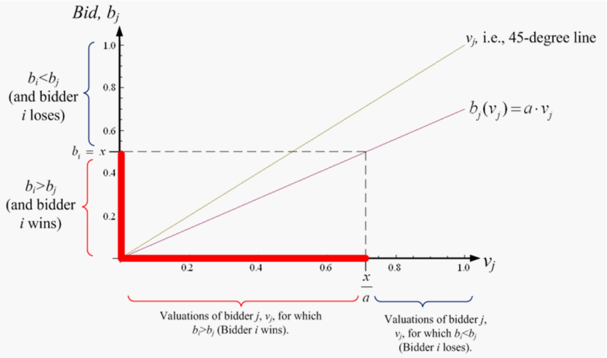
First-price auctions

- Given symmetry in the bidding function, bidder j can "recover" the valuation that produces a bid of exactly \$$x$.
  - From the vertical to the horizontal axis,
  - Solving for v_i in function $x = a \cdot v_i$, yields $v_i = \frac{x}{a}$



First-price auctions

- What is, then, the probability of winning when submitting a bid x is...
  - $prob(b_i > b_j)$ in the vertical axis, or
  - $prob(\frac{x}{a} > v_j)$ in the horizontal axis.



First-price auctions

- And since valuations are uniformly distributed...
  - $prob(\frac{x}{a} > v_j) = \frac{x}{a}$
  - which implies that the expected utility of submitting a bid x is...

$$EU_i(x|v_i) = \underbrace{\frac{x}{a}}_{prob(win)} (v_i - x)$$

- And simplifying...

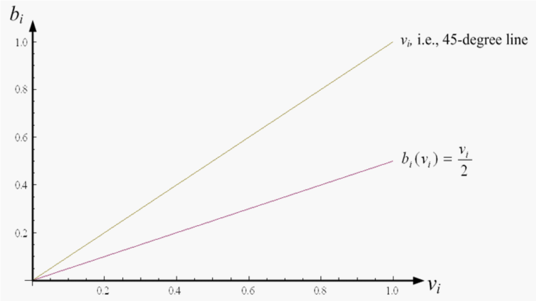
$$= \frac{xv_i - x^2}{a}$$

First-price auctions

- Taking first-order conditions of $\frac{xv_i - x^2}{a}$ with respect to x, we obtain
$$\frac{v_i - 2x}{a} = 0$$
and solving for x yields an optimal bidding function of
$$x(v_i) = \frac{1}{2}v_i.$$

Optimal bidding function in FPA

- $x(v_i) = \frac{1}{2}v_i$.



- *Bid shadding in half*:
  - for instance, when $v_i = 0.75$, his optimal bid is $\frac{1}{2}0.75 = 0.375$.

FPA with N bidders

- The expected utility is similar, but the probability of winning differs...

$$\begin{aligned} prob(win) &= \frac{x}{a} \cdot \ldots \cdot \frac{x}{a} \cdot \frac{x}{a} \cdot \ldots \cdot \frac{x}{a} \\ &= \left(\frac{x}{a}\right)^{N-1} \end{aligned}$$

- Hence, the expected utility of submitting a bid x is...

$$EU_i(x|v_i) = \left(\frac{x}{a}\right)^{N-1}(v_i - x) + \left[1 - \left(\frac{x}{a}\right)^{N-1}\right]0$$

FPA with N bidders

- Taking first-order conditions with respect to his bid, x, we obtain

$$-\left(\frac{x}{a}\right)^{N-1} + \left(\frac{x}{a}\right)^{N-2} \left(\frac{1}{a}\right) (v_i - x) = 0$$

- Rearranging,

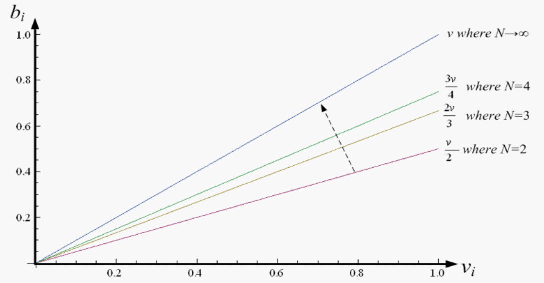
$$\left(\frac{x}{a}\right)^{N} \frac{a}{x^2} \left[(N-1)v_i - nx\right] = 0,$$

- and solving for x, we find bidder i's optimal bidding function,

$$x(v_i) = \frac{N-1}{N} v_i$$

FPA with N bidders

- Optimal bidding function $x(v_i) = \frac{N-1}{N} v_i$



- **Comparative statics:**
  - Bid shadding diminishes as N increases.
  - Bidding function approaches 45^0-line.

FPA - Generalization

- Let us now allow for valuations to be drawn from any cdf $F(v_i)$ (not necessarily uniform).
- First, note that, for a given bidding strategy $s : [0,1] \to \mathbb{R}_+$, i.e., $s(v_i) = x_i$, we can define its inverse $s^{-1}(x_i) = v_i$, implying that the cdf can be rewritten as

$$F(v_i) = F(s^{-1}(x_i)).$$

- Then bidder i's UMP becomes

$$\max_{x_i} \underbrace{\left[F(s^{-1}(x_i))\right]^{n-1}}_{\text{prob(win)}} (v_i - x_i)$$

FPA - Generalization

- Taking first-order conditions with respect to x yields

$$-\left[F\left(s^{-1}(x_i)\right)\right]^{n-1}+$$

$$(n-1)\left[F\left(s^{-1}(x_i)\right)\right]^{n-2}f\left(s^{-1}(x_i)\right)\frac{ds^{-1}(x_i)}{dx_i}(v_i-x_i)=0$$

- Since $s^{-1}(x_i)=v_i$ and $\frac{ds^{-1}(x_i)}{dx_i}=\frac{1}{s'\left(s^{-1}(x_i)\right)}$, the above expression becomes

$$-\left[F(v_i)\right]^{n-1}+(n-1)\left[F(v_i)\right]^{n-2}f(v_i)\frac{1}{s'(v_i)}(v_i-x_i)=0$$

FPA - Generalization

- Further rearranging, we obtain

$$
\begin{aligned}
& (n-1)\left[F(v_i)\right]^{n-2} f(v_i) v_i - (n-1)\left[F(v_i)\right]^{n-2} f(v_i) x_i \\
= \; & \left[F(v_i)\right]^{n-1} s'(v_i)
\end{aligned}
$$

or

$$
\begin{aligned}
& \left[F(v_i)\right]^{n-1} s'(v_i) + (n-1)\left[F(v_i)\right]^{n-2} f(v_i) v_i \\
= \; & (n-1)\left[F(v_i)\right]^{n-2} f(v_i) x_i
\end{aligned}
$$

The LHS is $\frac{d\left[\left[F(v_i)\right]^{n-1} s(v_i)\right]}{dv_i}$. Hence,

$$
\frac{d\left[\left[F(v_i)\right]^{n-1} s(v_i)\right]}{dv_i} = (n-1)\left[F(v_i)\right]^{n-2} f(v_i) x_i
$$

FPA - Generalization

- Integrating both sides yields

$$[F(v_i)]^{n-1}\, s(v_i) = \int_0^{v_i} (n-1)\,[F(v_i)]^{n-2}\, f(v_i) v_i \;\; dv_i \qquad (1)$$

Applying integration by parts on the RHS, we obtain

$$\int_0^{v_i} (n-1)\,[F(v_i)]^{n-2}\, f(v_i) v_i \;\; dv_i \qquad (2)$$

$$= \;\; [F(v_i)]^{n-1}\, v_i - \int_0^{v_i} [F(v_i)]^{n-1}\, dv_i \qquad (3)$$

Plugging that into the RHS of (1) yields

$$[F(v_i)]^{n-1}\, s(v_i) = [F(v_i)]^{n-1}\, v_i - \int_0^{v_i} [F(v_i)]^{n-1}\, dv_i \qquad (4)$$

- A note on integration by parts (next slide)

FPA - Generalization

- Recall integration by parts: You start from two functions g and h, so that $(gh)' = g'h + gh'$. Then, integrating both sides yields

$$g(x)h(x) = \int g'(x)h(x)dx + \int g(x)h'(x)dx$$

We can then reorder the terms in the above expression as follows

$$\int g'(x)h(x)dx = g(x)h(x) - \int g(x)h'(x)dx$$

FPA - Generalization

- In order to apply integration by parts in our auction setting, let $g'(x) \equiv (n-1)\left[F(v_i)\right]^{n-2} f(v_i)$ and $h(x) \equiv v_i$. That is

$$\int_0^{v_i} \underbrace{(n-1)\left[F(v_i)\right]^{n-2} f(v_i)}_{g'(x)} \underbrace{v_i}_{h(x)} dv_i$$

$$= \underbrace{\left[F(v_i)\right]^{n-1}}_{g(x)} \underbrace{v_i}_{h(x)} - \int_0^{v_i} \underbrace{\left[F(v_i)\right]^{n-1}}_{g(x)} \underbrace{1}_{h'(x)} dv_i$$

FPA - Generalization

- We can now rearrange expression (3). In particular, dividing both sides by $[F(v_i)]^{n-1}$ yields

$$s(v_i) = v_i - \frac{\int_0^{v_i} [F(v_i)]^{n-1} \, dv_i}{[F(v_i)]^{n-1}}$$

which is bidder i's optimal bidding function, $s(v_i)$.

- Intuitively, he shades his bid by the amount of ratio $\frac{\int_0^{v_i} [F(v_i)]^{n-1} dv_i}{[F(v_i)]^{n-1}}$.
- As a practice, note that when $F(v_i)$ is uniform, $F(v_i) = v_i$ implying that $[F(v_i)]^{n-1} = v_i^{n-1}$. Hence,

$$s(v_i) = v_i - \frac{\frac{1}{n} v_i^n}{v_i^{n-1}} = v_i - \frac{v_i^n}{n v_i^{n-1}} = v_i \left(\frac{n-1}{n} \right)$$

FPA with risk-averse bidders

- Utility function is concave in income, x, e.g., $u(x) = x^{\alpha}$,
  - where $0 < \alpha \leq 1$ denotes bidder i's risk-aversion parameter.
  - [Note that when $\alpha = 1$, the bidder is risk neutral.]
- Hence, the expected utility of submitting a bid x is

$$EU_i(x|v_i) = \underbrace{\frac{x}{a}}_{prob(win)} (v_i - x)^{\alpha}$$

FPA with risk-averse bidders

- Taking first-order conditions with respect to his bid, x,

$$\frac{1}{a}(v_i - x)^{\alpha} - \frac{x}{a}\alpha(v_i - x)^{\alpha-1} = 0,$$

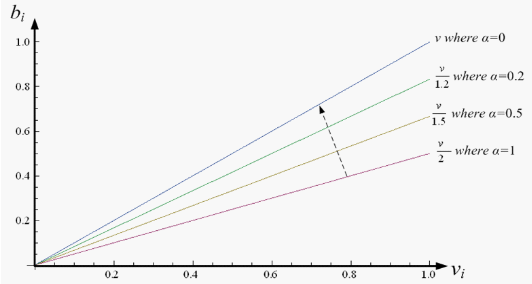
  and solving for x, we find the optimal bidding function,

$$x(v_i) = \frac{v_i}{1+\alpha}.$$

- Under risk-neutral bidders, $\alpha = 1$, this function becomes $x(v_i) = \frac{v_i}{2}$.
- But, what happens when α decreases (more risk aversion)?

FPA with risk-averse bidders

- Optimal bidding function $x(v_i) = \frac{v_i}{1+\alpha}$.



- Bid shading is *ameliorated* as bidders' risk aversion increases:
  - That is, the bidding function approaches the 45^0-line when α approaches zero.

FPA with risk-averse bidders

- **Intuition**: for a risk-averse bidder:
  - the **positive effect** of slightly lowering his bid, arising from getting the object at a cheaper price, is offset by...
  - the **negative effect** of increasing the probability that he loses the auction.
- Ultimately, the bidder's incentives to shade his bid are diminished.

Second-price auctions

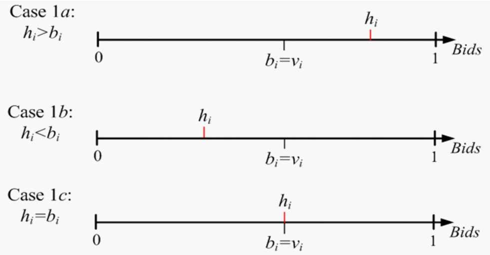
- Let's now move to second-price auctions.

Second-price auctions

- Bidding your own valuation, $b_i(v_i) = v_i$, is a weakly dominant strategy,
  - i.e., it yields a larger (or the same) payoff than submitting any other bid.
- In order to show this, let us find the expected payoff from submitting...
  - A bid that *coincides* with your own valuation, $b_i(v_i) = v_i$,
  - A bid that lies *below* your own valuation, $b_i(v_i) < v_i$, and
  - A bid that lies *above* your own valuation, $b_i(v_i) > v_i$.
- We can then compare which bidding strategy yields the largest expected payoff.

Second-price auctions

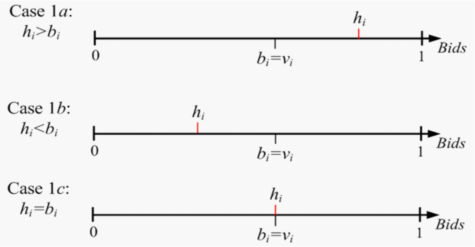
- Bidding your own valuation, $b_i(v_i) = v_i$...



- **Case 1a:** If his bid lies below the highest competing bid, i.e., $b_i < h_i$ where $h_i = \max_{j \neq i}\{b_j\}$,
  - then bidder i loses the auction, obtaining a zero payoff.

Second-price auctions

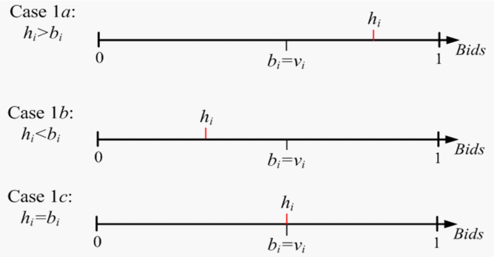
- Bidding your own valuation, $b_i(v_i) = v_i$...



- **Case 1b:** If his bid lies above the highest competing bid, i.e., $b_i > h_i$, then bidder i wins.
  - He obtains a net payoff of $v_i - h_i$.

Second-price auctions

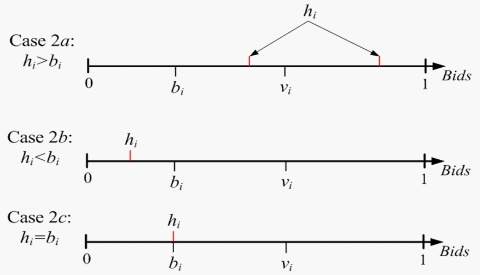
- Bidding your own valuation, $b_i(v_i) = v_i$...



- **Case 1c:** If, instead, his bid coincides with the highest competing bid, i.e., $b_i = h_i$, then a tie occurs.
  - For simplicity, ties are solved by randomly assigning the object to the bidders who submitted the highest bids.
  - As a consequence, bidder i's expected payoff becomes $\frac{1}{2}(v_i - h_i)$.

Second-price auctions

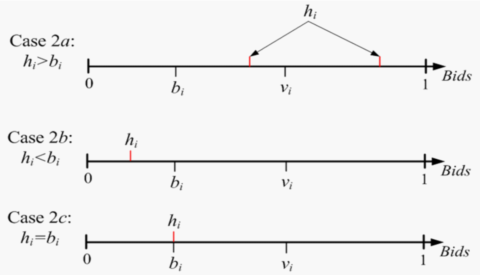
- Bidding *below* your valuation, $b_i(v_i) < v_i$...



- **Case 2a:** If his bid lies below the highest competing bid, i.e., $b_i < h_i$,
  - then bidder i loses, obtaining a zero payoff.

Second-price auctions

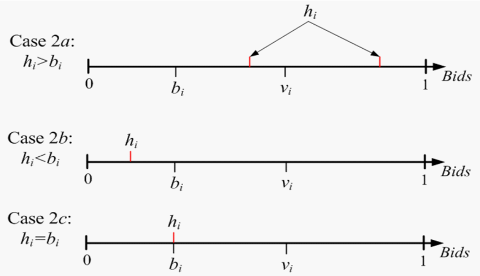
- Bidding *below* your valuation, $b_i(v_i) < v_i$...



- **Case 2b:** if his bid lies above the highest competing bid, i.e., $b_i > h_i$,
  - then bidder i wins, obtaining a net payoff of $v_i - h_i$.

Second-price auctions

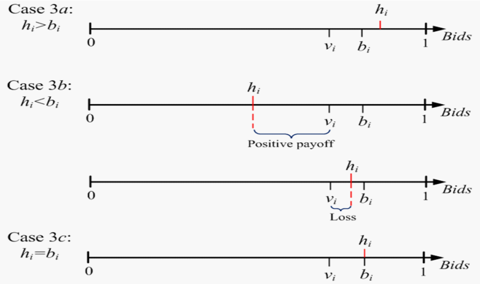
- Bidding *below* your valuation, $b_i(v_i) < v_i$...



- **Case 2c:** If, instead, his bid coincides with the highest competing bid, i.e., $b_i = h_i$, then a tie occurs,
  - and the object is randomly assigned, yielding an expected payoff of $\frac{1}{2}(v_i - h_i)$.

Second-price auctions

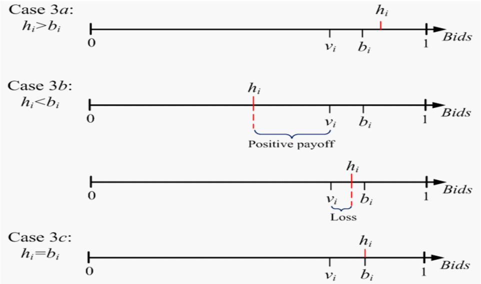
- Bidding *above* your valuation, $b_i(v_i) > v_i$...



- **Case 3a:** if his bid lies below the highest competing bid, i.e., $b_i < h_i$,
  - then bidder i loses, obtaining a zero payoff.

Second-price auctions

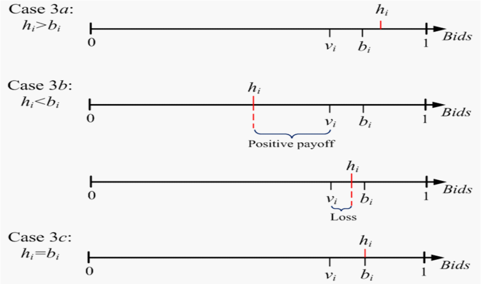
- Bidding *above* your valuation, $b_i(v_i) > v_i$...



- **Case 3b:** if his bid lies above the highest competing bid, i.e., $b_i > h_i$, then bidder i wins.
  - His payoff becomes $v_i - h_i$, which is positive if $v_i > h_i$, or negative otherwise.

Second-price auctions

- Bidding *above* your valuation, $b_i(v_i) > v_i$...



- **Case 3c:** If, instead, his bid coincides with the highest competing bid, i.e., $b_i = h_i$, then a tie occurs.
  - The object is randomly assigned, yielding an expected payoff of $\frac{1}{2}(v_i - h_i)$, which is positive only if $v_i > h_i$.

Second-price auctions

- **Summary:**
  - Bidder i's payoff from submitting a bid *above* his valuation:
    - either coincides with his payoff from submitting his own value for the object, or
    - becomes strictly lower, thus nullifying his incentives to deviate from his equilibrium bid of $b_i(v_i) = v_i$.
- Hence, there is no bidding strategy that provides a strictly higher payoff than $b_i(v_i) = v_i$ in the SPA.
- All players bid their own valuation, without shading their bids,
  - unlike in the optimal bidding function in FPA.

Second-price auctions

- **Remark:**
  - The above equilibrium bidding strategy in the SPA is unaffected by:
    - the number of bidders who participate in the auction, N, or
    - their risk-aversion preferences.

Efficiency in auctions

- The object is assigned to the bidder with the highest valuation.
  - Otherwise, the outcome of the auction cannot be efficient...
  - since there exist alternative reassignments that would still improve welfare.
  - FPA and SPA are, hence, efficient, since:
  - The player with the highest valuation submits the highest bid and wins the auction.
  - Lottery auctions are not necessarily efficient.

Common value auctions

- In some auctions all bidders assign the same value to the object for sale.
  - *Example*: Oil lease
  - Same profits to be made from the oil reservoir.

Common value auctions

- Firms, however, do not precisely observe the value of the object (profits to be made from the reservoir).
- Instead, they only observe an estimate of these potential profits:
    - from a consulting company, a bidder/firm's own estimates, etc.

Common value auctions

- Consider the auction of an oil lease.
- The true value of the oil lease (in millions of dollars) is $v \in [10, 11, ..., 20]$
- Firm A hires a consultant, and gets a signal s

$$s = \begin{cases} v + 2 \text{ with prob } \frac{1}{2} \text{ (overestimate)} \\ v - 2 \text{ with prob } \frac{1}{2} \text{ (underestimate)} \end{cases}$$

That is, the probability that the true value of the oil lease is v, given that the firm receives a signal s, is

$$prob(v|s) = \begin{cases} \frac{1}{2} \text{ if } v = s - 2 \text{ (overestimate)} \\ \frac{1}{2} \text{ if } v = s + 2 \text{ (underestimate)} \end{cases}$$

Common value auctions

- If firm A was not participating in an auction, then the expected value of the oil lease would be

$$\underbrace{\frac{1}{2}(s-2)}_{\text{if overestimation}} + \underbrace{\frac{1}{2}(s+2)}_{\text{if underestimation}} = \frac{s-2+s+2}{2} = \frac{2s}{2} = s$$

- Hence, the firm would pay for the oil lease a price $p < s$, making a positive expected profit.

Common value auctions

- What if the firm participates in a FPA for the oil lease against firm B?
- Every firm uses a different consultant...
  - but they don't know if their consultant systematically overestimates or underestimates the value of the oil lease.
- Every firm receives a signal s from its consultant,
  - observing its own signal, but not observing the signal the other firm receives, every firm submits a bid from $\{1, 2, ..., 20\}$.

Common value auctions

- We want to show that bidding $b = s - 1$ cannot be optimal for any firm.
- Notice that this bidding strategy seems sensible at first glance:
  - Bidding less than the signal, $b < s$.
    - So, if the true value of the oil lease was s, the firm would get some positive expected profit from winning.
  - Bidding is increasing in the signal that the firm receives.

Common value auctions

- Let us assume that firm A receives a signal of $s = 10$.
  - Then it bids $b = s - 1 = 10 - 1 = \$9$.
- Given such a signal, the true value of the oil lease is

$$v = \begin{cases} s + 2 = 12 \text{ with prob } \frac{1}{2} \\ s - 2 = 8 \text{ with prob } \frac{1}{2} \end{cases}$$

- In the first case (true value of 12)
  - firm A receives a signal of $s_A = 10$ (underestimation), and
  - firm B receives a signal of $s_B = 14$ (overestimation).
- Then, firms bid $b_A = 10 - 1 = 9$, and $b_B = 14 - 1 = 13$, and firm A loses the auction.

Common value auctions

- In the second case, when the true value of the oil lease is $v = 8$,
  - firm A receives a signal of $s_A = 10$ (overestimation), and
  - firm B receives a signal of $s_B = 6$ (underestimation).
- Then, firms bid $b_A = 10 - 1 = 9$, and $b_B = 6 - 1 = 5$, and firm A wins the auction.
  - However, the winner's expected profit becomes

$$\frac{1}{2}(8-9) + \frac{1}{2}0 = -\frac{1}{2}$$

- Negative profits from winning.
- Winning is a curse!!

Winner's curse

- In auctions where all bidders assign the same valuation to the object (common value auctions),
    - and where every bidder receives an inexact signal of the object's true value...
- The fact that you won...
    - just means that you received an overestimated signal of the true value of the object for sale (oil lease).
- How to avoid the winner's curse?
    - Bid $b = s - 2$ or less,
    - take into account the possibility that you might be receiving overestimated signals.

Winner's curse - Experiments I

- **In the classroom:** Your instructor shows up with a jar of nickels,
  - which every student can look at for a few minutes.

  - Paying too much for it!

Winner's curse - Experiments II

- **In the field:** Texaco in auctions selling the mineral rights to off-shore properties owned by the US government.
  - All firms avoided the winner's curse (their average bids were about 1/3 of their signal)...
  - Expect for Texaco:
    - Not only their executives fall prey of the winner's curse,
    - They submitted bids above their own signal!
    - They needed some remedial auction theory!